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Optimal Control of Quasilinear Elliptic Equations with non Differentiable Coefficients at the Origin

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ABSTRACT. In this paper we study some optimal control problems of systems governed by quasilinear elliptic equations in divergence form with non differentiable coefficients at the origin. We prove existence of solutions and derive the optimality conditions by considering a perturbation of the differential operator coefficients that removes the singularity at the origin. Regularity of optimal controls is also deduced.

1. INTRODUCTION

We will be considering optimal control problems involving the differential operator

$$Ay = -\operatorname{div}(\varphi(x, |\nabla y|)\nabla y) + \psi(x, y) \tag{1.1}$$

with φ: $\Omega\times(0,+\infty)\to(0,+\infty)$ and ψ: $\Omega\times\mathfrak{R}\to\mathfrak{R}$, where Ω is a bounded open subset of $\mathfrak{R}^N$ with Lipschitz continuous boundary Γ.

Authors have studied control problems associated with quasilinear elliptic operators in [2, 3, 5]. The novelty of this work is that the non differentiability of $\varphi(x,.)$ at 0 is allowed, which causes the non differentiability of state with respect to the control. This is not an obstacle to prove existence of optimal controls, but it becomes complicated to derive the optimality conditions. To overcome this difficulty, we introduce a family of approximating control problems that fall in the class of problems treated in [3,5].

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Operator A introduced above is less general than those ones studied in [2, 3, 5]. Nevertheless, most of the known quasilinear elliptic operators belong to this class. Next, we establish the hypotheses on A.

Let b: $\Omega\times[0,+\infty)\rightarrow[0,+\infty)$ be the function defined by

$$b(x,s)=\varphi(x,s)\,s$$

We will assume the following conditions

$$b\in C(\Omega\times[0,+\infty))\cap C^1(\Omega\times(0,+\infty)) \tag{1.2}$$

$$\begin{cases}\psi(.,s) \text{ is a measurable function on } \Omega\\ \psi(x,.) \text{ belongs to } C^1(\mathfrak{R})\end{cases} \tag{1.3}$$

$$\Lambda_1(k+s)^{\alpha-2}\leq\frac{\partial b}{\partial s}(x,s)\leq\Lambda_2(k+s)^{\alpha-2}\qquad\forall s\in(0,+\infty) \tag{1.4}$$

$$\sum_{i=1}^{N}\left|\frac{\partial b}{\partial x_i}(x,s)\right|\leq\Lambda_2(k+s)^{\alpha-2}s\qquad\forall s\in(0,+\infty) \tag{1.5}$$

$$0\leq\frac{\partial\psi}{\partial s}(x,s)\leq f(|s|)\qquad\forall s\in\mathfrak{R} \tag{1.6}$$

$$b(x,0)=\psi(x,0)=0 \tag{1.7}$$

for some $k\in[0,1]$, some $\alpha\in(1,+\infty)$, some strictly positive constants Λ_1, Λ_2, some positive and non decreasing function f and a. e. $x\in\Omega$.

Let us consider the boundary value problem:

$$\begin{cases}Ay=v & \text{in } \Omega\\ y=0 & \text{on } \Gamma\end{cases} \tag{1.8}$$

We make the following additional assumption on α

$$\alpha>N/2 \tag{1.9}$$

In the sequel, $W^{-1,\beta}(\Omega)$ will denote the dual of the usual Sobolev space $W_0^{1,\alpha}(\Omega)$ $(\frac{1}{\alpha}+\frac{1}{\beta}=1)$ and $\langle\ldots\rangle$ their duality product. Also, $D(\Omega)$ will denote the space of infinitely differentiable functions with a compact support in Ω.

We will prove that problem (1.8) has a unique solution $y_\nu \in W_0^{1,\alpha}(\Omega) \cap L^\infty(\Omega)$ for each $\nu \in L^2(\Omega)$. Hypothesis (1.9) is essential to deduce the boundedness of the solution y_ν.

Remark:

Hypothesis (1.7) can be weakened in the following form:

$$\psi(\cdot, 0) \in L^2(\Omega)$$

In this case it is enough to do the change

$$\overline{\psi}(x, s) = \psi(x, s) - \psi(x, 0)$$

Let us give some examples of the principal part of operators A that satisfy previous hypotheses with $b \notin C^1(\Omega \times [0, +\infty))$:

Example 1.—(Case $\alpha \geq 2$, $k \neq 0$)

$$\varphi(x, s) = \lambda(x) + \sin(\ln s) + s^{\alpha-2}$$

with $\lambda \in C^1(\overline{\Omega})$ such that $2 < \lambda(x) \quad \forall x \in \overline{\Omega}$.

Example 2.—(Case $\alpha < 2$, $k = 0$)

$$\varphi(s) = s^{\alpha-2}$$

Example 3.—(Case $\alpha < 2$, $k \neq 0$)

$$\varphi(x, s) = \sin^3 |\ln s|^{1/2} . \exp(-s) + \lambda(x)(k + s)^{\alpha-2}$$

with $\lambda \in C^1(\overline{\Omega})$ such that $0 < \lambda_0 < \lambda(x) \quad \forall x \in \Omega$ for a sufficiently large constant λ_0.

If $\alpha > 2$ and $k = 0$, it follows easily from (1.2), (1.5), (1.4) and (1.7) that $b \in C^1(\Omega \times [0, +\infty))$ with $\dfrac{\partial b}{\partial s}(x, 0) = 0$. Therefore, the operator A satisfies the hypotheses of papers [3, 5], as we will see later (lemma 2.2).

Let us introduce the following optimal control problem:

$$(P_\alpha)\begin{cases}\text{Minimize } J(v)\\ v\in\mathcal{K}\end{cases}\tag{1.10}$$

where $\mathcal{K}$ is a non empty, convex and closed subset of $L^2(\Omega)$ and $J: L^2(\Omega)\to\mathfrak{R}$ is the functional defined by

$$J(v)=\frac{1}{2}\int_\Omega |y_v-y_d|^2\,dx+\frac{\rho}{2}\int_\Omega |v|^2\,dx\tag{1.11}$$

with y_d a fixed element of $L^2(\Omega)$ and ρ a non negative constant. Let us remark that thanks to (1.9), it is verified the continuous imbedding

$$L^2(\Omega)\subset W^{-1,\beta}(\Omega)$$

Remark

If we suppress (1.9), it is possible to carry out the study of (P_α), substituting (1.6) by condition

$$0\leq\frac{\partial\psi}{\partial s}(x,s)\leq\Lambda_2(k+|s|)^{\alpha-2}\tag{1.6*}$$

for all $x\in\Omega$ and all $s\in\mathfrak{R}$.

Nevertheless, for $1<\alpha\leq\dfrac{2N}{N+2}$ and $N\geq3$, we must formulate (P_α) in a slight different form. Variations are motivated by the fact that in this case $W_0^{1,\alpha}(\Omega)$ is not imbedded in $L^2(\Omega)$ (see Casas and Fernández [2]).

The plan of the paper is as follows: in next section, we state some auxiliary lemmas about the differential operator and state equation; in Section 3, we prove existence of solutions and formulate the optimality necessary conditions for (P_α); Sections 4 and 5 are devoted to the proof of these optimality conditions; in last section, we obtain $H^1(\Omega)$ regularity (resp. $W^{1,\alpha}(\Omega)$ if $\alpha<2$) for optimal controls.

