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On the control measures of vector measures

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Abstract. If Σ is a σ-algebra and $\mathcal{X}$ a locally convex space we study conditions for a countably additive vector measures $\gamma : \Sigma \to \mathcal{X}$ to have a control measures μ. If Σ is the Borel σ-algebra of a metric space Ω we obtain necessary and sufficient conditions using the τ additivity of γ. We also give results for polymeasures.

Sobre las medidas de control de medidas vectoriales

Resumen. Si Σ es una σ-álgebra y $\mathcal{X}$ un espacio localmente convexo se estudian las condiciones para las cuales una medida vectorial σ-aditiva $\gamma : \Sigma \to \mathcal{X}$ tenga una medida de control μ. Si Σ es la σ-álgebra de Borel de un espacio métrico Ω, se obtienen condiciones necesarias y suficientes usando la τ aditividad de γ. También se dan estos resultados para las polimedidas.

1. Basic section

Following the usual notation, we write $ca(\Sigma; \mathcal{X})$ for the set of the countably additive measures defined on a σ-algebra Σ of subsets of Ω and taking values in a locally convex space (l.c.s.) $\mathcal{X}$. If $\mathcal{V}$ is an absolutely convex neighborhood (a.c.n.) of 0, we write $p_{\mathcal{V}}$ for the seminorm associated to $\mathcal{V}$ and we write $\mathcal{X}_{\mathcal{V}}$ for the quotient space $\mathcal{X}/p_{\mathcal{V}}^{-1}(0)$. As usual, we can endow $\mathcal{X}_{\mathcal{V}}$ with the norm $\|\cdot\|_{\mathcal{V}}$. defined by $\|g_{\mathcal{V}}(x)\|_{\mathcal{V}} = p_{\mathcal{V}}(x)$, where $g_{\mathcal{V}}$ is the canonical application $\mathcal{X} \to \mathcal{X}_{\mathcal{V}}$.

Theorem 1 *Let $\gamma : \Sigma \to \mathcal{X}$ where $\mathcal{X}$ is a metrizable space, or a space such that $\{0\}$ is a G_δ set. Then there exists a countably additive measure $\mu : \Sigma \to [0.l]$ such that*

$$\lim_{\mu(A)\to 0} \gamma(A) = 0,$$

that is, γ isμ-continuous. This mesure μ is called a control measure of γ.

PROOF. If $\mathcal{X}$ is a metrizable space then its topology can be defined by a sequence $\{\mathcal{V}_n\}$ of a.c.n. of 0. Then, since γ is countably additive, by [1, Corollary I.5.3], there exists a finite countably additive measure μ_n such that $g_{\mathcal{V}_n} \circ \gamma$ is μ_n-continuous. It follows that the measure

$$\mu(A) = \sum_n 2^{-n} \gamma_n(A)/\gamma_n(\Omega) \quad (A \in \Sigma)$$

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satisfies the required conditions because every measure $g_{\mathcal{V}_n} \circ \gamma$ is μ-continuous. In case $\{0\} = \bigcap_n \mathcal{V}_n$ we can proceed similarly, taking into account that $\mu(A) = 0$ implies that $\gamma(A) = 0$ and $g_{\mathcal{V}_n} \circ \gamma(A) = 0$ for every a.c.n. $\mathcal{V}$ and, hence, we can apply the theorem of Pettis [1, I.2]. ■

Corollary 1 *The previous Theorem is valid if $\mathcal{X}$ is an (LF) space and, more generally, if every bounded set of $\mathcal{X}$ is contained in a metrizable subspace of $\mathcal{X}$.* ■

Proposition 1 *There exists a reflexive and complete space $\mathcal{X}$ and a countably additive measure $\gamma : \Sigma \to \mathcal{X}$ so that there is no control measure μ such that $\mu(A) = 0$ implies $\gamma(A) = 0$.*

PROOF. Let Ω be an uncountable set and $\mathcal{X}$ the space $\mathbb{R}^\Omega$ endowed with the product topology. Then $\mathcal{X}$ is a reflexive complete Montel nuclear space. Let $\gamma = (\delta_x)_{x\in\Omega}$, where δ_x is Dirac's delta, and let Σ be the σ-algebra of all the subsets of $\mathcal{X}$. Let us suppose that there is one such measure μ. Then, since Ω is uncountable, there exists $x \in \Omega$ such that $\mu(\{x\}) = 0$ and, hence, $\gamma(\{x\}) = 0$ and $\delta_x(\{x\}) = 0$, against the definition of δ_x.

This measure γ is not diffuse. To define a diffuse measure we can consider $\Omega = \bigcup_{j\in J} I_j$, where $\{I_j\}_{j\in J}$ is a not countable disjoint family of copies of [0,1]. If μ_j is the Lebesgue measure of I_j we can define

$$\gamma(A) = (\mu_j(A \cap I_j))_{j\in J} \in \mathcal{X} = \mathbb{R}^J$$

for every set $A \subset \Omega$ such that $A \cap I_j$ is μ_j-measurable for every $j \in J$. ■

Definition 1 Let Ω be a topological space and $\gamma : \Sigma \to \mathcal{X}$ a Borel measure. Then we say that γ is a *fat measure* if every set of not null γ-measure contains a open set of not null measure. It is clear that every Borel measure on a space Ω endowed with the discreet topology is a fat measure. It is also easy to see that not null fat measure on a T_1 separable space, in particular, on a Lusin space, is an atomic measure.

