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On Some Weak Monomorphisms and Weak Epimorphisms of Pro-HTop*

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ABSTRACT. Related to Shape Theory, in a previous paper [6] we studied weak monomorphisms and weak epimorphisms in the category of pro-groups. In this note we give some intrinsic characterizations of the weak monomorphisms and the weak epimorphisms in pro-HTop* in the case when one of the two objects of such a morphism is a rudimentary system.

1. INTRODUCTION

If C is a category with zero-objects then a morphism $f: A \to B$ of C is a weak monomorphism if $f \circ u = 0$ implies u = 0. A morphism $f: A \to B$ is called a weak epimorphism if $u \circ f = 0$ implies u = 0.

Weakened versions of categorical notions of monomorphism and epimorphism have proved to be of some interest in pointed homotopy theory. A study of the comparison between weak monomorphism and monomorphism in homotopy theory was carried by T.Ganea [3] who, in

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particular, obtained examples of weak monomorphisms which are not monomorphisms. Examples of homotopy weak monomorphisms which are not homotopy epimorphisms have been given by J.Roitberg [7]. Certainly, the study of shape monomorphisms and epimorphisms and their weakened versions can be interesting (see, for the homotopy case, the recently papers of E.Dyer & J.Roitberg [2] and J.Dydak [1]). In [6] we characterized weak monomorphisms and weak epimorphisms in the category of pro-groups and we defined the notion of weakly exact sequence and we studied this notion in the category of pro-groups.

In this note we consider the pro-category of HTop*, the homotopy category of pointed topologal spaces, and we give some intrinsic characterizations of weak monomorphisms and weak epimorphisms $\underline{f}: X \to \underline{Y}$ in pro-HTop*, when \underline{X} or \underline{Y} is a rudimentary system. These results can be interesting (and maybe sufficient) so a shape morphism $F: X \to Y$ between topological spaces X and Y can be given by means of such morphisms $\underline{f}: \underline{X} \to \underline{Y}$ in pro-HTop* (approaching morphisms). The study of an arbitrary morphism $f: \underline{X} \to \underline{Y}$ of pro-HTop* is more complicated.

The notions and properties of pro-categories which are used in this paper are those of the book of S.Mardešić and J.Segal [4].

2. WEAK MONOMORPHISMS IN THE CATEGORY PROHTOP*

The category pro-HTop* is a category with zero objects. A zero-object is a single point rudimentary system.

If (X,*) is a rudimentary system in pro-HTop* and if $\underline{Y}=((Y_{\lambda},*),q_{\lambda\lambda'},\Lambda)$ is an arbitrary object in pro-HTop*, then the morphisms $\underline{f}=(f_{\lambda}): (X,*) \to \underline{Y}$ coincide with the morphisms in inv-HTop*, the category of inverse systems in HTop* [4, p.20]. This means that for each $\lambda \in \Lambda$ is given a morphism $f_{\lambda}: (X,*) \to (Y_{\lambda},*)$ in HTop* and for each pair $\lambda \leq \lambda'$ we have $q_{\lambda\lambda'}f_{\lambda'}=f_{\lambda}$.

Lemma 1. For a morphism $\underline{f}: (X,*) \to \underline{Y} = ((Y_{\lambda},*), q_{\lambda\lambda'}, \Lambda)$ in pro-HTop*, there exist an object $\underline{P} = ((P_{\lambda},*), r_{\lambda\lambda'}, \Lambda)$ and two morphisms $\underline{p} = (p_{\lambda}, 1_{\Lambda}): \underline{P} \to \underline{Y}, \underline{h} = (h_{\lambda}): (X,*) \to \underline{P}$ such that for each $\lambda \in \Lambda$:

(i) $h_{\lambda}: (X, *) \to (P_{\lambda}, *)$ is a pointed homotopy equivalence,

- (ii) $p_{\lambda}:(P,*)\to (Y_{\lambda},*)$ is a pointed fiber map,
- (iii) $f_{\lambda} = p_{\lambda} \circ h_{\lambda}$.