2. SOME AUXILIARY LEMMAS

In this section, we prove existence, uniqueness and continuous dependence of solutions of Dirichlet problem associated with operator A as well as some perturbed operators A_ε. We begin showing some properties about the coefficients:

Lemma 2.1. *Let us suppose (1.2), (1.4) and $b(x, 0)=0$. Then, there are positive constants Λ_3 and Λ_4 depending only on α, Λ_1 and Λ_2 such that*

$$\Lambda_3 (k+s)^{\alpha-2} \leq \varphi(x, s) \leq \Lambda_4 (k+s)^{\alpha-2} \qquad \forall x \in \Omega, \quad \forall s \in (0, +\infty)$$

Proof

Using (1.2) and the fact that $b(x, 0)=0$, we get

$$\varphi(x, s)\, s = b(x, s) = \int_0^s \frac{\partial b}{\partial s}(x, t)\, dt$$

In virtue of (1.4), we have that

$$\Lambda_1 \int_0^s (k+t)^{\alpha-2}\, dt \leq \int_0^s \frac{\partial b}{\partial s}(x, t)\, dt \leq \Lambda_2 \int_0^s (k+t)^{\alpha-2}\, dt$$

If $\alpha \geq 2$, it is clear that

$$\Lambda_1 \int_0^s (k+t)^{\alpha-2}\, dt = \Lambda_1 \frac{(k+s)^{\alpha-1} - k^{\alpha-1}}{\alpha-1} =$$

$$= \Lambda_1 \frac{k(k+s)^{\alpha-2} - k^{\alpha-1} + s(k+s)^{\alpha-2}}{\alpha-1} \geq \frac{\Lambda_1}{\alpha-1}(k+s)^{\alpha-2} s.$$

If $\alpha \leq 2$, applying the mean value theorem, it follows that

$$\Lambda_1 \frac{(k+s)^{\alpha-1} - k^{\alpha-1}}{\alpha-1} = \Lambda_1 (k+\theta s)^{\alpha-2} s \geq \Lambda_1 (k+s)^{\alpha-2} s$$

because $\theta \in (0, 1)$.

In any case, we obtain that

$$\Lambda_3 (k+s)^{\alpha-2} \leq \varphi(x, s) \qquad \forall x \in \Omega, \qquad \forall s \in (0, +\infty)$$

with $\Lambda_3 = \dfrac{\Lambda_1}{\alpha - 1}$ (*if* $\alpha \geq 2$) and $\Lambda_3 = \Lambda_1$ (if $\alpha \leq 2$).

For the upper bound, let us remark that if $\alpha \geq 2$, it is verified that

$$\Lambda_2 \int_0^s (k+t)^{\alpha-2}\, dt \leq \Lambda_2 (k+s)^{\alpha-2} s$$

and if $\alpha \leq 2$,

$$\Lambda_2 \int_0^s (k+t)^{\alpha-2}\, dt = \Lambda_2 \frac{(k+s)^{\alpha-1} - k^{\alpha-1}}{\alpha - 1} \leq \frac{\Lambda_2}{\alpha - 1} (k+s)^{\alpha-2} s$$

Thus, we deduce that $\Lambda_4 = \dfrac{\Lambda_2}{\alpha - 1}$ (if $\alpha \leq 2$) and $\Lambda_4 = \Lambda_2$ (if $\alpha \geq 2$). ∎

Next, we introduce the perturbed differential operator coefficients and state the coercivity and growth conditions.

Lemma 2.2. *Let us suppose (1.2), (1.4), (1.5) and $b(x, 0) = 0$. For each $\varepsilon \geq 0$, let us introduce $a^\varepsilon : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ defined by*

$$a^\varepsilon(x, \eta) = \varphi(x, \varepsilon + |\eta|)\, \eta$$

Then, $a^\varepsilon \in C^1(\Omega \times \mathbb{R}^N)$ if $\varepsilon > 0$ and $a^0 \in C(\Omega \times \mathbb{R}^N) \cap C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))$.

Moreover, there exist a positive constant Λ_5 depending only on N, α, Λ_1 and Λ_2 such that

$$\sum_{i,j=1}^N \frac{\partial a_j^\varepsilon}{\partial \eta_i}(x, \eta)\, \xi_i \xi_j \geq \Lambda_3 (k + \varepsilon + |\eta|)^{\alpha-2} |\xi|^2$$

$$\sum_{i,j=1}^N \left| \frac{\partial a_j^\varepsilon}{\partial \eta_i}(x, \eta) \right| \leq \Lambda_5 (k + \varepsilon + |\eta|)^{\alpha-2}$$

$$\sum_{i,j=1}^N \left| \frac{\partial a_j^\varepsilon}{\partial x_i}(x, \eta) \right| \leq N \Lambda_2 (k + \varepsilon + |\eta|)^{\alpha-2} |\eta|$$

for all $x \in \Omega$, for all $\xi \in \mathbb{R}^N$ and for all $\eta \in \mathbb{R}^N$ (resp. $\eta \in \mathbb{R}^N \setminus \{0\}$ if $\varepsilon = 0$).

Proof.

Given i, $j\in\{1,...,N\}$, it is immediate to verify that

$$\frac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)=\varphi(x,\varepsilon+|\eta|)\delta_{ij}+\frac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)\frac{\eta_i\eta_j}{|\eta|} \tag{2.1}$$

Therefore, we have that

$\dfrac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)\in C(\Omega\times\mathfrak{R}^N)$ if $\varepsilon>0$ (resp. $C(\Omega\times(\mathfrak{R}^N\setminus\{0\}))$ if $\varepsilon=0$) and

$$\sum_{i,j=1}^N\frac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)\xi_i\xi_j=\varphi(x,\varepsilon+|\eta|)|\xi|^2+\frac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)\frac{(\eta^T\xi)^2}{|\eta|}$$

Let us suppose that $\dfrac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)\geq 0$. Hence,

$$\sum_{i,j=1}^N\frac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)\xi_i\xi_j\geq\varphi(x,\varepsilon+|\eta|)|\xi|^2\geq\Lambda_3(k+\varepsilon+|\eta|)^{\alpha-2}|\xi|^2$$

thanks to lemma 2.1.

Otherwise, if $\dfrac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)<0$, using the Cauchy – Schwarz inequality and (1.4), we get

$$\sum_{i,j=1}^N\frac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)\xi_i\xi_j\geq\left(\varphi(x,\varepsilon+|\eta|)+\frac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)|\eta|\right)|\xi|^2\geq$$

$$\geq\frac{\partial b}{\partial s}(x,\varepsilon+|\eta|)|\xi|^2\geq\Lambda_1(k+\varepsilon+|\eta|)^{\alpha-2}|\xi|^2\geq\Lambda_3(k+\varepsilon+|\eta|)^{\alpha-2}|\xi|^2.$$

Before proving second inequality of lemma, let us note that from (1.4) and lemma 2.1, it follows

$$\left|\frac{\partial\varphi}{\partial s}(x,s)s\right|=\left|\frac{\partial b}{\partial s}(x,s)-\varphi(x,s)\right|\leq(\Lambda_2+\Lambda_4)(k+s)^{\alpha-2}\,\forall x\in\Omega,\forall s\in(0,+\infty)$$

Now, combining this expression with formula (2.1), we deduce

$$\sum_{i,j=1}^N\left|\frac{\partial a_j^\varepsilon}{\partial \eta_i}(x,\eta)\right|\leq N\left(|\varphi(x,\varepsilon+|\eta|)|+|\eta|\left|\frac{\partial\varphi}{\partial s}(x,\varepsilon+|\eta|)\right|\right)\leq$$

$$\leq \Lambda_5(k+\varepsilon+|\eta|)^{\alpha-2} \qquad \text{where } \Lambda_5 = N(2\Lambda_4+\Lambda_2)$$

Last inequality of lemma follows directly from (1.5). ■

Lemma 2.3. *Let us suppose (1.2)–(1.4), (1.6) and (1.7). Then, there exist positive constants Λ_6 and Λ_7 depending only on N, α, Λ_1 and Λ_2 such that*

$$\text{a)} \sum_{j=1}^{N} (a_j^\varepsilon(x,\eta) - a_j^\varepsilon(x,\eta'))(\eta_j - \eta_j') \geq \Lambda_6 \begin{cases} (1+|\eta|+|\eta'|)^{\alpha-2}|\eta-\eta'|^2 & \text{if } \alpha \leq 2 \\ |\eta-\eta'|^\alpha & \text{if } \alpha \geq 2 \end{cases}$$

$$b) \qquad \sum_{j=1}^{N} |a_j^\varepsilon(x,\eta)| \leq \Lambda_7(k+|\eta|)^{\alpha-2}|\eta|$$

Furthermore,

$$c) \qquad (\psi(x,s)-\psi(x,s'))(s-s') \geq 0$$

$$d) \qquad |\psi(x,s)| \leq |s| f(|s|)$$

for $\varepsilon \geq 0$, for all $x \in \Omega$, all $s, s' \in \mathfrak{R}$ and all $\eta, \eta' \in \mathfrak{R}^N$.