Theorem 2 *Let Σ be the σ-algebra of the Borel sets of a Lusin space Ω and $\mathcal{X}$ a l.c.s.. Then, every countably additive fat measure $\gamma : \Sigma \to \mathcal{X}$ has a control measure μ such that γ is μ-continuous.*

PROOF. Since Ω is a Lusin space there exists a strict web $(C_{n_1,\ldots,n_k})$ of Borel subsets of Ω such that $\Omega = \bigcup C_n$ and $C_{n_1,\ldots n_k} = \bigcup_n C_{n_1,\ldots,n_k n}$ and so that every open set is the union of disjoint sets of the web [5, p. 98 and 101].

Let $(\alpha_{n_1,\ldots,n_k})$ be a farnily of positive numbers whose total sum equals 1. Let us choose $x^*_{n_1,\ldots,n_k} \in \mathcal{X}^*$ such that

$$|x^*_{n_1,\ldots,n_k} \circ \gamma|(C_{n_1,\ldots,n_k}) = \alpha_{n_1,\ldots,n_k}$$

when $\gamma(C_{n_1,\ldots,n_k}) \neq 0$ and $x^*_{n_1,\ldots,n_k} = 0$ in the opposite case. Then, if

$$\mu(A) = \sum |x^*_{n_1,\ldots,n_k} \circ \gamma|(A \cap C_{n_1,\ldots,n_k}) \quad (\leq 1)$$

for every $A \in \Sigma$ with $\gamma(A) \neq 0$ there exists an open set $G \subset A$ such that $\gamma(G) \neq 0$ and therefore there exists a $C_{n_1,\ldots,n_k}$ with not null γ-measure. Hence $\mu(A) \geq \mu(G) \geq \mu(C_{n_1,\ldots,n_k}) > 0$ and so $\mu(A) = 0$ implies $\gamma(A) = 0$. Then, using the theorem of Pettis [1, I.21] we obtain that every measure $g_{\mathcal{V}} \circ \gamma$ is μ-continuous and, so, the measure γ is μ-continuous. ■

Definition 2 A l.c.s $\mathcal{X}$ is said to have the *property of the dual sequence* if for the closed linear span $\mathcal{Y}$ of every bounded sequence there exists a sequence $\{x^*_n\} \subseteq \mathcal{Y}^*$ such that, if $x \in \mathcal{Y}$ and $x^*_n(x) = 0$ for every $n \in \mathbb{N}$, then $x = 0$.

Proposition 2 *Let $\mathcal{Y}$ be the closed linear span of a bounded sequence. If for any such $\mathcal{Y}$ there exists a sequence $\{\mathcal{V}_n\}$ of neighborhoods of 0 such that $\bigcap_n \mathcal{V}_n \cap \mathcal{Y} = 0$ ($i.e.$ $\{0\}$ is a G_δ set in $\mathcal{Y}$), then $\mathcal{X}$ has the property of the dual sequence.*

PROOF. Let $\{x_n\}$ be a sequence dense in $\mathcal{Y}$ and $B_n = \mathcal{V}_n^0$ (the polar set of $\mathcal{V}_n$), where we can suppose $\mathcal{V}_n$ to be absolutely convex. Then, if p_n is the seminorming function of $\mathcal{V}_n$, for every x_n there exists $x_{nk}^* \in B_k$ such that $|x_{nk}^*(x_n)| \geq p_k(x_n)/2$. Let $x_{nk}^*(x) = 0$ with $x \in \mathcal{Y}$ for every pair (n, k) of whole numbers, then

$$p_k(x - x_n) \geq |x_{nk}^*(x - x_n)| = |x_{nk}^*(x_n)| \geq p_k(x_n)/2.$$

Since $\{x_n\}$ is dense in $\mathcal{Y}$, there exists a subsequence $\{x_{n_i}\}$ such that $p_k(x - x_n) \to 0$; it follows that $p_k(x_{n_i}) \to 0,\ p_k(x) = 0$ and $x \in \mathcal{V}_k$ for every $k \in \mathbb{N}$, hence $x = 0$. ■

We remark that if there exists such sequence $\{x_n^*\}$ with the required property, then the neighborhoods $\mathcal{V}_{nk} = \{x \in \mathcal{X} :\ |x_n^*(x)| < 1(k\}$ verify $\bigcap_{nk} \mathcal{V}_{nk} \cap \mathcal{Y} = \{0\}$.