Proof. The existence for each $\lambda \in \Lambda$ of a factorization (iii), satisfying (i) and (ii), is well known [5, p.249]. For a pair $\lambda \leq \lambda'$ in Λ we define $r_{\lambda\lambda'} = h_{\lambda} \circ h_{\lambda'}^{-1}$, for which is immediate that $\underline{P} = ((P_{\lambda}, *), r_{\lambda\lambda'}, \Lambda)$ is an inverse system in HTop* and that $\underline{h} = (h_{\lambda}) : (X, *) \to \underline{P}$ is a morphism in pro-HTop*. Also, from the relations $q_{\lambda\lambda'} \circ f_{\lambda'} \approx f_{\lambda}, f_{\lambda} = p_{\lambda}h_{\lambda}, f_{\lambda'} = p_{\lambda'} \circ h_{\lambda'}$, we deduce that $q_{\lambda\lambda'}p_{\lambda} \approx p_{\lambda'} \circ r_{\lambda\lambda'}$, which shows that $\underline{P} = (p_{\lambda}, 1_{\Lambda}) : \underline{P} \to \underline{Y}$ a morphism of pro-HTop*.

Remark 1. It is obvious from Lemma 1 that we can write the equality $\underline{f} = \underline{p} \circ \underline{h}$, in the category pro-HTop*, where \underline{h} is an isomorphism. Then it is clear that \underline{f} is a weak monomorphism if and only if \underline{p} is a weak monomorphism. We will refer to the morphism $\underline{p} : \underline{P} \to \underline{Y}$ as the fibred factor of the morphism f.

Remark 2. If $\underline{p}=(p_{\lambda},1_{\Lambda}):\underline{P}=((P_{\lambda},*),r_{\lambda\lambda'},\Lambda)\to\underline{Y}=((Y_{\lambda},*),p_{\lambda\lambda'},\Lambda)$ is a fibred factor in pro-HTop*, we can consider the *fiber* of this morphism by the object $\underline{F}=((F_{\lambda},*),r'_{\lambda\lambda'},\Lambda)$, where $F_{\lambda}=p_{\lambda}^{-1}(*)$ and $r'_{\lambda\lambda'}=r_{\lambda\lambda'}/F_{\lambda'}$, for $\lambda\leq\lambda'$. Then we can define a morphism $\underline{i}=(i_{\lambda},1_{\Lambda}):\underline{F}\to\underline{P}$, where i_{λ} is the inclusion of $(F_{\lambda},*)$ in $(P_{\lambda},*)$.

Definition 1. We will say that the fiber $\underline{F} = ((F_{\lambda}, *), r'_{\lambda\lambda'}, \Lambda)$ of the fibered factor $\underline{p} = (p_{\lambda}, 1_{\Lambda}) : \underline{p} = ((P_{\lambda}, *), r_{\lambda\lambda'}, \Lambda) \to \underline{Y} = ((Y_{\lambda}, *), p_{\lambda\lambda'}, \Lambda)$ is contractible in \underline{P} if for each $\lambda \in \Lambda$ there is a $\lambda' \geq \lambda$ such that $i_{\lambda} \circ r'_{\lambda\lambda'} \approx *$.

Theorem 1. A morphism $\underline{f} = (\underline{f_{\lambda}}) : (X, *) \to \underline{Y}$ is a weak monomorphism in the category pro-HTop* if and only if the fibre \underline{F} of every fibred factor $p = \underline{P} \to \underline{Y}$ is contractible in \underline{P} .

Proof. By Remark 1 it is sufficient to prove that \underline{p} is a weak monomorphism if and only if \underline{F} is contractible in \underline{P} .

