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Quadratic Lie algebras. Algorithms and (de)constructions

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Tesis Doctoral

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Autor: Jorge Roldán López

Directora: María del Pilar Benito Clavijo

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Contents

Abstract

In this dissertation, we study quadratic Lie algebras, with special interest in those which are 2-step nilpotent, and give algorithmic procedures to build a wide range of examples. After an overview of the known results in this matter, we start with a deconstruction process to reduce the study of general quadratic Lie algebras to the nilpotent ones. This is obtained undoing successive double extensions on quotients given from the location of some important ideals. The variety of nilpotent quadratic Lie algebras can be established from free nilpotent Lie algebras and their invariant bilinear forms. But this is a tough problem, so we focus ourselves in the 2-step case. We start by introducing a new method to obtain them using multilinear algebra. Later we prove this new method is equivalent to the two main classical techniques: double and T^*-extensions. In combination with trivectors, we end up giving a classification of these algebras up to dimension 17.

Once covered the 2-step nilpotent case, we start building larger and more general quadratic Lie algebras. This is achieved via double extensions using their skew-derivations, which can be described through the Universal Mapping Property of free nilpotent Lie algebras. After, we study the family of quadratic Lie algebras with only one maximal ideal: the local ones. These algebras have strong structural properties and include the well-known family of real oscillator algebras, which are the quadratic algebras attached to metric Lorentzian forms. The next part is devoted to the ideal structure of quadratic Lie algebras, specially those whose ideals form a chain by their inclusions.

Finally, we introduce and explain how to use a computational package we have developed. This software is supported on the thesis results and includes many tools used along this work.

Resumen

En esta tesis estudiamos las álgebras de Lie cuadráticas, con especial interés en aquellas nilpotentes de índice 2, dando métodos algorítmicos para construir una amplia gama de ejemplos. Después de una visión general de los resultados conocidos sobre este tema, comenzamos con un proceso de deconstrucción que nos permite reducir el estudio de las álgebras de Lie cuadráticas generales a tan solo aquellas nilpotentes. Esta reducción se obtiene deshaciendo sucesivas doble extensiones sobre cocientes, una vez es conocida la ubicación de algunos ideales importantes. La variedad de álgebras de Lie cuadráticas nilpotentes se puede establecer a partir de las álgebras de Lie nilpotentes libres y sus formas bilineales invariantes. Pero hacerlo es difícil, así que nos enfocamos en el caso donde el índice de nilpotencia es 2. Empezamos presentando un nuevo método para obtener dichas álgebras empleando técnicas de álgebra multilineal, para luego demostrar que este nuevo método es equivalente a las dos técnicas clásicas principales: dobles extensiones y T^*-extensiones. En combinación con trivectores, terminamos dando una clasificación de estas álgebras hasta dimensión 17.

Una vez cubierto el caso nilpotente de índice 2, comenzamos la construcción álgebras de Lie cuadráticas más grandes y generales. Esto se logra mediante dobles extensiones usando sus derivaciones antisimétricas, que se pueden describir a través de la propiedad universal para álgebras de Lie nilpotentes libres. Después, estudiamos la familia de álgebras de Lie cuadráticas con un único ideal maximal: las álgebras locales. Estas tienen propiedades estructurales sólidas e incluyen a la conocida familia de álgebras osciladoras reales, que son las álgebras cuadráticas asociadas a formas métricas Lorentzianas. La

siguiente parte está dedicada a la estructura que presentan los ideales de álgebras de Lie cuadráticas, especialmente aquellas cuyos ideales forman una cadena por inclusión.

Finalmente, explicamos cómo usar un paquete computacional que hemos desarrollado. Este software está respaldado por los resultados de la tesis e incluye muchas herramientas utilizadas a lo largo de esta memoria.

1 Introduction

CHAPTER

Lie algebras are non-associative algebraic structures defined over vector spaces. These structures are heavily related to Lie groups of differential varieties, groups where product and inversion are differentiable. Apart from this geometric relation, Lie algebras are also important in the study of differential equations, for example in physics.

These algebras take their name from the Norwegian mathematician Marius Sophus Lie (1842–1899), which discovered an alternative form of studying what now are called Lie groups. He proposed analysing their tangent vector fields which behave like a Lie algebra. It was Hermann Weyl (1885–1955) who applied this theory to the study of groups that, nowadays, model symmetries in quantum mechanics.

Apart from Lie studies in 1870s, in Germany, in 1880s, and independently, Wilhelm Karl Joseph Killing (1847–1923) defined Lie algebras. Although his work was less rigorous in a mathematical sense, he made great progress classifying simple Lie algebras of finite dimension, as well as giving several conjectures which later became proved as true.

The next step forward was made by Élie Joseph Cartan (1869–1951) in France. He took the works of Wilhelm Killing and Friedrich Engel (1861–1941) and was able to complete the simple Lie algebras classification, identifying the four main families and the five exceptional algebras.