Examples. Every metrizable space and, in general, every l.c.s. such that $\{0\}$ is a G_δ set, have the property of the dual sequence. The space $\mathcal{D}^*(\Omega)$ of the distributions on an open set of $\mathbb{R}^n$ have the property of the dual sequence, because its dual is separable, and it is not metrizable. $\mathcal{D}(\Omega)$ also has the property of the dual sequence. In general, every normal space of distributions [3, 4.2, p. 3l9] has the property of the dual sequence.

Theorem 3 *If Ω is a Lusin space and $\mathcal{X}$ is a l.c.s with property of the dual sequence, then every Borel measure $\gamma :\ \Sigma \to \mathcal{X}$ has a control measure.*

PROOF. With the notations of Theorem 2, let $\mathcal{Y}$ be the closed linear span of the range of γ. If $x* \in \mathcal{Y}^*$ verifies $x* \circ \gamma(C_{n_1,\ldots,n_k}) = 0$ for any $n_1, \ldots, n_k$, then $x^*(G) = 0$ fon every open set G, because G is the union of disjoint sets $C_{n_1,\ldots,n_k}$. Therefore, $x* \circ \gamma(A) = 0$ for every Borel set A, because every Lusin space is a Radon space [5, p. l22] and, so, $x^* = 0$. Then, the numerable set $\gamma(C_{n_1,\ldots,n_k})$ is a total set in $\mathcal{Y}$ and there exists a sequence $\{x_n^*\} \subset \mathcal{X}^*$ such that, if $x_n^*(x) = 0$ for every $n \in \mathbb{N}$ then $x = 0$. It is clear that, if B is the range of γ, we can take the $x_n^* \in B^0$. Then, since $\{x^* \circ \gamma(A)\}$ is a bounded sequence for every $A \in \Sigma$, it follows from the theorem of Nikodym [1, I.3] that $\{|x^* \circ \gamma|(\Omega)\}$ is also bounded. Then

$$\mu(A) = \sum_n 2^{-n} |x_n^* \circ \gamma|(A) \quad (A \in \Sigma)$$

is a finite Borel measure such that if $\mu(A) = 0$ then $x_n^* \circ \gamma(A) = 0$ for every $n \in \mathbb{N}$ with $\gamma(A) \in \mathcal{Y}$. From here it follows that $\gamma(A) = 0$.

Let $\mathcal{V}$ be an a.c.n. of 0 in $\mathcal{X}$ and let $g_\mathcal{V} :\ \mathcal{X} \to \mathcal{X}_\mathcal{V}$ be the canonical application. Then, applying the theorem of Pettis [1, I.2] to $g_\mathcal{V} \circ \gamma$ the theorem follows immediately. ■

In orden to study the theorem of Rybakov [1, IX.2] we give the following

Definition 3 A l.c.s. $\mathcal{X}$ is said to have the *property of the sequence* if, for every bounded sequence $\{x_n\}$ of not null elements of $\mathcal{X}$, there exists an $x^* \in \mathcal{X}^*$ such that $x^*(x_n) = 0$ for every $n \in \mathbb{N}$.

Proposition 3 *Every normed space $\mathcal{X}$ and, hence, every direct sum of normed spaces, has the property of the sequence. The same holds true for every (LB) space and every 1. c. s. such that every bounded sequence of $\mathcal{X}$ is contained in a normed space of $\mathcal{X}$.*

PROOF. Let $\mathcal{X}$ be a normed space and $H_n = \{x^* :\ x^*(x) = 0\}$. Since $\mathcal{X}^*$ is a Banach space and every $H_n \neq \mathcal{X}^*$, it follows from the theorem of Baire that $\bigcup_n H_n \neq \mathcal{X}^*$ and, hence, there exists $x^* \in \mathcal{X}^* \backslash \bigcup_n H_n$ such that $x^*(x) \neq 0$ for every $n \in \mathbb{N}$.

Finally, it $\mathcal{X}$ is the direct sum of normed spaces it can be easily proved that $\mathcal{X}$ has the property of the sequence. (It $\mathcal{X}$ is the topological product of an infinite family of Banach spaces, then $\mathcal{X}$ has not the property of the sequence.) ■

Examples: 1. Let Ω be a locally compact space and $\mathcal{C}_{00}(K)$ the space of the scalar continuous functions with support contianed in the compact $K \subset \Omega$, endowed with the usual norm. Then we write $\mathcal{C}_{00}(\Omega)$ (or $K(\Omega)$) for the strict inductive limit of the spaces $\mathcal{C}_{00}(K)$.