Proof

For a) and b), see lemma 1 of Tolksdorf [13]. Conditions c) and d) follow immediately from the hypotheses. ■

Lemma 2.4. *Let us suppose (1.2), (1.4) and $b(x,0)=0$. Assume $\alpha \leq 2$. Then, for each $\varepsilon \geq 0$ we have*

$$a) \quad \int_\Omega (a^\varepsilon(x,\nabla y) - a^\varepsilon(x,\nabla y'))(\nabla y - \nabla y')\,dx \geq$$
$$\Lambda_6 \|\nabla y - \nabla y'\|^2_{L^\alpha(\Omega)} \| 1+|\nabla y|+|\nabla y'| \|^{\alpha-2}_{L^\alpha(\Omega)}$$

Moreover, there exist positive constants Λ_8 and Λ_9 depending only on N, α, Λ_1 and Λ_2 such that

$$\text{b)} \qquad \sum_{j=1}^{N} a_j^\varepsilon(x,\eta)\eta_j \geq \begin{cases} \Lambda_8|\eta|^\alpha - \Lambda_9 \\ \Lambda_8(|\eta|^\alpha - |\eta|) \end{cases}$$

for $\varepsilon \geq 0$, for all $x \in \Omega$ and all $\eta \in \mathfrak{R}^N$.

Proof

a) It is a simple consequence of Lemma 2.3-a) and Hölder's inequality applied with $p=2/\alpha$ and $p'=2/(2-\alpha)$:

$$\int_\Omega |\nabla y-\nabla y'|^\alpha\,dx \leq$$

$$\leq \left(\int_\Omega |\nabla y-\nabla y'|^2(1+|\nabla y|+|\nabla y'|)^{\alpha-2}\,dx\right)^{\alpha/2}\left(\int_\Omega (1+|\nabla y|+|\nabla y'|)^\alpha\,dx\right)^{(2-\alpha)/2}$$

b) It is enough to take into account Lemma 2.3-a) again, hypothesis (1.7) and to distinguish the cases $|\eta|\geq 1$ and $|\eta|<1$. ∎

Now, we are ready to derive existence and uniqueness of solution for the Dirichlet problems. First, for each $\varepsilon>0$, let us introduce the perturbed differential operator

$$A_\varepsilon y=-\operatorname{div}(a^\varepsilon(x,\nabla y))+\psi(x,y)=-\operatorname{div}(\varphi(x,\varepsilon+|\nabla y|)\nabla y)+\psi(x,y)$$

and the correponding Dirichlet problems

$$\begin{cases} A_\varepsilon y=\nu & \text{in } \Omega\\ y=0 & \text{on } \Gamma\end{cases} \tag{2.2}$$

Utilizing previous lemmas 2.2 — 2.4, we can apply the result of Rakotoson [10] to deduce that, given $\nu\in L^2(\Omega)$, there exists a unique $y_\varepsilon(\nu)\in W_0^{1,\alpha}(\Omega)\cap L^\infty(\Omega)$ solution of (2.2) for each $\varepsilon>0$ (resp. there exists a unique $y_\nu\in W_0^{1,\alpha}(\Omega)\cap L^\infty(\Omega)$ solution of (1.8), for $\varepsilon=0$)

In the following result, we show continuous dependence with respect to the data for this type of equations.

Lemma 2.5. *Let us suppose (1.2)—(1.4), (1.6)—(1.7) and (1.9). Given $\varepsilon\geq 0$, let $y_\varepsilon\in W_0^{1,\alpha}(\Omega)$* be the solution of

$$\begin{cases} A_\varepsilon y=\nu & \text{in } \Omega\\ y=0 & \text{on } \Gamma\end{cases}$$

and, for each $m\in\mathbb{N}$, let $y_\varepsilon^m\in W_0^{1,\alpha}(\Omega)$ satisfy

$$\begin{cases} A_\varepsilon y=\nu_m & \text{in } \Omega\\ y=0 & \text{on } \Gamma\end{cases}$$

with $A_0 = A$. Assume that $v_m \rightarrow v$ weakly in $L^2(\Omega)$ as $m \rightarrow +\infty$.

Then, $y_\varepsilon^m \rightarrow y_\varepsilon$ in $W_0^{1,\alpha}(\Omega)$ as $m \rightarrow +\infty$.

Proof

First, let us remark that $v_m \rightarrow v$ in $W^{-1,\beta}(\Omega)$, because $L^2(\Omega) \subset W^{-1,\beta}(\Omega)$ with compact imbedding (Adams [1]).

From the relations satisfied by y_ε and y_ε^m, it follows that

$$\int_\Omega (\psi(x, y_\varepsilon^m) - \psi(x, y_\varepsilon))(y_\varepsilon^m - y_\varepsilon)\, dx + \int_\Omega (a^\varepsilon(x, \nabla y_\varepsilon^m) - a^\varepsilon(x, \nabla y_\varepsilon))(\nabla y_\varepsilon^m - \nabla y_\varepsilon)\, dx = (v_m - v, y_\varepsilon^m - y_\varepsilon)$$

Suppose $\alpha \geq 2$. Applying a) and c) of lemma 2.3, we get

$$\Lambda_6 \|\nabla y_\varepsilon^m - \nabla y_\varepsilon\|^\alpha_{L^\alpha(\Omega)} \leq \|v_m - v\|_{W^{-1,\beta}(\Omega)} \|y_\varepsilon^m - y_\varepsilon\|_{W_0^{1,\alpha}(\Omega)}$$

Finally, using the equivalent norm in $W_0^{1,\alpha}(\Omega)$ (see [1]) and the hypotheses, we obtain

$$y_\varepsilon^m \rightarrow y_\varepsilon \quad \text{in} \quad W_0^{1,\alpha}(\Omega)$$

In the case $\alpha < 2$, argumentation is similar using lemma 2.4-a). ∎

3. EXISTENCE OF SOLUTION AND OPTIMALITY CONDITIONS

We begin showing existence of solutions of problem (P_α) defined in (1.10)–(1.11):

Theorem 3.1. *Let us suppose (1.2)–(1.4), (1.6)–(1.7) and (1.9). Assume that*

Either $\mathcal{K}$ is bounded in $L^2(\Omega)$ or $\rho > 0$.

Then, there exists (at least) one solution of (P_α).

Proof

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$ be a minimizing sequence and $\{y_n\}_{n\in\mathbb{N}}$ the sequence of associated states. By the hypothesis, there exists $\bar{u}\in\mathcal{K}$ and a subsequence (again denoted by $\{u_n\}$) such that

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } L^2(\Omega)$$

$\mathcal{K}$ is a convex and closed subset of $L^2(\Omega)$. Therefore, $\mathcal{K}$ is weakly closed in $L^2(\Omega)$ and $\bar{u}\in\mathcal{K}$.