Following these classification attempts, we have to mention one of the greatest advances: Levi's Theorem. Despite being conjectured by Killing and Cartan, it was Eugenio Elia Levi (1883–1917) who proved it in 1905. This theorem says every finite dimensional Lie algebra can be decomposed as a direct sum of two subalgebras: one semisimple and another one solvable. It is the solvable part the one which remains open and leads several studies. In 1945, Anatoly Ivanovich Maltsev (1907–1967) reduced the solvable case to the study of nilpotent Lie algebras and their derivations and automorphisms (see [Maltsev, 1945]).

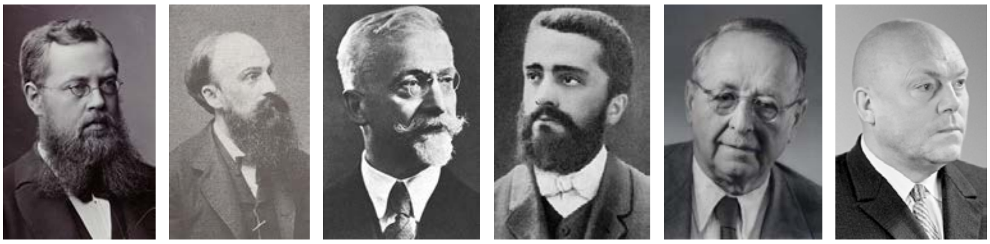

Figure 1.1: From left to right, and in chronological order: Sophus Lie, Wilhelm Killing, Élie Cartan, Eugenio Levi, Hermann Weyl and Anatoly Maltsev.

But this thesis is focused on only certain Lie algebras, the quadratic ones. Quadratic Lie algebras, also named as metrizable, were introduced in 1955 by Shou-Town Tsou, it was in his PhD Thesis titled "On Metrisable Lie Groups and Lie Algebras" (see [Yuan, 1963]). The main results from this dissertation appeared in an article written by S-T. Tsou and A.G. Walker (see margin note for more information about authors and check [Tsou and Walker, 1957] for reference). Here, metric Lie algebras appear as real Lie algebras of Lie groups admitting a Riemannian metric invariant under all translations of the group. In fact, according to [Milnor, 1976, Lemmas 7.1 and 7.2] (see also [Medina, 1985, Lemma 2.1]), the connected Lie groups admitting a bi-invariant Riemannian metric are those Lie groups for which their Lie algebras are quadratic. In [Tsou and Walker, 1957] several decompositions and existence theorems are given, and it is shown that every metrizable algebra decomposes as an orthogonal sum of an abelian algebra and a finite number of non-decomposable reduced ones. The family of quadratic Lie algebras is quite large and contains reductive Lie algebras, and also infinitely many non-semisimple examples. Their structure has essential patterns which can be used to decode the one of some Lie groups. Riemannian geometry makes

Arthur Geoffrey Walker (1909–2001), on the left, was a British geometer. He was famous for its contributions in physics. On the right, his Chinese student Shou-Town Tsou (1918-1993) who spent 2 years in Liverpool (1953–1955) obtaining his PhD before returning to Hong Kong.

this class of algebras visible, but they also play an important role in many other branches of mathematics and physics from Cartan's Criterion up to completely integrable Hamiltonian systems (see [Bordemann, 1997, Section 1]).

Due to the tough classification, along the 2000s, many efforts were made on small dimension classifications of quadratic Lie algebras or in deepening the structure of certain families related to symmetric spaces (Kath, Olbrich, Duong, Benayadi, Hilgert, Neeb). From then until today (see [Ovando, 2016] and references therein), we find papers on classification by using non-classical procedures (Duong, Ovando, Benito, Laliena, de-la-Concepción, Kath, Olbrich) or relating quadratic algebras, geometric structures and their applications (Bajo, Benayadi, Albuquerque, Salgado, Rodriguez Vallarte, Cornulier, del Barco, Conti and Rossi).

Multilinear algebra is closely related to quadratic Lie algebras as ilustrated in the work of Tsou and Walker, or more recently in [Noui and Revoy, 1997], where 2-step quadratic Lie algebras are related to trivectors. On the other hand, their relation with Riemannian geometry (check [Milnor, 1976]) reflects that adjoint and coadjoint representations of their attached Lie algebras are isomorphic. This is why quadratic Lie algebras are also named self-dual. In [Medina, 1985], it is shown the existence of Lie-Poisson structures on the variety of quadratic Lie algebras. The author also describes the basic structural properties of real and non-simple Lie algebras endowed with a Lorentzian metric. This type of algebras is known in quantum mechanic as oscillator algebras because they are related to the motion of n-uncoupled harmonic oscillators near an equilibrium position (see [Ovando, 2006, Ovando, 2007a]). At the same date, Hilgert and Hofmann arrived at the oscillator Lie algebras in their study of Lorentzian semialgebras in [Hilgert and Hofmann, 1985]. Real oscillator algebras are one-dimensional split extensions of Generalized Heisenberg algebras and can be doubly extended to a large series of mixed quadratic Lie algebras (see [Benito and Roldán-López, 2022c]). But quadratic families can be found more easily. For instance, the Killing form turns simple Lie algebras into quadratic. So, reductive Lie algebras, which are direct sum of simple and one-dimensional ideals, so are.