If Ω is a countable union of compacts sets then $\mathcal{C}_{00}(\Omega)$ is an (LB) space [3, 2.12, p.164] and it follows from Proposition 3 that $\mathcal{C}_{00}(\Omega)$ has the property of the sequence. We will give a direct proof of this fact assuming that Ω is a σ-compact set. In that case, since Ω is a locally compact space there exists an increasing sequence $\{K_n\}$ of compact subsets of Ω such that every compact set $K \subset \Omega$ is contained in one of them. Then, every bounded sequence $\{\varphi_n\} \subset \mathcal{C}_{00}(\Omega)$ is contained in a subspace $\mathcal{C}_{00}(K)$ and, so, $\mathcal{C}_{00}(\Omega)$ has the property of the sequence. Indeed, let us suppose that $A = \bigcup_n \operatorname{supp} \varphi_n$ is not relatively compact. Then, for every $n \in \ N$, there exists $x_n \in \operatorname{supp} \varphi_n$ such that $\varphi_{k_n}(x) \neq 0$. Let

$$\mu = \sum_n n|\varphi_{k_n}(x_n)|^{-1}\delta_{x_n},$$

then every compact set $K \subseteq \Omega$ has only a finite number of points x_n because K is contained in a K_k and, therefore, $\mu \in \mathcal{C}^*_{00}(\Omega)$. Since $|\mu(|\varphi_{k_n}|)| \geq n$ we get that the sequence $\{\varphi_n\}$ is not bounded, a contradiction.

Let us now see that the result we just obtained remains valid without any restriction. Let $\{\varphi_n\} \subset \mathcal{C}_{00}(\Omega)$ be a bounded sequence. Let G_n be a relatively compact open neighborhood of $A_n = \operatorname{supp} \varphi_n$ and let ψ_n be a continuous function on $\overline{G}_n$ that takes the value 1 on A_n and the value 0 on the border of $\overline{G}_n$. Then $G'_n = \{x \in G_n : \ \psi_n(x) \neq 0\}$ is a σ-compact open neighborhood of A_n and $G' = \bigcup_n G'_n$ is a σ-compact open neighborhood of $A = \bigcup_n A_n$. Since G' is a locally compact space there exists a measure $\nu \in \mathcal{C}_{00}(G')$ such that

$$\nu(\varphi_n) \neq 0 \quad \text{for every } n \in \mathbb{N}. \tag{1}$$

We are not done because the canonical application $\mathcal{C}_{00}(\Omega) \to \mathcal{C}_{00}(G')$ need not be onto. Let $\{K_n\}$ be an increasing sequence of compact sets whose union is G'. Let $\{c_k\}$ be a positive sequence such that $\sum_k c_k|\nu|(K_n \setminus K_{n-1}) \leq 1$ ($K_0 = \emptyset$) and, for every $\alpha = \{\alpha_k\} \in \ell_\infty$ and $\varphi \in \mathcal{C}_{00}(G')$ let us set

$$\mu^\alpha(\varphi) = \sum_k \alpha_k c_k \nu(\varphi\chi_{K_n\setminus K_{n-1}}).$$

Let $H_n = \{\alpha \in \ell_\infty : \ \mu^\alpha(\varphi_n) = 0\}$. It follows from (1) that every $H_n \neq \ell_\infty$, so the theorem of Baire states that there exists an $\alpha \in \ell_\infty \setminus \bigcup_n H_n$ such that $\mu^\alpha(\varphi_n) \neq 0$ for every $n \in \mathbb{N}$. Since $\mu^\alpha \in \mathcal{C}^*_{00}(G')$ is a finite measure, the Borel measure defined by $\mu(A) = \mu^\alpha(A \cap G')$ solves the matter, since belong to $\mathcal{C}^*_{00}(\Omega)$ and it verifies $\mu(\varphi_n) \neq 0$ for every $n \in \ N$.

2. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $\mathcal{D}(\Omega)$ the set of the infinitely differentiable functions with compact support $K \subseteq \Omega$, with its usual topology [3]. If $\{\varphi_n\}$ is a bounded sequence of not null functions of $\mathcal{D}(\Omega)$, then their supports are contained in a fixed compact set $K \subseteq \Omega$. Let $j : \ \mathcal{D}(K) \to \mathcal{C}_{00}(K)$ be the natural injection. Since $\mathcal{C}_{00}(K)$ has the property of the sequence, there exists a measure $\mu \in \mathcal{C}^*_{00}(K)$ such that $j\mu(\varphi_n) = \mu(j\varphi_n) \neq 0$ for every $n \in \mathbb{N}$ and so we get that $\mathcal{D}(\Omega)$ has the property of the sequence.

In general, if $\mathcal{X}$ has the property of the sequence and j is an injective continuous linear application from a l.c.s. $\mathcal{Y}$ into $\mathcal{X}$, then $\mathcal{Y}$ also has the property of the sequence.

3. Let $\mathcal{D}^*(\Omega)$ be the dual of $\mathcal{D}(\Omega)$, that is, the space of the distributions. We will prove that $\mathcal{D}^*(\Omega)$ has not the property of the sequence. Let $\{x_n\} \subset \Omega$ be a sequence not contained in any compact set of Ω and let $T_n = \delta_{x_n}$. Then, $\{T_n\}$ is a bounded sequence of not null elements of $\mathcal{D}^*(\Omega)$ and, for every $\varphi \in \mathcal{D}^{**}(\Omega) \subset \mathcal{D}^*(\Omega)$ there exists an $n \in \mathbb{N}$ such that $T_n(\varphi) = \varphi(x_n) = 0$ and it follows that $\mathcal{D}^*(\Omega)$ has not the property of the sequence.

