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On an Inequality of Gauss

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ABSTRACT. In this note we prove a new extension and a converse of an inequality due to Gauss.

The following theorem which is important in Statistics (see [1, p. 183 and p. 256]) is due to C.F. Gauss:

If $f:[0,\infty)\rightarrow\mathfrak{R}$ is a decreasing function, then we have for all real numbers $k>0$:

$$k^2\int_k^\infty f(x)\,dx\leq\frac{4}{9}\int_0^\infty x^2 f(x)\,dx. \tag{1}$$

Interesting generalizations of this result were given by V. N. Volkov [3] and, recently, by J. E. Pečarić [2].

The aim of this paper is two-fold. On the one hand we establish a new extension of inequality (1) and on the other hand we present a converse of (1) which provides (under the additional assumption that f is nonnegative) a lower bound for

$$k^2\int_k^\infty f(x)\,dx.$$

First we prove the following proposition:

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Theorem. *Let $g:[a,b]\to\mathfrak{R}$ be strictly increasing, convex and differentiable, and let $f:I\to\mathfrak{R}$ be decreasing. Then*

$$\int_a^b f(s(x))\,g'(x)\,dx \le \int_{g(a)}^{g(b)} f(x)\,dx \le \int_a^b f(t(x))\,g'(x)\,dx \tag{2}$$

where

$$s(x)=\frac{g(b)-g(a)}{b-a}(x-a)+g(a)$$

and

$$t(x)=g'(x_0)\,(x-x_0)+g(x_0)\,,\ x_0\in[a,b]\,.$$

($I\subset\mathfrak{R}$ is an interval containing a, b, $g(a)$, $g(b)$, $t(a)$ and $t(b)$.)

If either g is concave (instead of convex) or f is increasing, then the reversed inequalities hold.

Proof. Let g be convex and let f be decreasing. We denote by h the function

$$h(x)=f(g(x));$$

then h is also decreasing. Since g is convex we obtain for all $x\in[a,b]$:

$$t(x)\le g(x)\le s(x)\,.$$

This implies

$$g^{-1}(t(x))\le x\le g^{-1}(s(x))$$

and

$$h(g^{-1}(t(x)))\ge h(x)\ge h(g^{-1}(s(x)))\,,$$

where g^{-1} designates the inverse function of g.

Because of $g'\ge 0$ we conclude

$$h(g^{-1}(t(x)))\,g'(x)\ge h(x)\,g'(x)\ge h(g^{-1}(s(x)))\,g'(x)$$

and integration yields

$$\int_a^b h(g^{-1}(t(x)))\,g'(x)\,dx \geq \int_a^b h(x)\,g'(x)\,dx$$

$$\geq \int_a^b h(g^{-1}(s(x)))\,g'(x)\,dx\,. \tag{3}$$

Finally, from (3) and

$$\int_a^b h(x)\,g'(x)\,dx = \int_{g(a)}^{g(b)} h(g^{-1}(y))\,dy$$

(which follows immediately from the substitution $y=g(x)$) we get

$$\int_a^b f(t(x))\,g'(x)\,dx \geq \int_{g(a)}^{g(b)} f(x)\,dx \geq \int_a^b f(s(x))\,g'(x)\,dx\,.$$

Similarly we can verify, if either g is concave (instead of convex) or f is increasing, then in the last inequalities we have to replace "$\geq$" by "$\leq$". This completes the proof. □

An application of the Theorem leads to a new proof and to a converse of inequality (1).

Corollary. If $f\colon [0,\infty)\to\mathfrak{R}$ *is decreasing, then we have for all real positive real numbers* k:

$$k^2\int_k^\infty f(x)\,dx \leq \frac{4}{9}\int_0^\infty x^2 f(x)\,dx\,,$$

and, under the additional assumption that f *is nonnegative we obtain*

$$3\int_0^k x^2 f(x+k)\,dx \leq k^2\int_k^\infty f(x)\,dx\,, \tag{4}$$

where the constant 3 cannot be replaced by a greater number.

Proof. Let $a=0$, $b\geq x_0=k/2^{1/3}$ and $g(x)=(1/k^2)\,x^3+k$.

Then we have

$$t(x)=(3/4^{1/3})\,x$$

and from the right-hand side of (2) we conclude

$$\int_k^{k+b^3/k^2} f(x)\,dx \leq \frac{3}{k^2}\int_0^b x^2 f(3x/4^{1/3})\,dx = \frac{4}{9k^2}\int_0^{3b/4^{1/3}} x^2 f(x)\,dx\,.$$

If b tends to ∞ we obtain inequality (1).

In order to prove (4) we set $a=0$, $b=k$ and $g(x)=(1/k^2)\,x^3+k$.

Then we get

$$s(x)=x+k$$

and from the first inequality of (2) we obtain

$$\frac{3}{k^2}\int_0^k x^2 f(x+k)\,dx \leq \int_k^{2k} f(x)\,dx\,. \tag{5}$$

Since f is nonnegative we conclude

$$\int_k^{2k} f(x)\,dx \leq \int_k^{\infty} f(x)\,dx \tag{6}$$

such that (5) and (6) imply inequality (4).

If we put

$$f(x)=\begin{cases} 1, & 0\leq x\leq 2k\,,\\ 0, & 2k<x\,,\end{cases}$$

then equality holds in (4). Therefore, the constant 3 is best possible. □

References

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