In order to obtain large amounts of quadratic Lie algebras, there are two main construction procedures. The double extension is a classical method introduced in the 1980's to build specifically quadratic Lie algebras. The approach to the method is based on fundamental ideas appeared in two exercises in [Kac, 1983], proposed by the author to his undergraduate students. This observation is made in [Favre and Santharoubane, 1987], where the authors show how to reconstruct a quadratic solvable algebra by double extensions using an isotropic central element. We can found these elements in any quadratic and non-reductive Lie algebra. This is a multi-step process which can be used iteratively to obtain any quadratic Lie algebra (see [Medina and Revoy, 1985]). For indecomposable algebras, the authors justify the claim by deconstructing the algebra using a maximal ideal and its minimal orthogonal ideal. The finite dimensionality ensures that the process ends after a few steps. A pretty explanation of the deconstruction-and-construction procedure is given in [Figueroa-O'Farrill and Stanciu, 1996]. Figueroa and Stanciu refine the Medina and Revoy structure theorem and describe briefly some applications of quadratic Lie algebras to Conformal field Theory and String Theory. In 1997, M. Bordemann in his work [Bordemann, 1997] introduced another approach: the T*-extension method, which produces even-dimensional quadratic Lie algebras but in a single step and works for other non-associative algebras. In fact, the algebras that appear as T*-extensions are those that have a lagrangian ideal. This is the case for the indecomposable 2-step quadratic Lie algebras.

Figure 1.2: From left to right: John Milnor, Karl Heinrich Hofmann, Joachim Hilgert, Ines Kath, Karl-Hermann Neeb, Gabriela Paola Ovando.

Previous methods are good to build examples but they presents difficulties when dealing with the classification problem. The handling of isomorphisms is also not a simple task. Indeed, up to now, only classifications of small dimension have been obtained. In 1987, Favre and Santhorouban easily listed the nilpotent Lie algebras up to dimension 7 (four indecomposable in the list). Until 2007, only the complete list of nilpotent real quadratic Lie algebras up to dimension 10 has been achieved in [Kath, 2007]. The classification of non-nilpotent solvable Lie algebras of dimension greater than six is unknown. In 2012, Duong and Pham described the solvable Lie algebras up to dimension 6. Duong and Ushirobira discussed the classification of solvable Lie algebras up to dimension 8 in their 2014 preprint (see [Duong and Ushirobira, 2017], we have not found any journal reference so far). In 2014, Elduque and Benayadi get the list of complex and real mixed Lie algebras up to dimension 13 (see [Benayadi and Elduque, 2014]). The latest classification is accomplished by employing tools such as representation theory of three-dimensional simple algebras and the structure of small-dimensional Jordan algebras. Given the challenging task of classifying quadratic Lie algebras in low dimension, over time alternative structural proposals have emerged (for a more detailed description, see [Ovando, 2006]). Amalgamated products in the case of quadratic nilpotent found at [Favre and Santharoubane, 1987], bi-extensions in [Keith, 1984] and inflactions in [Hofmann and Keith, 1986] for quadratic mixed. Twofold quadratic extensions in [Kath and Olbrich, 2004, Kath and Olbrich, 2006], and the use of invariant bilinear forms of free nilpotent Lie algebras in [Benito et al., 2017] were given to offer constructions and alternative schemes of classification.

Seeking for isotropic ideals to build quadratic algebras is inherent in the essence of classical constructions. For quadratic indecomposable algebras, solvable and nilpotent radicals are totally isotropic. From [Kath and Olbrich, 2006, Section 3], since socle and Jacobson radical are orthogonal each other, a series of increasing and decreasing ideals can be obtained inside a quadratic Lie algebra. Using both chains, the notion of balanced extensions of quadratic algebras shows up as an alternative of deconstruction of quadratic algebras. The authors pointed out their proposal follows the ideas of the hand-written notes (unpublished) by Berard Bergery "Structure des espaces symetriques pseudo-riemanniens". This is not an unusual situation because the presence of a non-degenerate invariant form exerts a constraining influence on the ideal structure of the algebra (see [Hofmann and Keith, 1986, Corollary 1.4]). In fact, lattices of ideals of quadratic Lie algebras are self-dual by orthogonality. So, the aforementioned chains and their placement follow established patterns.

The idea of deconstructing connects with Maltsev's proposal for general Lie algebras in [Maltsev, 1945] (see also [Oniščik and Hakimdžanov, 1975, Propositions 3 and 4])In a similar way, ideals, nilpotent quadratic Lie algebras, derivations and automorphisms arise as main objects in the theory of general quadratic Lie algebras. And multilinear algebra emerge as a powerful tool. Despite the complete classification of quadratic Lie algebras is hopeless, the study of this variety of algebras, their main structural patterns, relationships with other structures (complex structures, symmetric spaces and Lie triple systems, Manin triples and pairs among others, see chronology in Section 2.3) and applications to theoretical physics and other mathematical branches is an interesting challenge.