4. Let $\mathcal{C}(\Omega)$ be the space of continuous functions endowed with the topology of the uniform convergence on the compact subsets of the locally compact space Ω. Then, using the fact that every measure $\mu \in \mathcal{C}^*(\Omega)$ has compact support, we are going to prove that $\mathcal{C}^*(\Omega)$ has the property of the sequence.

First of all, to prove that every positive measure $\mu \in \mathcal{C}^*(\Omega)$ has compact support, let us take into account that there exists a sequence $\{K_n\}$ of compact subsets of Ω such that $\mu(\Omega \setminus \bigcup_n K_n) = 0$ and every K_n is contained in the interior G_{n+1} of K_{n+1}. Then, if the support of μ is not compact, we can suppose, taking subsequences it necessary, that $\mu(K_n) < \mu(G_{n+1})$ for every $n \in \mathbb{N}$. So, for every $n \in \mathbb{N}$ there exists a positive function $\varphi_n \in \mathcal{C}(\Omega$ such that $\mu(\varphi_n) = 1$ and $\operatorname{supp} \varphi_n \subseteq G_{n+1} \setminus K_n$. It follows that $\varphi = \sum_n \varphi_n \in \mathcal{C}(\Omega)$ a contradiction with $\mu(\varphi) = \lim_n \mu(\sum_{k\leq n} \varphi_k) = \infty$.

Now, if $\{\varphi_n\}$ is a bounded sequence in $\beta(\mathcal{C}^*(\Omega), \mathcal{C}(\Omega))$ of not null measures, there exists a constant $M > 0$ such that $|\mu_n(\varphi)| \leq M\|\varphi\|_\infty$ for every bounded function $\varphi \in \mathcal{C}(\Omega)$ and every $n \in \mathbb{N}$. Hence, $|\mu_n|(\Omega) \leq M$ and $\mu = \sum_n 2^{-n}|\mu_n|$ is a finite Radon measure. Moreover, it $\mathcal{K}$ is the class of the compact sets of Ω, then the sets

$$B = \{\varphi \in \mathcal{C}(\Omega) : \ \|\varphi\|_K \leq M_K, \ \forall K \in \mathcal{K}\}$$

are fundamental system of bounded sets of $\mathcal{C}(\Omega)$. So,

$$M = \sup\{|\mu_n(\varphi)| : \ n \in \mathbb{N}, \ \varphi \in \ B\} < \infty$$

and

$$\sup\{|\mu_n|(\varphi) : \ n \in \mathbb{N}, \ \varphi \in \ B\} \leq M.$$

Therefore, $\{|\mu_n|\}$ is also a bounded sequence in $\beta(\mathcal{C}^*(\Omega), \mathcal{C}(\Omega))$ and $\mu = \sum_n 2^{-n}|\mu_n| \in \mathcal{C}^*(\Omega)$ and $K = \operatorname{supp} \mu$ is a compact set.

Then, there exists a sequence $\{f_n\}$ in $L_1(\mu)$, with $f_n \neq 0, \ a.e.$ and such that $\mu_n(A) = \int_A f_n d\mu$ for every Borel set $A \subseteq \Omega$. Since $L_1(\mu)$ is a Banach space, it follows that there exists a bounded Borel function $g \in L_\infty(\mu)$ with $\operatorname{supp} g \subseteq K$ and, hence, $g \in \mathcal{C}^{**}(\Omega)$, and such that $\langle \mu_n, g\rangle = \int g f_n d\mu \neq 0$ for every $n \in \mathbb{N}$. Then $\mathcal{C}^*(\Omega)$ has the property of the sequence.

Let us now prove that if Ω is σ-compact and non compact, then $\mathcal{C}(\Omega)$ has not the property of the sequence. In the hypothesis there exists an increasing sequence $\{K_n\}$ of compact subsets of Ω such that every compact subset of Ω is contained in a K_n. Let $\varphi_n \neq 0$ be a continuous with support in $\Omega \setminus K_n$ and $\mu \in \mathcal{C}^*(\Omega)$. Then, since the support of μ is a compact set $K \subset \Omega$, there exists a $K_n \supseteq K$ and, therefore, $\mu(\varphi_k) = 0$ for $k \geq n$, because the support of φ_k is disjoint with K. It follows that the sequence $\{\varphi_n\}$ is bounded and $\mathcal{C}(\Omega)$ has not the property of sequence.

Moreover, if Ω is an infinite set endowed with the discreet topology, then $\mathcal{C}(\Omega) = \mathbb{R}^\Omega$ has not the property of the sequence.

5. Let $H(\Omega)$ be the space of the holomorphic functions on an open set $\Omega \subseteq \mathbb{C}$ endowed with the topology of the uniform convergence oven the compact subsets of Ω. Then, $H(\Omega)$ has the property of the sequence. Indeed, let $\{\varphi_n\}$ be sequence of not null functions in $H(\Omega)$. Then, since the set of zeros of every φ_n is countable, there exists a point $z \in \Omega$ such that $\varphi_n(z) = 0$ for every $n \in \mathbb{N}$. On the other hand, if $j : \ H(\Omega) \to \mathcal{C}(\Omega)$ is the natural injection, then ${}^{t_j}(\delta_z) \in H^*(\Omega)$ and $\langle \varphi_n, {}^{t_j}(\delta_z)\rangle = \langle j(\varphi_n), \delta_z\rangle = \varphi_n(z) \neq 0$, which proves what we wanted.