Suppose that $\underline{p}: \underline{P} \to \underline{Y}$ is a weak monomorphism in the category pro-HTop*. For the morphism $\underline{i}: \underline{F} \to \underline{P}$ from Remark 2 we have $\underline{p} \circ \underline{i} = \underline{*}$ and by hypothesis it follows $\underline{i} = \underline{*}$. If $\underline{*} = (*, \Phi)$ then we have an equivalence $(i_{\lambda}, 1_{\Lambda}) \sim (*, \Phi)$ [4, \underline{p} .6] which implies that for each

 $\lambda \in \Lambda$ there is $\lambda' \geq \lambda$ (and $\lambda' \geq \Phi(\lambda)$) such that the following diagram in HTop* commutes

$$F_{\Phi(\lambda)} \leftarrow F_{\lambda'} \qquad F_{\lambda'} \\ * \downarrow \qquad \qquad \downarrow_{r_{\lambda\lambda'}} \\ P_{\lambda} \leftarrow F_{\lambda}$$

This implies that $i_{\lambda} \circ r'_{\lambda \lambda'}$ is pointed null-homotopic, i.e. $i_{\lambda} \circ r'_{\lambda \lambda'} \approx *$. Thus, \underline{F} is contractible in \underline{P} .

Conversely, suppose that the libre \underline{F} is contractible in \underline{P} and let $\underline{u} = (u_{\lambda}, \Phi) : \underline{Z} = ((Z_{\mu}, *), s_{\mu\mu'}, M) \to \underline{P} = ((P_{\lambda}, *), r_{\lambda\lambda'}, \Lambda)$ be a morphism, such that $\underline{p} \circ \underline{u} = \underline{*}$. But $\underline{p} \circ \underline{u} = (p_{\lambda} \circ u_{\lambda}, \Phi)$, with the function $\Phi : \Lambda \to M$, and $p_{\lambda} \circ u_{\lambda} : Z_{\Phi(\lambda)} \to \overline{P_{\lambda}} \to Y_{\lambda}$. This relation implies that each $\lambda \in \Lambda$ admits $\mu \in M$, $\mu \geq \Phi(\lambda)$ such that $p_{\lambda} \circ u_{\lambda} \circ s_{\Phi(\lambda)\mu} \approx *$, by a pointed homotopy $H_{\lambda} : Z_{\mu} \times [0,1] \to Y_{\lambda}$. Then, by the homotopy covering property of p_{λ} , there exists a pointed homotopy $K_{\lambda} : Z_{\mu} \times [0,1] \to P_{\lambda}$ such that $K_{\lambda}(\cdot,0) = u_{\lambda} \circ s_{\Phi(\lambda)\mu}$ and $p_{\lambda} \circ K_{\lambda} = H_{\lambda}$. Thus we have $u_{\lambda} \circ s_{\Phi(\lambda)\mu} \approx K_{\lambda}(\cdot,1)$ and $Im K_{\lambda} \subseteq F_{\lambda}$. By the proof of Lemma 1 and since the index sets are directed, for each $\lambda \in \Lambda$ we can choose the indices $\lambda' \in \Lambda$ and $\mu, \mu' \in M$ such that $i_{\lambda} \circ r_{\lambda \lambda'} \approx *$ and the following diagram commutes

This means that $u_{\lambda}s_{\phi(\lambda)\mu} \approx i_{\lambda}r_{\lambda\lambda'}K_{\lambda'}(\cdot,1)s_{\mu'\mu} \approx *$, i.e. $(u_{\lambda},\Phi) \sim (*,\Phi')$ for satisfactory function $\Phi': \Lambda \to M$. Thus we obtained $\underline{u} = \underline{*}$, what finishes the proof of the theorem.

Remark 3. If $f:(X,*) \to (Y,*)$ is a pointed continuous map then f is a weak monomorphism in HTop* if and only if it is a weak monomorphism in pro-HTop*. Theorem 1 generalizes the usual result for pointed continuous map [7, Prop. 2.2, (ii)].

3. WEAK EPIMORPHISMS IN THE CATEGORY PROHTOP*

In this section we consider only morphisms of the form $\underline{f}: \underline{X} \to (Y,*)$, where \underline{X} is an arbitrary inverse system in HTop*. In fact the morphism \underline{f} can be represented by a continuous map $f_{\lambda}: (X_{\lambda},*) \to (Y,*)$, if $\underline{X} = ((X_{\lambda},*), p_{\lambda\lambda'}, \Lambda)$, and two such maps $f_{\lambda_1}, f_{\lambda_2}$ define the same morphism \underline{f} if there is $\lambda \geq \lambda_1, \lambda_2$ such that $f_{\lambda}p_{\lambda_1\lambda} = f_{\lambda_2}p_{\lambda_2\lambda}$ in HTop*.