1.1 Objectives

The aim of this dissertation is to study the structure of quadratic 2-step nilpotent Lie algebras and beyond, and to develop algorithms based on quadratic Lie structural results. This main goal comes with smaller targets:

- Reach structural results in order to reduce the study of general quadratic Lie algebras to the nilpotent case.
- Present different approaches to get quadratic nilpotent Lie algebras.
- Classification of quadratic nilpotent algebras of nilpotency index 2 up to dimension 17.
- Development of general constructions of quadratic Lie algebras, both solvable and non-solvable, based on previously known and new methods and classifications.

Apart from it, and as a natural consequence of the development, we have achieved other results:

- Study of the ideal structure of quadratic Lie algebras.
- Computational algorithms applied to quadratic Lie algebras.

1.2 Thesis structure

The memoir is divided into six chapters, including this first one, where we are now, which is the introduction. The second chapter, in its first section, explains the classical well-known concepts and results on general Lie algebras we are going to use through the thesis as basic tools. In this chapter, we also have a second section devoted to Lie quadratic structures. This section is essential for the following chapters because it reviews, reformulates and expands main research on quadratic Lie algebras obtained in the course of time by various authors, which is summarize in a timeline at the end of chapter.

The next three chapters are the ones with the main results explained in the objectives (deconstructions, new methods for obtaining quadratic 2-step nilpotent Lie algebras, equivalence among the different methods for 2-step algebras, classifications of these algebras, derivations, automorphisms and ideal structure of quadratic Lie algebras). At the end of each of these chapters, we can find a summary of the main covered results. Most of the presented results have been published or accepted for future publication in different papers.

The final and sixth chapter introduces a computational package which serves as an aid through all the thesis to generate examples and to obtain conjectures before proving new results.

It is important to notice that, at the very end of the memoir, we can find a list of terms and symbols, just after the bibliography. This enumeration can be extremely useful when trying to remember some notation or concepts previously explained.

As a final remark, in this dissertation, all vector spaces are considered finite dimensional over a field $\mathbb{F}$ of characteristic zero. Although, it is worth mentioning many of the results can be established in characteristic different from two.

Background

2

CHAPTER

In this chapter, we will explain the important terms and results which will be the support for the rest of the thesis. We will start from what is a Lie algebra, to continue seeing their types, including quadratic Lie algebras which are the main topic. For these ones, we will describe the classical construction methods, as well as a chronology about how they have been studied and which advancements have been made through time.

Experts on Lie algebras can omit Section 2.1 as it only includes basic well-known definitions and results.

2.1 General Lie algebras

Basics notions and facts on Lie algebras of this section follows from [Jacobson, 1979], [Elduque, 2015] and [Humphreys, 1997].

To start with, we have the most general and basic concept in this thesis, an algebra.

Definition 2.1.1. An *algebra* A is a vector space over a field $\mathbb{F}$ endowed with a binary bilinear product $f\colon A \times A \to A$.

In general algebras, this product $f(x, y)$ is usually denoted as xy. When $(xy)z = x(yz)$ or $xy = yx$ for every $x, y, z \in A$ we have an *associative algebra* or *commutative algebra* respectively. Any field $\mathbb{F}$, its polynomials in one or several variables $\mathbb{F}[x_1, \ldots, x_n]$ and $d \times d$ matrices $\mathcal{M}_d(\mathbb{F})$ provides examples of associative algebras, where the first one is also commutative. But Lie algebras, in general, are neither associative nor commutative. They are defined as follows:

Definition 2.1.2. A *Lie algebra* L is a vector space over a field $\mathbb{F}$ endowed with a binary bilinear product

$$\begin{aligned}[\cdot,\cdot]\colon L \times L &\to L\\ (x,y) &\mapsto [x,y]\end{aligned}$$

satisfying the following two properties:

1. $[x, x] = 0$ for every $x \in L$,
2. for all $x, y, z \in L$ we have the *Jacoby Identity*:

$$J(x, y, z) = [[x, y], z] + [[z, x], y] + [[y, z], x] = 0. \tag{2.1}$$

This product is known as Lie bracket or commutator. Note, that when several Lie brackets appear together, we will add a subindex $[\cdot,\cdot]_L$ to specify to which algebra belongs each.

Remark 2.1.1. Being bilinear in combination with the first property implies the product is skew-symmetric. In fact, when the characteristic of the field is different from two, this fact is equivalent to that first condition.

We can easily find examples of Lie algebras. The most straightforward one is taking any vector space with a null bracket. This type of algebras is called *abelian* and, in them, identity (2.1) becomes trivial. Another well-known example of a Lie algebra is the usual cross product of two vectors u, v denoted as $u \wedge v$ in the euclidean space $\mathbb{R}^3$.

But, in order to obtain more general Lie algebras, we can take the following constructions. Let A be an associative algebra with product xy, if we twist the product in the form

$$[x, y] = xy - yx, \tag{2.2}$$

the new binary product is skew-symmetric and satisfies identity (2.1). The Lie algebra obtained is denoted as A^-.