6. Let $\mathcal{S}$ be the space of the functions φ infinitely differentiable on such that $(1+|x|^2)^k \partial^p \varphi(x)$ equals 0 in infinite for every $k \in \mathbb{N}$ and $p \in \mathbb{N}^n$, and let $\mathcal{S}^*$ be its dual of the temperate distributions. Then, since $\mathcal{S}$ is a reflexive and Fréchet space, the Proposition 4 states that $\mathcal{S}^*$ has the property of the sequence. On the other hand, since $\mathcal{S}$ is a subspace of $\mathcal{S}^*$ and the injection $j : \ \mathcal{S} \to \mathcal{S}^*$ is continuous, it follows that $\mathcal{S}$ also has the property of the sequence.

Proposition 4 *If $\mathcal{X}^*$ is a Baire space for the $\beta(\mathcal{X}^*, \mathcal{X})$ topology, then $\mathcal{X}$ has the property of the sequence. Therefore, if $\mathcal{X}^*$ is a Fréchet space, then $\mathcal{X}$ has the property of the sequence.*

PROOF. Let $\{x_n\} \subset \mathcal{X}$ be a sequence of not null elements and $H_n = \{x^* \in \mathcal{X}^* : \ x^*(x_n) = 0\}$. Then $H_n \subset \mathcal{X}^*$ is closed for the $\sigma(\mathcal{X}^*, \mathcal{X})$ and $\beta(\mathcal{X}^*, \mathcal{X})$ topologies. Since $\mathcal{X}$ is a Baire space, we have that $\mathcal{X}^* \neq \bigcup_n H_n$. If we now take $x^* \in \mathcal{X}^* \setminus \bigcup_n H_n$ it follows that $x^*(x_n) \neq 0$ for $n \in \mathbb{N}$. ■

Theorem 4 *In the conditions of Theorem 2, if $\mathcal{X}$ is a l.c.s. with property of the sequence, there exists $x^* \in \mathcal{X}^*$ such that $\mu = |x^* \circ \gamma|$ is a control measure of γ.*

PROOF. It suffices to consider that, in virtue of the previous definition, there exists $x^* \in \mathcal{X}^*$ such that $x^* \circ \gamma(C_{n_1,\ldots,n_k}) \neq 0$ when $\gamma(C_{n_1,\ldots,n_k}) \neq 0$ and then reason as in Theorem 2. ■

Remark 1 It $\Omega = \mathbb{N}$, Σ is the σ-algebra of the subsets of Ω, $\mathcal{X} = \mathbb{R}^{\mathbb{N}}$ and δ_n is the corresponding Dirac's delta, then the measure $\gamma = \{\delta_n\}_{n\in\mathbb{N}}$ has the control measure $\mu = \sum_n 2^{-n}|e_n^* \circ \gamma| = \sum_n 2^{-n}\delta_n$ but there exists no $x^* \in \mathcal{X}^*$ such that $|x^* \circ \gamma|$ is a control measure of γ. We notice that $\mathcal{X}$ so chosen is a Fréchet space, and also Montel and nuclear space. ■

Theorem 5 *If $\mathcal{X}$ is a sequentially complete l.c.s. without the property of the sequence, then there exists a countably additive measure $\gamma : \ \Sigma \to \mathcal{X}$ such that for none $x^* \in \mathcal{X}^*$ is the measure $|x^* \circ \gamma|$ a control measure for γ.*

PROOF. According to the hypothesis, there exists a bounded sequence $\{x_n\} \subset \mathcal{X}$ for which the property of the sequence fails. Let $\Omega = \mathbb{N}$, Σ the σ-algebra of the subsets of Ω and $\gamma(A) = \sum_{n\in A} 2^{-n}x_n$ for every $A \in \Sigma$ (γ is well defined because $\mathcal{X}$ is sequentially complete and $\{x_n\}$ is bounded). It can be easily proved that $\gamma : \ \Sigma \to \mathcal{X}$ is a countably measure. Since the property fails for $\{x_n\}$, for every $x^* \in \mathcal{X}^*$ there exists $n \in \mathbb{N}$ such that $|x^* \circ \gamma|(\{n\}) = 2^{-n}|x^*(x_n)| = 0$. Since $\gamma(\{n\}) = 2^{-n}x_n \neq 0$, it follows that no measure $|x^* \circ \gamma|$ can be a control measure for γ. ■

Remark 2 If $\Omega = \mathbb{N}$, Σ the σ-algebra of the subsets of Ω, $\mathcal{X}$ is a l.c.s. with the property of the sequence and $\gamma : \ \Sigma \to \mathcal{X}$ is a measure. Then there exists $x^* \in \mathcal{X}^*$ such that $|x^* \circ \gamma|$ a control measure for γ(see, Theorem 4). ■

Definition 4 If Σ is the σ-algebra of the Borel sets of a topological space Ω and $\gamma : \ \Sigma \to \mathcal{X}$ is a countably additive measure with values in a l.c.s. $\mathcal{X}$, we say that γ is *τ-additive* if, for every family $\{G_i\}_{i\in I}$ of open sets of Ω there exists a countably set $J \subseteq I$ such that, when A is a Borel subset of $\bigcup_{i\in I} G_i \setminus \bigcup_{i\in J} G_i$ then $\gamma(A) = 0$.