Let $\bar{y}$ be the associated state of $\bar{u}$. From lemma 2.5, we obtain that

$$y_n \to \bar{y} \quad \text{in } W_0^{1,\alpha}(\Omega)$$

The lower semicontinuity of J in the weak topology of $L^2(\Omega)$ and the imbedding $W_0^{1,\alpha}(\Omega)\subset L^2(\Omega)$, completes the proof. ∎

Optimality conditions for problem (P_α) can be formulated as follows:

Theorem 3.2. *Let us suppose (1.2)–(1.7) and (1.9) with $k\neq 0$ if $\alpha>2$. Assume that $\mathcal{K}$ is a bounded subset of $L^\infty(\Omega)$. Let $\bar{u}$ be a solution of (P_α), $\bar{y}$ the associated state and $\Omega_0=\{x\in\Omega\colon\ |\nabla\bar{y}(x)|>0\}$. Then, there exists $\bar{p}\in H_0^1(\Omega)$ (resp. $W_0^{1,\alpha}(\Omega)$ if $\alpha<2$) such that*

$$\begin{cases} -div\,(\varphi(x,|\nabla\bar{y}|)\nabla\bar{y}))+\psi(x,\bar{y})=\bar{u} & in\ \Omega \\ \bar{y}=0 & on\ \Gamma \end{cases} \tag{3.1}$$

$$-\mathrm{div}\left(\left(\varphi(x,|\nabla\bar{y}|)\,I+\frac{\partial\varphi}{\partial s}(x,|\nabla\bar{y}|)\frac{\nabla\bar{y}\cdot\nabla\bar{y}^T}{|\nabla\bar{y}|}\right)\nabla\bar{p}\right)+$$
$$+\frac{\partial\psi}{\partial s}(x,\bar{y})\bar{p}=\bar{y}-y_d \quad \text{in } \Omega_0 \tag{3.2}$$

$$\int_\Omega (\bar{p}+\rho\bar{u})(v-\bar{u})\,dx\geq 0 \qquad \forall v\in\mathcal{K} \tag{3.3}$$

where I denotes the identity matrix $N\times N$ and $\nabla\bar{y}\cdot\nabla\bar{y}^T$ denotes the $N\times N$ matrix with coefficients $\dfrac{\partial\bar{y}}{\partial x_i}\cdot\dfrac{\partial\bar{y}}{\partial x_j}$ $1\leq i,j\leq N$.

Proof of this theorem requires a rather long development and it will be carry out in Sections 4 and 5.

Remarks

1) Remember that if $\alpha>2$ and $k=0$, it follows from the hypotheses that $b\in C^1(\Omega\times[0,+\infty))$ with $\dfrac{\partial b}{\partial s}(x,0)=0$ and we apply the results of [3,5].

2) In the case $\alpha<2$ and $k=0$, we can obtain some additional information about the adjoint state $\overline{p}$:

$$\nabla\overline{p}(x)=0 \qquad \text{a.e. } x\in\Omega\setminus\Omega_0$$

See the end of section 5 for the proof.

3) Introduction of set Ω_0 was suggested by the work [6] of A. Friedman.

4. AN APPROXIMATING FAMILY OF PROBLEMS (P_α^ε)

Let $\overline{u}$ be a solution of (P_α) (see theorem 3.1). In order to derive the optimality system for $\overline{u}$, we introduce the following family of control problems:

$$(P_\alpha^\varepsilon)\begin{cases}\text{Minimize } J_\varepsilon(v)\\ \quad v\in\mathcal{K}\end{cases}$$

where the cost functional is given by

$$J_\varepsilon(v)=\frac{1}{2}\int_\Omega|y_\varepsilon(v)-y_d|^2\,dx+\frac{\rho}{2}\int_\Omega|v|^2\,dx+\frac{1}{2}\int_\Omega|v-\overline{u}|^2\,dx$$

and $y_\varepsilon(v)$ is the solution of (2.2).

Following result can be proved arguing as in theorem 3.1, with the aid of lemma 2.5.

Theorem 4.1. *Let us suppose (1.2)–(1.4), (1.6)–(1.7) and (1.9). Then, for each $\varepsilon>0$, there exists (at least) one solution of (P_α^ε).*

Before deriving the optimality conditions for (P_α^ε), we need to define some functional spaces.

Given $y\in W_0^{1,\alpha}(\Omega)$, let $H_0^{\alpha,y}(\Omega)$ be the space completed of $D(\Omega)$ respect to the norm

$$\|z\|_{H_0^{\alpha,y}(\Omega)}=\left(\int_\Omega(1+|\nabla y|)^{\alpha-2}|\nabla z|^2\,dx\right)^{1/2}$$

It may be easily verified that $H_0^{\alpha,y}(\Omega)$ is a Hilbert space with the inner product

$$(z_1, z_2)=\int_\Omega (1+|\nabla y|)^{\alpha-2}\nabla z_1\,\nabla z_2\,dx$$

Moreover, we have

$$W_0^{1,\alpha}(\Omega)\subset H_0^{\alpha,y}(\Omega)\subset H_0^1(\Omega) \qquad \text{if } \alpha\geq 2$$

$$H_0^1(\Omega)\subset H_0^{\alpha,y}(\Omega)\subset W_0^{1,\alpha}(\Omega) \qquad \text{if } \alpha\leq 2$$

with continuous imbeddings.

More general spaces of this type have been studied by Murthy and Stampacchia [9], Coffman *et al.* [4] and Trudinger [15].

Since operator A_ε satisfies the hypotheses of [3, 5] (see lemma 2.2), we deduce the following results which are analogue to [5, theorems 3.2 and 3.7].

Theorem 4.2. *Let us suppose (1.2)−(1.4), (1.6)−(1.7) and one of the following conditions:*

i) $\alpha\geq 2, \alpha>N/2$ *ii)* $\alpha<2$ *and* $N=1$

For each $\varepsilon>0$, *let* u_ε *be a solution of* (P_α^ε) *and* $y_\varepsilon\in W_0^{1,\alpha}(\Omega)$ *the associated state. Then there exists a unique* $p_\varepsilon\in H_0^{\alpha,y_\varepsilon}(\Omega)$ *such that*

$$\begin{cases} -\operatorname{div}\,(a^\varepsilon(x,\nabla y_\varepsilon))+\psi(x,y_\varepsilon)=u_\varepsilon & \text{in } \Omega \\ y_\varepsilon=0 & \text{on } \Gamma \end{cases} \tag{4.1}$$

$$\begin{cases} -\operatorname{div}\left(\dfrac{\partial a^\varepsilon}{\partial\eta}(x,\nabla y_\varepsilon)\nabla p_\varepsilon\right)+\dfrac{\partial\psi}{\partial s}(x,y_\varepsilon)\,p_\varepsilon=y_\varepsilon-y_d & \text{in } \Omega \\ p_\varepsilon=0 & \text{on } \Gamma \end{cases} \tag{4.2}$$

$$\int_\Omega (p_\varepsilon+\rho u_\varepsilon+u_\varepsilon-\overline{u})(v-u_\varepsilon)\,dx\geq 0 \qquad \forall v\in\mathcal{K} \tag{4.3}$$