Example 2.1.1. Let V be a vector space over a field $\mathbb{F}$. If we consider the set of linear endomorphisms[1] of V, $\mathrm{End}_{\mathbb{F}} V$, with the product obtained from natural compositions

$$[x, y] = x \circ y - y \circ x, \tag{2.3}$$

for $x, y \in \mathrm{End}_{\mathbb{F}} V$ we obtain a Lie algebra. This is known as the *general linear Lie algebra*, and it is usually denoted as $\mathfrak{gl}(V)$ or $(\mathrm{End}_{\mathbb{F}} V)^-$.

It is quite common to take a basis and identify endomorphisms with coordinate matrices. In this case, product defined in equation (2.3) becomes like the one in equation (2.2) where xy denotes this time the usual matrix product. Here, if V is a n-dimensional vector space the algebra is referred as $\mathfrak{gl}(n, \mathbb{F})$, $\mathfrak{gl}_n\mathbb{F}$ or simply $\mathfrak{gl}_n$.

General linear Lie algebras play an important role as any Lie algebra of finite dimension can be embedded into a general linear Lie algebra using its adjoint representation (see Definition 2.1.9) as stated by Ado's Theorem (characteristic 0) and Ivasawa's Theorem (prime characteristic), see [Jacobson, 1979, Chapter VI, Section 2 and 3].

Apart from the A^- construction, given an associative algebra with an *anti-involution*[2] $(A, \star)$ we can obtain two different non-associative algebras considering eigenvalues 1 and -1 as $\star^2 = \mathrm{Id}$:

$$\mathrm{Skew}(A, \star) = \{a \in A \mid a^\star = -a\},$$
$$\mathrm{Sym}(A, \star) = \{a \in A \mid a^\star = a\},$$

The first one, under the product given in expression (2.2), is a Lie algebra (subalgebra of A^-). While the last one, with the commutative product $x \cdot y = xy + yx$, becomes a Jordan algebra as it satisfies Jordan identity

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x).$$

This algebra is commutative and non-associative. Both subspaces lead to the $\mathbb{Z}_2$-graded decomposition considering A^- product (see Section 2.1.2.3)

$$A = A_0 \oplus A_1 = \mathrm{Skew}(A, \star) \oplus \mathrm{Sym}(A, \star). \tag{2.4}$$

[1] A morphism from a mathematical object to itself. In the case of vector spaces, it is simply a linear map.

[2] A linear function that is its own inverse and $(xy)^\star = y^\star x^\star$ for every $x, y \in A$.

Example 2.1.2. In the particular case $A = \mathcal{M}_d(\mathbb{F})$, expression (2.4) is the natural direct sum decomposition in skew-symmetric and symmetric matrices.

Next, we are introducing the Lie algebra of derivations.

Definition 2.1.3. Let A be an algebra over a field $\mathbb{F}$. A *derivation* is an endomorphism $d\colon A \to A$ such that for any $a, b \in A$

$$d(a\,b) = a\,d(b) + d(a)\,b.$$

Given two derivations d_1 and d_2, its commutator from equation (2.2) produces another derivation. This way, from any algebra the Lie algebra L we obtain $\mathrm{Der}(L)$, which is the set of derivations considering the bracket product.

In relation with derivations, we can define left and right multiplications. Given any element $x \in L$ Lie algebra, we define

$$\begin{aligned} l_x &= [x, \cdot\,], \\ r_x &= [\,\cdot\,, x]. \end{aligned}$$

Both are derivations given the Jacobi Identity. They are called *inner derivations*. This way we have

$$\mathrm{Inner}(L) := \{\mathrm{ad}\, x : x \in L\},$$

where $\mathrm{ad}\, x = l_x = -r_x$ is called the *adjoint* of x. $\mathrm{Inner}\, L$ is also a Lie algebra, as it is closed under the commutator, so we have the chain

$$\mathrm{Inner}\, L \subseteq \mathrm{Der}\, L \subseteq \mathfrak{gl}(L).$$

Finally, a usual way to describe a Lie algebra is taking a basis and giving the product of any pair of elements in that basis. This way, for a basis $\{e_1, \ldots, e_n\}$, we can define the product by simply giving for $i < j$

$$[e_i, e_j] = \sum_{k=1}^{n} a_{ijk} e_k,$$

where elements a_{ijk} are called *structure constants*. The other way around, a family of scalars $\{a_{ijk}\}$ define uniquely a Lie algebra over the vector space generated by $\mathrm{span}\langle e_1, \ldots, e_n \rangle$ if and only if for every subindex i, j, k, r and m the following conditions are satisfied

(a) $a_{iik} = 0$,

(b) $a_{ijk} + a_{jik} = 0$, and

(c) $a_{ijr}a_{krm} + a_{jkr}a_{irm} + a_{kir}a_{jrm} = 0$.

These scalars are indeed the procedure used in the final section of Chapter 4 to list all algebras in the given classification and they also play an important role in the equivalence of the three construction methods in that same chapter.

2.1.1 Substructures and mappings

2.1.1.1 Subalgebras and ideals

Subalgebras and ideals are main structures in algebras. It is easy to define both notions in general nonassociative algebras, but we will refer only to Lie algebras from now on, in order to have a clearer notation and as it will only be needed there.