Lemma 2. For a morphism $\underline{f}: \underline{X} = ((X_{\lambda}, *), p_{\lambda\lambda'}, \Lambda) \to (Y, *)$ there exist an object $\underline{M} = ((M_{\lambda}, *), r_{\lambda\lambda'}, \Lambda), \Phi: \Lambda' \hookrightarrow \Lambda$ and two morphisms $\underline{j} = (j_{\lambda}, \Phi): \underline{X} \to \underline{M}, \underline{h}: \underline{M} \to (Y, *),$ such that for each $\lambda \in \Lambda'$:

- (i) $h_{\lambda}: (M_{\lambda}, *) \to (Y, *)$ is a pointed homotopy equivalence,
- (ii) $j_{\lambda}: (X_{\lambda}, *) \to (M_{\lambda}, *)$ is a pointed cofiber inclusion map,
- (iii) $f_{\lambda} = h_{\lambda} \circ j_{\lambda}$.

Proof. Denote by Λ' the subset of Λ such that an index λ is in Λ' if and only if there is a map $f_{\lambda}: (X_{\lambda}, *) \to (Y, *)$ defining f.

The existence for each $\lambda \in \Lambda'$ of a factorization (iii) satisfying (i) and (ii) is well known [5, p.246]. For a pair $\lambda \leq \lambda'$ in Λ' define $r_{\lambda\lambda'} = h_{\lambda}^{-1}$ o $h_{\lambda'}$, from which is immediate that $\underline{M} = (M_{\lambda}, *), r_{\lambda\lambda'}, \Lambda'$) is an inverse system in HTop* and that all maps $h_{\lambda}, \lambda \in \Lambda'$ define the same morphism $\underline{h} : \underline{M} \to (Y, *)$. Finally, if $\Phi : \Lambda' \hookrightarrow \Lambda$ is the inclusion function, then $\underline{j} = (j_{\lambda}, \Phi) : \underline{X} \to \underline{M}$ is a morphism of pro-HTop*.

Remark 4. It is obvious from Lemma 2 that we can write $\underline{f} = \underline{h} \circ \underline{j}$, in the category pro-HTop*, where \underline{h} is an isomorphism. Then it is clear that \underline{f} is a weak epimorphism if and only if \underline{j} is a weak epimorphism. We will refer to the morphism $\underline{j}: \underline{X} \to \underline{M}$ as the *cofibred factor* of the morphism \underline{f} .

Remark 5. Let $\underline{j}=(j_{\lambda},\Phi):\underline{X}=((X_{\lambda},*),p_{\lambda\lambda'},\Lambda)\to\underline{M}=((M_{\lambda},*),r_{\lambda\lambda'},\Lambda')$ be a cofibred factor in pro-HTop*. Then for each $\lambda\in\Lambda'$ we can consider the pointed quotient space M_{λ}/X_{λ} with the pointed identification map $\pi_{\lambda}:M_{\lambda}\to M_{\lambda}/X_{\lambda}$. We can consider the

inverse system $\underline{M}/\underline{X}=((M_{\lambda}/X_{\lambda},*),\bar{r}_{\lambda\lambda'},\Lambda')$ and the morphism $\underline{\pi}=(\pi_{\lambda},1_{\Lambda'}):\underline{M}\to\underline{M}/\underline{X}$. For the morphism $\underline{f}:\underline{X}\to(Y,*)$ we will say that (Y,*) is contractible in the cofibred factor of f if each $\lambda\in\Lambda'$ admits a $\lambda'\geq\lambda$ such that $\pi_{\lambda}\circ r_{\lambda\lambda'}\approx*$.