Theorem 4.3. *Let us suppose (1.2)−(1.4), (1.6)−(1.7) and one of the following conditions:*

i) $1<\alpha<2$ *and* $N=2$ *ii)* $\frac{3}{2}<\alpha<2$ *and* $N=3$

For each $\varepsilon>0$, let u_ε be a solution of (P_α^ε) and $y_\varepsilon \in W_0^{1,\alpha}(\Omega)$ the associated state. Then there exists $p_\varepsilon \in W_0^{1,\alpha}(\Omega)$ satisfying (4.1)–(4.3) and

$$\int_\Omega \nabla p_\varepsilon^T \frac{\partial a^\varepsilon}{\partial \eta}(x, \nabla y_\varepsilon) \nabla p_\varepsilon \, dx + \int_\Omega \frac{\partial \psi}{\partial s}(x, y_\varepsilon) p_\varepsilon^2 \, dx \leq \int_\Omega (y_\varepsilon - y_d) p_\varepsilon \, dx \tag{4.4}$$

Remarks

1) There exists a unique solution in $H_0^{\alpha, y_\varepsilon}(\Omega)$ of problem (4.2): it is enough to consider the bilinear form defined in $H_0^{\alpha, y_\varepsilon}(\Omega)$ by

$$B(z_1, z_2) = \int_\Omega \nabla z_1^T \frac{\partial a^\varepsilon}{\partial \eta}(x, \nabla y_\varepsilon) \nabla z_2 \, dx + \int_\Omega \frac{\partial \psi}{\partial s}(x, y_\varepsilon) z_1 z_2 \, dx$$

and to apply the Lax-Milgram theorem.

2) In theorem 4.3 (case $\alpha<2$ and $N>1$), we can only prove that p_ε belongs to $W_0^{1,\alpha}(\Omega)$ and satisfies the equation in the distribution sense. In general we can not guarantee the uniqueness of p_ε. In relation with this question see Serrin [11].

Before stating in what sense the problem (P_α) is approximated by the problems (P_α^ε) we need to prove two previous lemmas:

Lemma 4.4. *Let us suppose (1.2)–(1.4), (1.6)–(1.7) and (1.9). For each $\varepsilon>0$, let $(y_\varepsilon(v_\varepsilon), v_\varepsilon)$ belong to $(W_0^{1,\alpha}(\Omega) \cap L^\infty(\Omega)) \times L^2(\Omega)$ and satisfy*

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x, \nabla y_\varepsilon(v_\varepsilon)) + \psi(x, y_\varepsilon(v_\varepsilon)) = v_\varepsilon & \text{in } \Omega \\ y_\varepsilon(v_\varepsilon) = 0 & \text{on } \Gamma \end{cases} \tag{4.5}$$

Let us assume that $\{v_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega)$, then there exists $C>0$ such that

$$\|y_\varepsilon(v_\varepsilon)\|_{W_0^{1,\alpha}(\Omega)} + \|y_\varepsilon(v_\varepsilon)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all } \varepsilon>0$$

Proof

The boundedness of $\{y_\varepsilon(v_\varepsilon)\}_{\varepsilon>0}$ in $W_0^{1,\alpha}(\Omega)$ is a simple consequence of lemmas 2.3 and 2.4.

We will prove that $\{y_\varepsilon(\nu_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^\infty(\Omega)$.

First, given $r>0$, we consider

$$y_\varepsilon^r(\nu_\varepsilon)=\max\{y_\varepsilon(\nu_\varepsilon)-r,0\} \text{ and } A_\varepsilon(r)=\{x\in\Omega : y_\varepsilon(\nu_\varepsilon)(x)\geq r\}$$

We have $y_\varepsilon^r(\nu_\varepsilon)\in W_0^{1,\alpha}(\Omega)$ and

$$\nabla y_\varepsilon^r(\nu_\varepsilon)(x)=\begin{cases} 0 & \text{if } x\notin A_\varepsilon(r) \\ \nabla y_\varepsilon^r(\nu_\varepsilon)(x) & \text{if } x\in A_\varepsilon(r)\end{cases} \tag{4.6}$$

for almost every x in Ω.

Using Hölder's inequality and lemma 2.3 or lemma 2.4, we deduce from (4.5) and (4.6) that

$$C\Big(\|\nabla y_\varepsilon^r(\nu_\varepsilon)\|^\alpha_{L^\alpha(A_\varepsilon(r))}-m(A_\varepsilon(r))^{1\ \beta}\|\nabla y_\varepsilon^r(\nu_\varepsilon)\|_{L^\alpha(A_\varepsilon(r))}\Big)\leq$$

$$\leq C\Big(\|\nabla y_\varepsilon^r(\nu_\varepsilon)\|^\alpha_{L^\alpha(A_\varepsilon(r))}-\|\nabla y_\varepsilon^r(\nu_\varepsilon)\|_{L^1(A_\varepsilon(r))}\Big)\leq$$

$$\leq\int_{A_\varepsilon(r)}\varphi(x,\varepsilon+|\nabla y_\varepsilon^r(\nu_\varepsilon)|)|\nabla y_\varepsilon^r(\nu_\varepsilon)|^2\,dx+\int_{A_\varepsilon(r)}\psi(x,y_\varepsilon(\nu_\varepsilon))\,y_\varepsilon^r(\nu_\varepsilon)\,dx=$$

$$=\int_\Omega\varphi(x,\varepsilon+|\nabla y_\varepsilon(\nu_\varepsilon)|)\,\nabla y_\varepsilon(\nu_\varepsilon)\nabla y_\varepsilon^r(\nu_\varepsilon)\,dx+\int_\Omega\psi(x,y_\varepsilon(\nu_\varepsilon))\,y_\varepsilon^r(\nu_\varepsilon)\,dx=$$

$$=\int_\Omega\nu_\varepsilon y_\varepsilon^r(\nu_\varepsilon)\,dx=\int_{A_\varepsilon(r)}\nu_\varepsilon y_\varepsilon^r(\nu_\varepsilon)\,dx\leq\|\nu_\varepsilon\|_{L^2(A_\varepsilon(r))}\,\|y_\varepsilon^r(\nu_\varepsilon)\|_{L^2(A_\varepsilon(r))}$$

Thus, by the continuity of the imbedding $W_0^{1,\alpha}(\Omega)\subset L^\mu(\Omega)$, with $\mu=\dfrac{N\alpha}{N-\alpha}$ (if $\alpha<N$) or $\mu>2\alpha$ (if $\alpha=N$), see Adams [1], and the hypothesis, we get

$$\|y_\varepsilon^r(\nu_\varepsilon)\|_{W_0^{1,\alpha}(\Omega)}\Big(\|y_\varepsilon^r(\nu_\varepsilon)\|^{\alpha-1}_{W_0^{1,\alpha}(\Omega)}-m(A_\varepsilon(r))^{1\ \beta}\Big)\leq C_1\|y_\varepsilon^r(\nu_\varepsilon)\|_{L^2(A_\varepsilon(r))}\leq$$

$$\leq C_1\,m(A_\varepsilon(r))^{1\ 2-1\ \mu}\|y_\varepsilon^r(\nu_\varepsilon)\|_{L^\mu(A_\varepsilon(r))}\leq C_2\,m(A_\varepsilon(r))^{1\ 2-1\ \mu}\|y_\varepsilon^r(\nu_\varepsilon)\|_{W_0^{1,\alpha}(\Omega)}$$

and hence,

$$\|y_\varepsilon^r(\nu_\varepsilon)\|^{\alpha-1}_{L^\mu(A_\varepsilon(r))}\leq C_3\,m(A_\varepsilon(r))^{1\ 2-1\ \mu} \tag{4.7}$$

Now, let $s>r>0$; then $A_\varepsilon(s)\subset A_\varepsilon(r)$ and moreover

$$(s-r)\ m(A_\varepsilon(s))^{1\ \mu}\leq\|y_\varepsilon^r(\nu_\varepsilon)\|_{L^\mu(A_\varepsilon(s))}\leq\|y_\varepsilon^r(\nu_\varepsilon)\|_{L^\mu(A_\varepsilon(r))} \tag{4.8}$$

From (4.7) and (4.8) it follows

$$m(A_\varepsilon(s)) \leq \frac{C_4}{(s-r)^\mu} m(A_\varepsilon(r))^{\frac{\mu}{\alpha-1}\left(\frac{1}{2}-\frac{1}{\mu}\right)}$$

Finally, applying lemma 4.1 of Stampacchia [12] to the function $\phi_\varepsilon(t) = m(A_\varepsilon(t))$ for $t > 0$, and noting that thanks to (1.9)

$$\frac{\mu}{\alpha-1}\left(\frac{1}{2}-\frac{1}{\mu}\right) > 1,$$

it follows the existence of a constant $C_5 < +\infty$ independent of ε such that $y_\varepsilon(\nu_\varepsilon)(x) \leq C_5$ a.e. $x \in \Omega$.