Definition 2.1.4. Let L be a Lie algebra and M a subspace of L, we say M is a *subalgebra* if $[x, y] \in M$ for every $x, y \in M$.

Usually, throughout this paper, we will write for simplicity

$$[U, V] = \operatorname{span}\langle [u, v] : u \in U, v \in V \rangle,$$

where span denotes all $\mathbb{F}$-linear combinations of elements in the set. Apart from span, when there is only one element, we will write $\mathbb{F}x = \operatorname{span}\langle x \rangle$. Therefore, M is a subalgebra if $[M, M] \subseteq M$.

In any Lie algebra L we have trivial subalgebras as the zero-dimensional space, any 1-dimensional vector space, or the total algebra itself. In $\mathfrak{gl}(n, \mathbb{F})$ we have other well-known subalgebras as upper or lower triangular matrices, skew-symmetric matrices (check Example 2.1.2), diagonal matrices or the special linear algebra $\mathfrak{sl}(n, \mathbb{F})$ defined as the subset of matrices with zero trace. In the more general $\mathfrak{gl}(A)$ we have already seen the subalgebra $\operatorname{Der} A$.

On the same way, we have ideals, a type of subalgebra with a stronger condition. In contrast with other algebras, we do not have left or right ideals as Lie algebras are skew-symmetric.

Definition 2.1.5. A subspace I of L is said to be an *ideal* when $[I, L] \subseteq I$.

Again, as before, the zero-dimensional or total algebra are also always ideals. But probably, the most wide-known examples are found next:

Definition 2.1.6. Let L be a Lie algebra, we defined its *centre* as

$$Z(L) = \{x \in L : [x, y] = 0 \,\forall\, y \in L\}.$$

Ant its *derived algebra*[3] as $L' = L^2 = [L, L]$.

Both the centre and derived algebra are ideals. Indeed, every subspace of the centre, and every subspace which contains the derived algebra inside are also ideals. The derived algebra plays an important role in some Lie algebras and will appear in Section 2.1.2.1.

Now, whenever we have some subalgebras and ideals, we are ready to produce new ones. Given I, J ideals and A, B subalgebras of the same Lie algebra L we have

- $I + J$, $I \cap J$, $[I, J]$ are ideals,
- $I + A$, $A \cap B$ are subalgebras, and
- $I \cap A$ is an ideal of A.

Here, given U and V subspaces, $U + V$ denotes the usual subspace sum

$$U + V := \{x + y : x \in U, y \in V\}.$$

At the same time, we have the inner direct sum $U \oplus V$ when $U \cap V = 0$. This can also be identified with the more general outer sum $U \times V := \{(x, y) : x \in U, y \in V\}$ taking commutator

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2]).$$

This direct sum prepares us for the following definition:

Definition 2.1.7. A Lie algebra is *decomposable* if it can be expressed as a direct sum of proper ideals. Otherwise, it is called *indecomposable*.

Directly from ideals we also get quotients of Lie algebras in the usual way.

[3]Not confuse derived algebra with algebra of derivations. They are completely different terms.

2.1.1.2 Lie algebras mappings

Immediately after the definition of an algebra, there appear linear mappings that preserve the product.

Definition 2.1.8. Let L_1 and L_2 be Lie algebras, and $\varphi\colon L_1 \to L_2$ a mapping. We say φ is a *Lie algebra homomorphism* if it is linear and for every $x, y \in L_1$

$$\varphi([x,y]_{L_1}) = [\varphi(x), \varphi(y)]_{L_2}.$$

Here, $\ker\varphi$ is an ideal of L_1, and $\operatorname{Im}\varphi$ is a subalgebra of L_2.

The set of homomorphisms from L_1 to L_2 will be written as $\operatorname{Hom}(L_1, L_2)$.

Inside these homomorphisms we can find isomorphisms (bijective), endomorphisms (from some Lie algebra to itself) or automorphisms (bijective endomorphisms). The group of these last morphisms will be denoted as $\operatorname{Aut}(L)$.

One of the main morphisms is the adjoint one, which we have briefly see at the beginning of this chapter when seeing left and right products.

Definition 2.1.9. Let L be a Lie algebra, the *adjoint homomorphism* $\operatorname{ad}\colon L \to \mathfrak{gl}(L)$ is defined as $(\operatorname{ad} x)(y) = [x, y]$.

Whenever there could be some confusion, we will write $\operatorname{ad}_L x$. And, as said earlier, this allows us to embed any algebra inside the general linear one. With these adjoints we have defined $\operatorname{Inner} L$, which is an ideal of $\operatorname{Der} L$ as $[d, \operatorname{ad} x] = \operatorname{ad} d(x)$ for any $d \in \operatorname{Der} L$ and $x \in L$. Even more, from the first out of three classic isomorphisms theorems, which can be easily rewritten in terms of Lie algebras, we get

$$L/Z(L) \cong \operatorname{ad}_L L = \operatorname{Inner} L \tag{2.5}$$

as $\ker \operatorname{ad}_L = \{x \in L : [x, L] = 0\} = Z(L)$.