Theorem 2. A morphism $\underline{f}: \underline{X} \to (Y, *)$ of pro-HTop* is a weak epimorphism if and only if (Y, *) is contractible in every cofibred factor.

Proof. Suppose that \underline{f} is a weak epimorphism, what is equivalent to the fact that the morphism $\underline{j}:\underline{X}\to\underline{M}$ is a weak epimorphism. Since $\underline{\pi}\circ j=\underline{*}$, the hypothesis implies $\underline{\pi}=\underline{*}$ in pro HTop*. This means $(\pi_{\lambda},1_{\Lambda'})\sim(*,\Phi')$, i.e. each $\lambda\in\Lambda'$ admits a $\lambda'\geq\lambda$ such that $\pi_{\lambda}\circ\tau_{\lambda\lambda'}=*$ in HTop*. Thus, (Y,*) is contractible in the cofibred factor $j:\underline{X}\to\underline{M}$ of f.

Conversely, suppose that (Y, *) is contractible in the cofibred factor of f. It is sufficient to prove that f is a weak epimorphism. For this, suppose that for a morphism $\underline{u} = (u_{\nu}, \Psi) : \underline{M} \to \underline{Z} = ((Z_{\nu}, *), s_{\nu\nu'}, N)$ we have $\underline{u} \circ \underline{j} = \underline{*}$. This implies that for each $\nu \in N$ there is a pointed homotopy $H_{\nu}: u_{\nu} \circ j_{\Psi(\nu)} \approx *$. Then, by the pointed homotopy extension property of the pair $(M_{\Psi(\nu)}, X_{\Psi(\nu)})$ there exists a pointed homotopy K_{ν} : $M_{\Psi(\nu)} \times [0,1] \to Z_{\nu}$, such that $K_{\nu}(\cdot,0) = u_{\nu}$ and $K_{\nu}/K_{\Psi(\nu)} \times [0,1] = H_{\nu}$. Now, if we consider the pointed map $\varphi_{\nu}:\ M_{\Psi(\nu)} \to Z_{\nu}, \varphi_{\nu}=K_{\nu}(\cdot,1),$ then we have $\varphi_{\nu}/X_{\Psi(\nu)}=K_{\nu}/X_{\Psi(\nu)}\times\{1\}=H_{\nu}(\cdot,1)=*$. Therefore, we can define the pointed map $\tilde{\varphi}_{\nu}$: $M_{\Psi(\nu)}/X_{\Psi(\nu)} \to Z_{\nu}$, such that $ilde{arphi}_
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u$ and the pointed homotopy $ilde{arphi}_
u \circ F_{\Psi(
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ightarrow$ $M_{\Psi(\nu)}/X_{\Psi(\nu)} o Z_{
u},$ where $F_{\Psi(\nu)}:\ \pi_{\Psi(
u)}\circ r_{\Psi(
u)\lambda'}pprox *,$ for a convenient $\lambda' \geq \Psi(\nu)$. For this we have $\tilde{\varphi}_{\Psi(\nu)} \circ F_{\Psi(\nu)} : \varphi_{\nu} \circ r_{\Psi(\nu)\lambda} \approx * \text{ in Top*. On}$ the other hand K_{ν} is a pointed homotopy, $K_{\nu}: u_{\nu} \circ r_{\Psi(\nu)\lambda} \approx \varphi_{\nu} \circ r_{\Psi(\nu)\lambda}$, and therefore $\tilde{\varphi}_{\nu} \circ F_{\Psi(\nu)} \circ K_{\nu} : u_{\nu} r_{\Psi(\nu)\lambda} \approx *$. This proves the equivalence $(u_{\nu}, \Psi) \sim (*, \Psi)$ for every $\nu \in N$ and therefore $\underline{u} = \underline{*}$, what finishes the proof of the theorem.

Remark 6. If $f:(X,*) \to (Y,*)$ is a pointed continuous map then f is a weak epimorphism in HTop* if and only if it is a weak epimorphism in pro-HTop*. Particularly, Theorem 2 generalizes the usual intrinsic characterization of a weak epimorphism in HTop* [7, Prop. 2.2 (i)].

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