Theorem 6 *Theorem 3 is valid when $\mathcal{X}$ is a l.c.s. with the property of the dual sequence and Ω is a subset of a Lusin space and, in particular, when Ω is a separable metric space or it is a metric space but the vector mesure γ is τ-additive. Moreover, we can choose a control measure with the form $\mu = \sum_n |x_n^* \circ \gamma|$ with $x_n^* \in \mathcal{X}^*$. If $\mathcal{X}$ has not the property of the sequence we can not assure the existence of a control measure for γ of the form $\mu = |x^* \circ \gamma|$ with $x^* \in \mathcal{X}^*$. Nevertheless, if every bounded sequence of $\mathcal{X}$ is contained in a Banach subspace, the existence of a control measure of the form $\mu = |x^* \circ \gamma|$ with $x^* \in \mathcal{X}^*$ follows as a consequence of the theorem of Rybakov [1, IX.2, p. 168].*

PROOF. It suffices to prove that, if γ is τ-additive and Ω is a metric space, then γ is supported in a separable space. Let F be the support of γ, $i.e.$, the set of all the points $\omega \in \Omega$ such that the restriction $\gamma_{\mathcal{V}}$ of γ to every neighborhood $\mathcal{V}$ of ω is not null. First of all, let us prove that every open neighborhood $\mathcal{V}$ of any point $\omega \in F$ contains an open set $G \subseteq \Omega$ such that $\gamma(G) \neq 0$. Indeed, there exists Borel set $A \subseteq \mathcal{V}$ such that $\gamma(A) \neq 0$ and an $x^* \in \mathcal{X}^*$ which verities that $x^* \circ \gamma(A) \neq 0$ and, for every $\varepsilon > 0$, there exists an open set G such that $A \subseteq G \subseteq \mathcal{V}$ and $|x^* \circ \gamma(G) - x^* \circ \gamma(A)| \leq |x^* \circ \gamma(G \setminus A)| \leq \varepsilon$. It follows immediately the existence of an open set G which verifies $A \subseteq G \subseteq \mathcal{V}$ and $x^* \circ \gamma(G) \neq 0$ and $\gamma(G) \neq 0$. Using the τ-additivity of γ it follows that every family of disjoint balls with their centers in points of F is countable. Therefore, for every n there exists a sequence $\{B_{nk}\}_k$ of balls with radius $1/n$ which covers F, and, so, F is separable. Moreover, the τ-additivity of γ implies that γ is supported in F, $i.e.$, $\gamma(A) = 0$ if $A \subseteq \Omega \setminus F$, and the support F is proper. ■

Remark 3 Opposity to what happens, in general, in the scalar case, there exists Borel vector measures γ defined on a space of fixed density bigger than $\aleph_0$ which are not τ-additive. To prove this, it suffices to consider the measure $\gamma = \{\delta_x\}_{x\in\Omega}$ of Proposition 1, endowing Ω with metric ϱ defined by $\varrho(x, y) = 1$ if $x \neq y$.

If the density of the metric space Ω is of measure zero, for a countably measure $\gamma : \ \Sigma \to \mathcal{X}$ to have control measure μ it is necessary that γ is τ-additive. To see this, if there exists such control measure μ it must be τ-additive, because the density of Ω is of measure zero. Therefore, if $\{G_i\}_{i\in I}$ is a family of open sets of Ω, there exists a countable set $J \subseteq I$ such that $\mu(\bigcup_{i\in I} G_i \setminus \bigcup_{j\in J} G_i) = 0$. So, if A is a Borel subset of $\bigcup_{i\in I} G_i \setminus \bigcup_{j\in J} G_i$, then $\mu(A) = 0$ and $\gamma(A) = 0$ and, therefore, γ is τ-additive.

If Ω is a metric space whose density is of measure zero and $\mathcal{X}$ is a metrizable space (or (LF)), it follows from this result and Theorem 1 that every Borel measure $\gamma :\ \Sigma \to \mathcal{X}$ is τ-additive.

Many of these results for metric spaces Ω are obviously true when Ω is a topological space such that there exists a metric space Ω' so that Ω and Ω' have the same Borel sets.

Theorem 6 can be extended in a different direction when the measure γ is supported in a Lusin space $\Omega_0 \subseteq \Omega$. This condition is not very restrictive because, it Ω is a Radon space of type $\{\mathcal{K}_m\}$ where $\mathcal{K}_m$ is the class of the compact metrizable sets of Ω, it is necessary that γ is supported in a Lusin subspace for that γ to have a control measure. ■

2. Extension to polymeasures

We are going to see now that the preceding results can be extended to polymeasures [2].