2.1.1.3 Lie algebras representation

In order to examine an abstract Lie algebra in different ways, we can use representation theory. This allows us to view it as a subalgebra or a quotient inside the general linear algebra $\mathfrak{gl}(V)$.

Definition 2.1.10. Let L be a Lie algebra and V a vector space. A *representation* of L is a Lie algebra homomorphism $\varphi\colon L \to \mathfrak{gl}(V)$ and the vector space V is called *L-module*.

When $\ker\varphi = 0$ we say the representation is *faithful*, and then $L \cong \varphi(L)$. The adjoint representation introduced in Definition 2.1.9 is a classic example, which is faithful whenever $Z(L) = 0$. Another example, related to this one and that we will use in this thesis, is the coadjoint representation.

Definition 2.1.11. Let L be a Lie algebra, the *coadjoint representation* $\mathrm{ad}^*\colon L \to \mathfrak{gl}(L^*)$ is defined as $(\mathrm{ad}^* x)(\beta) = -\beta \circ \mathrm{ad}\, x$, so $(\mathrm{ad}^* x)(\beta)(y) = -\beta([x, y])$ for $x, y \in L$, $\beta \in L^*$ dual space of L.

On the other hand, an equivalent way to define representations is doing it throughout modules:

Definition 2.1.12. Let L be a Lie algebra and V a finite dimensional vector space. A Lie *module* for L or L-module is an action

$$\begin{aligned} \phi\colon L \times V &\to V \\ (x, v) &\mapsto x \cdot v \end{aligned}$$

satisfying for $x, y \in L$, $v, w \in V$ and $\lambda, \mu \in \mathbb{F}$:

- $(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v)$
- $x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w)$
- $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

Both definitions are equivalent concepts as given a representation φ we can define a module $\phi(x, v) := \varphi(x)(v)$, and, on the other way, we define $\varphi(x) := \phi(x, \cdot)$.

And, as it happens in algebras, we also have submodules and factor modules or quotients of modules.

Definition 2.1.13. Let V be a L-module, we say W is a *submodule* if $L \cdot W \subseteq W$.

Definition 2.1.14. Let W be a L-submodule of V. We define the *quotient of a module* or *factor module* as

$$\begin{aligned} \phi\colon L \times V/W &\to V/W \\ (x, v + W) &\mapsto x \cdot (v + W) = (x \cdot v) + W \end{aligned}$$

For examples, ideals produce submodules in the adjoint representation. And, again, given two L-modules U and V, we can easily obtain new combining them:

- summing them, $U + V$ (or $U \oplus V$)
- intersecting them, $U \cap V$,
- and tensorazing them $U \otimes V$. In this case the action is defined as

$$x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v).$$

Definition 2.1.15. A Lie algebra module is called *irreducible* if it is not zero and their unique submodules are itself and the null-space.

Definition 2.1.16. A Lie algebra module U is called *indecomposable* if there is no non-null submodules V and W such that $U = V \oplus W$. Otherwise it is called *decomposable*.

Note irreducible implies indecomposable, but the reciprocal is not true.

Going to the extreme, we arrive at the next definition:

Definition 2.1.17. A Lie algebra module U is called *completely reducible* if

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_k,$$

with U_i irreducible modules for $i = 1, \ldots, k$.

This definition comes in use when describing Weyl's Theorem in Theorem 2.1.23.

Finally, we will see modules homomorphisms which end up defining isomorphism, and the corresponding classical isomorphism theorems.

Definition 2.1.18. Let L be a Lie algebra, V and W L-modules via actions ρ_V and ρ_W respectively. A L-module *homomorphism* is a lineal mapping $\theta\colon V \to W$ such that for each $x \in L$ and $v \in V$ we have

$$\theta(\rho_V(x, v)) = \rho(x, \theta(v)).$$

These morphisms in combination with both representations seen here (adjoint in Definition 2.1.9 and coadjoint in Definition 2.1.11) lead us directly to the following concept:

Definition 2.1.19. An algebra L is called *self-dual* if the adjoint and coadjoint representations are isomorphic, i.e. there is a bijective L-module homomorphism from L to L^*.

These elements in $\mathrm{Hom}_L(L, L^*)$ consists on the linear maps $f\colon L \to L^*$ such that for all $x, y \in L$

$$f([x, y]) = -f(y) \circ \mathrm{ad}\, x. \tag{2.6}$$

Here, we use notation $\mathrm{Hom}_L(V, W)$ to denote the set of L-module homomorphisms from V to W.

With a module and a Lie algebra we can obtain new Lie algebras. This can be done through the *semidirect product* of Lie algebras.

Definition 2.1.20. Let L and M two Lie algebras and $\rho\colon L \to \mathfrak{gl}(M)$ a representation. The vector space $L \times M$ with product

$$[(x, m), (x', m')] = [x, x']_L + \rho(x)(m') - \rho(x')(m) + [m, m']_M,$$

is denoted as $L \ltimes_\rho M$.

Lemma 2.1.2. *$L \ltimes_\rho M$ is a Lie algebra if and only if $\rho(L) \subseteq \mathrm{Der}\, M$.*

This lemma can be easily checked just verifying the Jacobi identity and can be found as Lemma 1.1. on [Benito and de-la-Concepción, 2013].