In the same way, taking $y_\varepsilon^r(\nu_\varepsilon) = \min\{y_\varepsilon(\nu_\varepsilon) + r, 0\}$, we derive the existence of $C_6 > -\infty$ such that $y_\varepsilon(\nu_\varepsilon)(x) \geq C_6$ a.e. $x \in \Omega$ for every $\varepsilon > 0$. ∎

Lemma 4.5. *Let us suppose (1.2)–(1.4), (1.6)–(1.7) and (1.9). Assume that $\nu_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0$. Then, $y_\varepsilon(\nu_\varepsilon) \rightharpoonup y_u$ in $W_0^{1,\alpha}(\Omega)$ as $\varepsilon \to 0$.*

Proof

From previous lemma we derive that $\{y_\varepsilon(\nu_\varepsilon)\}_{\varepsilon>0}$ is bounded in $W_0^{1,\alpha}(\Omega)$. Thus, there exist a subsequence (again denoted $\{y_\varepsilon(\nu_\varepsilon)\}_{\varepsilon>0}$) and $y \in W_0^{1,\alpha}(\Omega)$ such that

$$y_\varepsilon(\nu_\varepsilon) \rightharpoonup y \qquad \text{weakly in } W_0^{1,\alpha}(\Omega)$$

Furthermore, $y_\varepsilon(\nu_\varepsilon)$ is the solution of (2.2) with $\nu = \nu_\varepsilon$ and then we have

$$\int_\Omega \varphi(x, \varepsilon + |\nabla y_\varepsilon(\nu_\varepsilon)|)\nabla y_\varepsilon(\nu_\varepsilon)\nabla\phi\,dx + \int_\Omega \psi(x, y_\varepsilon(\nu_\varepsilon))\phi\,dx = \int_\Omega \nu_\varepsilon\phi\,dx \tag{4.9}$$

for all $\phi \in W_0^{1,\alpha}(\Omega)$.

For proving that $y = y_u$ it is sufficient to pass to the limit in (4.9) as $\varepsilon \to 0$.

In virtue of b), d) of lemma 2.3 and lemma 4.4 we have

$$\int_\Omega |\varphi(x, \varepsilon + |\nabla y_\varepsilon(\nu_\varepsilon)|)\nabla y_\varepsilon(\nu_\varepsilon)|^\beta\,dx \leq C_1 \int_\Omega (k + \varepsilon + |\nabla y_\varepsilon(\nu_\varepsilon)|)^\alpha\,dx \leq C_2$$

$$\int_\Omega |\psi(x, y_\varepsilon(\nu_\varepsilon))|^\beta\,dx \leq C_3$$

Therefore, we may infer that there exist a subsequence (denoted in the same way) and $\chi \in (L^\beta(\Omega))^N$ such that

$$\varphi(x, \varepsilon + |\nabla y_\varepsilon(\nu_\varepsilon)|)\nabla y_\varepsilon(\nu_\varepsilon) \to \chi \quad \text{weakly in } (L^\beta(\Omega))^N$$

$$\psi(x, y_\varepsilon(\nu_\varepsilon)) \to \psi(x, y) \qquad \text{weakly in } L^\beta(\Omega)$$

Let us introduce the element L of $W^{-1,\beta}(\Omega)$ defined by the formula

$$L(\phi) = \int_\Omega \chi \nabla\phi\, dx$$

Letting ε tend to 0 in (4.9) we deduce

$$L(\phi) = \int_\Omega (u - \psi(x, y))\phi\, dx \tag{4.10}$$

for all $\phi \in W_0^{1,\alpha}(\Omega)$. Moreover, by (4.10) and the strong convergence of $y_\varepsilon(\nu_\varepsilon)$ to y in $L^2(\Omega)$ and $L^\alpha(\Omega)$

$$\limsup_\varepsilon \left(\int_\Omega \varphi(x, \varepsilon + |\nabla y_\varepsilon(\nu_\varepsilon)|)\, |\nabla y_\varepsilon(\nu_\varepsilon)|^2\, dx\right) =$$

$$= \limsup_\varepsilon \left(\int_\Omega \nu_\varepsilon y_\varepsilon(\nu_\varepsilon)\, dx - \int_\Omega \psi(x, y_\varepsilon(\nu_\varepsilon))\, y_\varepsilon(\nu_\varepsilon)\, dx\right) =$$

$$= \int_\Omega (u - \psi(x, y))\, y\ dx = L(y)$$

Since the operator $\mathcal{A}: W_0^{1,\alpha}(\Omega) \to W^{-1,\beta}(\Omega)$ defined by

$$(\mathcal{A}y, w) = \int_\Omega \varphi(x, |\nabla y|)\nabla y \nabla w\, dx$$

satisfies M-property (Lions [8, pp. 171-187]) and $\mathcal{A}y_\varepsilon(\nu_\varepsilon) \to L$ weakly in $W^{-1,\beta}(\Omega)$, it is verified that

$$\int_\Omega \varphi(x, |\nabla y|)\nabla y \nabla\phi\, dx + \int_\Omega \psi(x, y)\phi\, dx = \int_\Omega u\phi\, dx \qquad \forall \phi \in W_0^{1,\alpha}(\Omega)$$

Hence, $y = y_u$ and then

$$y_\varepsilon(\nu_\varepsilon) \to y_u \qquad \text{weakly in } W_0^{1,\alpha}(\Omega)$$

Finally, from the above results and lemma 2.3, we conclude in the case $\alpha \geq 2$ that

$$\limsup_\varepsilon \Lambda_6 \|\nabla y_\varepsilon(\nu_\varepsilon) - \nabla y_u\|^\alpha_{L^\alpha(\Omega)} \leq$$

$$\leq \limsup_{\varepsilon} \int_{\Omega} (\varphi(x, \varepsilon+|\nabla y_\varepsilon(v_\varepsilon)|)\nabla y_\varepsilon(v_\varepsilon) - \varphi(x, \varepsilon+|\nabla y_u|)\nabla y_u)(\nabla y_\varepsilon(v_\varepsilon) - \nabla y_u)\, dx =$$

$$= \limsup_{\varepsilon} \int_{\Omega} v_\varepsilon (y_\varepsilon(v_\varepsilon) - y_u)\, dx - \int_{\Omega} \psi(x, y_\varepsilon(v_\varepsilon))(y_\varepsilon(v_\varepsilon) - y_u)\, dx = 0.$$

In the case $\alpha < 2$, the argument is the same, utilizing lemma 2.4.

Hence, $y_\varepsilon(v_\varepsilon) \to y_u$ strongly in $W_0^{1,\alpha}(\Omega)$, as $\varepsilon \to 0$. ■

Theorem 4.6. *Let us suppose (1.2)–(1.4), (1.6)–(1.7) and (1.9). Let u_ε be a solution of (P_α^ε). Set $\bar{y} = y_{\bar{u}}$ and $y_\varepsilon = y_\varepsilon(u_\varepsilon)$. Then, we have*

$$u_\varepsilon \to \bar{u} \quad in \quad L^2(\Omega) \tag{4.11}$$

$$y_\varepsilon \to \bar{y} \quad in \quad W_0^{1,\alpha}(\Omega) \tag{4.12}$$

$$J_\varepsilon(u_\varepsilon) \to J(\bar{u}) \tag{4.13}$$

as $\varepsilon \to 0$.