Definition 5 A countably additive polymeasure

$$\gamma :\ \Sigma_1 \times \cdots \Sigma_d \to \mathcal{X}$$

is *uniform in the variable i (or in Σ_i)* if the measures

$$\gamma(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \cdots \times A_d)$$

are uniformly countably additive in the sets $A_j \in \Sigma_j,\ j \neq i$.

Theorem 7 *Let $\gamma :\ \Sigma_1 \times \cdots \Sigma_d \to \mathcal{X}$ be a countably additive polymeasure and $\mathcal{X}$ a metrizable space, or a space such that its origin is a G_δ. Then, if γ is uniform in Σ_1, there exists a countably additive measure $\mu_1 :\ \Sigma \to [0,1]$ such that*

$$\lim_{\mu_1(A)\to 0} \gamma(A, A_2, \ldots, A_d) = 0$$

uniformly in $(A_2, \ldots, A_d) \in (\Sigma_2 \times \cdots \times \Sigma_d)$. ■

Theorem 8 *Let $\gamma :\ \Sigma_1 \times \cdots \Sigma_d \to \mathcal{X}$ be polymeasure where the σ-algebra Σ_i is the class of the Borel sets of a Lusin space Ω_i. Then, if γ is uniform in Σ_1 and $\mathcal{X}$ is a 1.c.s. with property of the dual sequence, there exists a countably additive measure $\mu_1 :\ \Sigma_1 \to [0,1]$ such that*

$$\lim_{\mu_1(A)\to 0} \gamma(A, A_2, \ldots, A_d) = 0$$

uniformly in $(A_2, \ldots, A_d) \in (\Sigma_2 \times \cdots \times \Sigma_d)$.

PROOF. First of all, same as in Theorem 3 we can prove that the closed linear span of the range B of γ is the closed linear span of a bounded sequence. So, since $\mathcal{X}$ the property of the dual sequence, there exists a sequence $x_n^* \in B^0$ such that $x_n^* \circ \gamma(A_1, \ldots, A_d) = 0$ for every $n \in \mathbb{N}$ implies $\gamma(A_1, \ldots, A_d) = 0$.

On the other hand, since γ is uniformly countably additive in Σ_1, every application

$$\Sigma_1 \ni A \to (x_n^* \circ \gamma(A, A_2, \ldots, A_d)) \in \ell_\infty(\Sigma_2 \times \cdots \times \Sigma_d)$$

is a countably additive and, therefore, since $x_n^* \in B^0$, the measure

$$\Sigma_1 \ni A \to \left\{\left(2^{-n} x_n^* \circ \gamma(A, A_2, \ldots, A_d)\right) :\ n \in \mathbb{N},\ A_i \in \Sigma_i\ \forall i\right\}$$

is also countably additive. Corollary 3 [1, I.5] implies now the existence of a countably additive measure $\mu_1 : \Sigma_1 \to [0,1]$ such that $\mu_1(A) = 0$ implies $x_n^* \circ \gamma(A, A_2, \ldots, A_d) = 0$ for every $n \in \mathbb{N}$ and, hence, $\gamma(A, A_2, \ldots, A_d) = 0$.

Let $\mathcal{V}$ be an absolutely convex neighborhood of 0 in $\mathcal{X}$ and $g_{\mathcal{V}}$ the canonical application $\mathcal{X} \to \mathcal{X}_{\mathcal{V}}$. The measure

$$G(A) = (g_{\mathcal{V}} \circ \gamma(A, A_2, \ldots, A_d), \quad (A_2, \ldots, A_d) \in \Sigma_2 \times \cdots \times \Sigma_d \quad (A \in \Sigma_1)$$

is countably additive for the norm such that

$$\|G(A)\| = \sup \{\|g_{\mathcal{V}} \circ \gamma(A, A_2, \ldots, A_d)\|_{\mathcal{V}} : \ A_i \in \Sigma_i \ \forall i\}.$$

So, it follows easily from the theorem of Pettis [1, I.2] that

$$\lim_{\mu_1(A)\to 0} \gamma(A, A_2, \ldots, A_d) = 0$$

uniformly in $(A_2, \ldots, A_d) \in (\Sigma_2 \times \cdots \times \Sigma_d)$. ■

Remark 4 Using this theorem, the first part of Theorem 6 can be generalized te polymeasures. Also, the control measure $\mu_1 : \Sigma_1 \to [0,1]$ can be taken so that

$$\mu_1(A) = \sum_n |x_n^* \circ \gamma|_1(A, A_2^n, \ldots, A_d^n), \quad (A \in \Sigma_1)$$

where $|x_n^* \circ \gamma|_1$ is the variation of the measure $A \to x_n^* \circ \gamma(A, A_2^n, \ldots, A_d^n)$ and $A_k^n \in \Sigma_k$ for $k = 1, \ldots, d$. ■

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