Proof

Applying previous lemma to $v_\varepsilon = \bar{u} \quad \forall \varepsilon > 0$, we deduce

$$y_\varepsilon(\bar{u}) \to \bar{y} \quad \text{in } W_0^{1,\alpha}(\Omega)$$

Since $\bar{u} \in \mathcal{K}$, it follows that for all $\varepsilon > 0$

$$\frac{1}{2} \|u_\varepsilon - \bar{u}\|^2_{L^2(\Omega)} \leq J_\varepsilon(u_\varepsilon) \leq J_\varepsilon(\bar{u}) \leq C \tag{4.14}$$

Thus, $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega)$ and selecting a subsequence, if necessary, we may infer that there exists $u \in \mathcal{K}$ such that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega)$$

Using once more lemma 4.5, we obtain

$$y_\varepsilon \to y_u \quad \text{in } W_0^{1,\alpha}(\Omega)$$

From (4.14) and the lower semicontinuity of J_ε, we get

$$\frac{1}{2} \|y_u - y_d\|^2_{L^2(\Omega)} + \frac{\rho}{2} \|u\|^2_{L^2(\Omega)} + \frac{1}{2} \|u - \bar{u}\|^2_{L^2(\Omega)} \leq \liminf_{\varepsilon} J_\varepsilon(u_\varepsilon) \leq$$

$$\leq \limsup_{\varepsilon} J_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon} J_\varepsilon(\overline{u}) = \frac{1}{2}\|\overline{y}-y_d\|^2_{L^2(\Omega)} + \frac{\rho}{2}\|\overline{u}\|^2_{L^2(\Omega)} = J(\overline{u})$$

Since $\overline{u}$ is solution of (P_α), we know that $J(\overline{u}) \leq J(u)$, thus $u=\overline{u}$ and

$$J_\varepsilon(u_\varepsilon) \to J(\overline{u}) \quad \text{as } \varepsilon \to 0$$

Moreover, u_ε converges strongly towards $\overline{u}$, because

$$\frac{1}{2}\limsup_{\varepsilon} \|u_\varepsilon - \overline{u}\|^2_{L^2(\Omega)} \leq \limsup_{\varepsilon}\left(J_\varepsilon(u_\varepsilon) - \frac{1}{2}\|y_\varepsilon - y_d\|^2_{L^2(\Omega)} - \frac{\rho}{2}\|u_\varepsilon\|^2_{L^2(\Omega)}\right) =$$

$$= \frac{\rho}{2}\left(\|\overline{u}\|^2_{L^2(\Omega)} - \liminf \|u_\varepsilon\|^2_{L^2(\Omega)}\right) = 0. \blacksquare$$

5. PROOF OF THEOREM 3.2

In the preceding section, given $\overline{u}$ a solution of (P_α), we have obtained the optimality conditions for the solutions of the perturbed problems. Hereafter, our purpose is to pass to the limit in conditions (4.1)—(4.3) with the aid of the last results. We will distinguish two cases:

5.1. Case $\alpha \geq 2$ and $k \neq 0$

Let $\{p_\varepsilon\}_{\varepsilon>0} \subset H_0^{\alpha, y_\varepsilon}(\Omega) \subset H_0^1(\Omega)$ be given as in theorem 4.2.

Applying (4.2) to p_ε, we derive

$$\int_\Omega \nabla p_\varepsilon^T \frac{\partial a^\varepsilon}{\partial \eta}(x, \nabla y_\varepsilon)\nabla p_\varepsilon \, dx + \int_\Omega \frac{\partial \psi}{\partial s}(x, y_\varepsilon) p_\varepsilon^2 \, dx = \int_\Omega (y_\varepsilon - y_d) p_\varepsilon \, dx$$

By lemma 2.2, (1.6) and Hölder's inequality, we have

$$\Lambda_3 k^{\alpha-2}\int_\Omega |\nabla p_\varepsilon|^2 dx \leq \Lambda_3 \int_\Omega (k+\varepsilon+|\nabla y_\varepsilon|)^{\alpha-2}|\nabla p_\varepsilon|^2 dx \leq C_0 \|p_\varepsilon\|_{L^2(\Omega)} \tag{5.1}$$

In particular, since $k \neq 0$, it follows from (5.1) that $\{p_\varepsilon\}_{\varepsilon>0}$ is a bounded sequence in $H_0^1(\Omega)$ and there exist a subsequence (again denoted p_ε) and an element $\overline{p} \in H_0^1(\Omega)$ such that

$$p_\varepsilon \to \overline{p} \text{ weakly in } H_0^1(\Omega) \tag{5.2}$$

It is immediate to obtain (3.3) taking into account (4.11) and passing to the limit in (4.3) as $\varepsilon \to 0$.

It remains to verify that $\bar{p}$ satisfies equation (3.2):

Since $\bar{u}\in\mathcal{K}\subset L^\infty(\Omega)$ and thanks to lemma 2.2, we can apply a Tolksdorf's result [13] and deduce that the optimal state $\bar{y}$ belongs to $C^{1,\mu}(\Omega)$ for some $0<\mu<1$. Therefore, $\Omega_0=\{x\in\Omega: |\nabla\bar{y}(x)|>0\}$ is an open set of $\mathbb{R}^N$.

Let $\phi\in D(\Omega_0)$ and let us denote sop $\phi=\overline{\Omega}'\subset\Omega$. Our purpose is to pass to the limit as $\varepsilon\to 0$ in the following expression

$$\int_\Omega \nabla\phi^T \frac{\partial a^\varepsilon}{\partial\eta}(x,\nabla y_\varepsilon)\nabla p_\varepsilon\, dx+\int_\Omega \frac{\partial\psi}{\partial s}(x,y_\varepsilon)p_\varepsilon\phi\, dx=\int_\Omega (y_\varepsilon-y_d)\phi\, dx \tag{5.3}$$

Utilizing again the Tolksdorf's result [13], it follows the existence of constants C_1 and C_2 depending only on N, α, Λ_1, Λ_2, $d(\Omega',\Gamma)$, $\|u_\varepsilon\|_{L^\infty(\Omega)}$ and $\|y_\varepsilon\|_{L^\infty(\Omega)}$ such that

$$|\nabla y_\varepsilon(x)|\le C_1 \qquad \forall x\in\Omega'$$

$$|\nabla y_\varepsilon(x)-\nabla y_\varepsilon(x')|\le C_2|x-x'|^\mu \qquad \forall x,x'\in\Omega'$$

By hypotheses, $\mathcal{K}$ is bounded in $L^\infty(\Omega)$. Furthermore, we know that $\|y_\varepsilon\|_{L^\infty(\Omega)}$ is uniformly bounded by a constant independent of ε (see lemma 4.4). Then, we can apply Ascoli – Arzelá theorem to deduce the existence of a subsequence (denoted in the same way) such that

$$\nabla y_\varepsilon(x)\to\nabla\bar{y}(x) \quad \text{uniformly in } \overline{\Omega}' \tag{5.4}$$

By other hand, in virtue of lemma 2.2 and taking $\varepsilon\in(0,1)$ we obtain that

$$\left\|\frac{\partial a^\varepsilon}{\partial\eta}(x,\nabla y_\varepsilon)\right\|\le\Lambda_5(k+\varepsilon+|\nabla y_\varepsilon|)^{\alpha-2}\le\Lambda_5(k+1+C_1)^{\alpha-2} \quad \forall x\in\Omega'$$

Taking into account (1.6) and lemma 4.4, it follows that

$$\left|\frac{\partial\psi}{\partial s}(x,y_\varepsilon)\right|\le f(\|y_\varepsilon\|_{L^\infty(\Omega)})\le C_3 \qquad \forall x\in\Omega \qquad \forall\varepsilon>0$$

Thanks to the hypothesis (1.3), lemma 2.2, convergence (5.4), (4.12) and the Dominated Convergence Theorem, we deduce that as $\varepsilon\to 0$

$$\frac{\partial\psi}{\partial s}(x,y_\varepsilon)\to\frac{\partial\psi}{\partial s}(x,\bar{y}) \qquad \text{in } L^r(\Omega)$$

$$\frac{\partial a^\varepsilon}{\partial\eta}(x,\nabla y_\varepsilon)\to\frac{\partial a^0}{\partial\eta}(x,\nabla\bar{y}) \quad \text{in } (L^r(\Omega'))^{N\times N}$$

for all $1\le r<+\infty$.

Now, we can pass to the limit in (5.3) with the aid of (5.2) and (4.12). ∎

5.2. Case $\alpha<2$

Let $\{p_\varepsilon\}\subset W_0^{1,\alpha}(\Omega)$ be given as in theorem 4.2 (if $N=1$) or theorem 4.3 (if $N>1$).

Using the same argument as in the proof of lemma 2.4—a) and the conclusion of lemma 4.4, we get

$$\|\nabla p_\varepsilon\|^2_{L^\alpha(\Omega)}\leq C_1\left(\int_\Omega \frac{|\nabla p_\varepsilon|^2}{(k+\varepsilon+|\nabla y_\varepsilon|)^{2-\alpha}}\,dx\right) \tag{5.5}$$

Combining lemma 2.2—a), (1.6), (5.5) and taking into account that p_ε is solution of (4.2) if $N=1$ or using (4.4) if $N>1$, we deduce

$$\|p_\varepsilon\|^2_{W_0^{1,\alpha}(\Omega)}\leq C_2\left(\int_\Omega \nabla p_\varepsilon{}^T\frac{\partial a^\varepsilon}{\partial\eta}(x,\nabla y_\varepsilon)\nabla p_\varepsilon\,dx+\int_\Omega\frac{\partial\psi}{\partial s}(x,y_\varepsilon)p_\varepsilon^2\,dx\right)\leq$$

$$\leq C_2\int_\Omega (y_\varepsilon-y_d)p_\varepsilon\,dx\leq C_3\|y_\varepsilon-y_d\|_{L^2(\Omega)}\|p_\varepsilon\|_{W_0^{1,\alpha}(\Omega)}$$

Remind that $W_0^{1,\alpha}(\Omega)\subset L^2(\Omega)$ thanks to (1.9).

Thus, $\{p_\varepsilon\}_{\varepsilon>0}$ is bounded in $W_0^{1,\alpha}(\Omega)$ and it is possible to choose a subsequence $\varepsilon(n)\to 0$ such that

$$p_{\varepsilon(n)}\to\bar{p}\qquad\text{weakly in } W_0^{1,\alpha}(\Omega) \tag{5.6}$$

for some $\bar{p}\in W_0^{1,\alpha}(\Omega)$.

Rest of the theorem follows exactly as in the previous proof. ■

We conclude this section proving the following additional property about $\bar{p}$ in the case $\alpha<2$ and $k=0$:

$$\nabla\bar{p}(x)=0\qquad\text{a.e. } x\in\Omega\setminus\Omega_0 \tag{5.7}$$

Let K be a compact subset of $\Omega\setminus\Omega_0$. Then, we know that

$$\nabla y_\varepsilon(x)\to\nabla\bar{y}(x)=0\qquad\text{uniformly in } K$$

Given $\delta>0$, there exists $\varepsilon'>0$ such that

$$(\varepsilon+|\nabla y_\varepsilon(x)|)^{2-\alpha}<\delta\qquad\forall\varepsilon<\varepsilon'\qquad\forall x\in K$$

Now, thanks to (5.5), we deduce

$$\frac{1}{\delta}\int_K |\nabla p_\varepsilon|^2\, dx \leq \int_K \frac{|\nabla p_\varepsilon|^2}{(\varepsilon + |\nabla p_\varepsilon|)^{2-\alpha}}\, dx \leq C_4$$

and we can conclude that

$$\nabla p_{\varepsilon(n)} \rightharpoonup \nabla \overline{p} \quad \text{weakly in } (L^2(K))^N$$

Hence, we have

$$\int_K |\nabla \overline{p}|^2\, dx \leq \liminf_n \int_K |\nabla p_{\varepsilon(n)}|^2\, dx \leq C_4\,\delta \qquad \forall \delta > 0$$

which implies $\nabla \overline{p}(x) = 0$ a.e. $x \in K$.

Since $\Omega \setminus \Omega_0$ can be written as a countable union of compact sets (except a set of measure zero), assertion (5.7) holds.

6. REGULARITY OF THE OPTIMAL CONTROL

In this last section, we deduce some qualitative properties about optimal controls, using the optimality conditions.

Theorem 6.1. *Les us suppose that ρ is strictly positive in (1.11) and*

$$\mathcal{K} = \{ v \in L^2(\Omega) : m \leq v(x) \leq M \ a.e.\ x \in \Omega \}$$

with $-\infty < m < M < +\infty$. *Then,* $\overline{u} \in \begin{cases} H^1(\Omega) & \text{if } \alpha \geq 2 \\ W^{1,\alpha}(\Omega) & \text{if } \alpha < 2 \end{cases}$

Proof

Inequality (3.3) characterizes $\overline{u}$ as the projection of $-\dfrac{\overline{p}}{\rho}$ on $\mathcal{K}$. Hence, it follows that

$$\overline{u}(x) = \max\left\{ m,\ \min\left\{ -\frac{1}{\rho}\,\overline{p}(x),\ M \right\}\right\} \qquad \text{a.e. } x \in \Omega$$

We conclude the proof noting that function $f: \mathfrak{R} \rightarrow \mathfrak{R}$ defined by

$$f(s) = \max\{ m,\ \min\{ s,\ M\}\}$$

is uniformly Lipschitz, using lemma 28.1 of Treves [14, pg. 261] and the fact

that $\bar{p}\in\begin{cases} H_0^1(\Omega) & \text{if } \alpha\geq 2 \\ W_0^{1,\alpha}(\Omega) & \text{if } \alpha\leq 2 \end{cases}$. ∎

Remarks

1) If $0\in[m, M]$, moreover we obtain that $\bar{u}_{|_\Gamma}=0$.

2) If $\rho=0$, it follows from (3.3) that

$$\begin{cases} \bar{u}(x)\in[m, M] & \text{if } \bar{p}(x)=0 \\ \bar{u}(x)=m & \text{if } \bar{p}(x)>0 \\ \bar{u}(x)=M & \text{if } \bar{p}(x)<0 \end{cases}$$

When $\bar{p}(x)\neq 0$ a.e. $x\in\Omega$, we have that $\bar{u}$ is «bang-bang».

In the conditions of theorem 6.1 and if $N=1$, the Hölder continuity of $\bar{u}$ in $\bar{\Omega}$ is a consequence of the Sobolev imbedding theorem. In general ($N>1$), assuming that $y_d\in L^{\rho'}(\Omega_0)$ with $\rho'>N/2$, we can apply Theorem 14.1 of Ladyzhenskaya-Ural'tseva [7, p. 201] and deduce that $\bar{p}$ is Hölder continuous in Ω_0 (and then $\bar{u}$ too), see [3, 5].

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