One trivial example of this construction appears when $L \subseteq \mathrm{Der}\, M$ and ρ is the identity. Another trivial example is obtained when M is an abelian algebra as its derivations are $\mathfrak{gl}(M)$. This case is usually referred as the *split extension* of L by a module M. This semidirect product will be used later in the double extension method.

2.1.2 Types of algebras

We can distinguish two big families of Lie algebras: solvable and semisimple. Later, we will see that in general a Lie algebra is neither solvable, neither semisimple. Instead, it is a mix of both types.

2.1.2.1 Solvable and nilpotent Lie algebras

Before starting, we need a definition and a first lemma:

Definition 2.1.21. Let L be a Lie algebra, we define the *derived Lie algebra* as the product $L' = [L, L]$. This ideal can be extended to form a chain recursively defining $L^{(0)} = L$ and $L^{(k)} := [L^{(k-1)}, L^{(k-1)}]$ obtaining

$$L = L^{(0)} \supseteq L' = L^{(1)} \supseteq L^{(2)} \supseteq \cdots \supseteq L^{(k)}.$$

This chain, called *derived series*, is precisely the one needed to define a solvable Lie algebra:

Definition 2.1.22. A Lie algebra L is said to be *solvable* if there exist $m \geq 1$ such that $L^{(m)} = 0$. The smallest m such that $L^{(m)} \neq 0$ and $L^{(m+1)} = 0$ is called *solvable index*.

This concept comes with some lemmas attached:

Lemma 2.1.3. *If L is a Lie algebra where theres exist a chain of ideals*

$$L = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m-1} \supseteq I_m = 0$$

such that I_{k-1}/I_k is abelian for $1 \leq k \leq m$, then L is solvable.

Indeed, the derived series is one of these chains. Moreover:

Lemma 2.1.4. *Let L be a Lie algebra, then:*

- *If L is solvable, then every subalgebra and homomorphic image of L is solvable.*
- *If L contains a solvable ideal I such that L/I is also solvable, then L must be solvable.*
- *If I and J are solvable ideals of L, then $I + J$ is solvable as well.*

It is precisely this last property which involves a sum the one we need to introduce the next key concept.

Corollary 2.1.5. *Let L be a finite dimensional Lie algebra, then there exist a unique solvable ideal which contains every other solvable ideal of L. This ideal is called* radical ideal *and it is denoted as* $\operatorname{Rad} L$, $R(L)$ *or simply* R.

Analogously, we have another series which introduces the family of nilpotent Lie algebras.

Definition 2.1.23. The *descending central series* (*DCS*), also known as lower central series, of a Lie algebra is defined with terms $L^1 = L$ and $L^k = [L, L^{k-1}]$ for $k \geq 2$. This produces the following chain of ideals

$$L \supseteq L' = L^2 \subset L^3 \supseteq \dots$$

The central name refers to $L^k/L^{k+1} \subseteq Z(L/L^{k+1})$ and also defines an ascending counterpart:

Definition 2.1.24. The *ascending central series* (*ACS*), also known as upper central series, of a Lie algebra is defined from $Z_0(L) = 0$ and $Z_k(L)$ as the ideal of L such that

$$Z(L/Z_{k-1}(L)) = Z_k(L)/Z_{k-1}(L),$$

for $k \geq 1$. This generates the next chain of ideals

$$0 = Z_0(L) \subseteq Z(L) = Z_1(L) \subseteq Z_2(L) \subseteq \dots \subseteq L.$$

Definition 2.1.25. A Lie algebra L is said nilpotent if there exists $t \geq 2$ such that $L^t = 0$. The smallest t such that $L^t \neq 0$ and $L^{t+1} = 0$ is called *nilpotency index* or *nilindex*. Moreover, an algebra whose nilpotency index is t is also called *t-step*.

Remark 2.1.6. Every nilpotent Lie algebra is also solvable, but the converse does not hold.

Abelian and 2-step (also called *metabelian*) Lie algebras are the easiest examples. These last algebras, the 2-step ones, are one our main focus for classification.

Lemma 2.1.7. *For every L Lie algebra such that $L \neq 0$ we have the next results:*

- *If L is nilpotent, then $Z(L) \neq 0$ and all of their subalgebras and quotients are nilpotent too.*
- *If $L/Z(L)$ is nilpotent, then L is nilpotent.*
- *For any I ideal of a nilpotent Lie algebra L, $I \cap Z(L) \neq 0$.*

- *If I and J are nilpotent ideals of L then $I + J$ is also nilpotent.*

And, again, this last sum property gives a maximal nilpotent ideal:

Corollary 2.1.8. *Let L be a finite dimensional Lie algebra, then there exists a unique nilpotent ideal which contains every other nilpotent ideal of L. This ideal is called* nilradical ideal *and it is denoted as* $\operatorname{Nil} L$, $N(L)$ *or simply* N.

Originally, the definition was given as

$$N(L) = \{x \in L \mid \operatorname{ad}_L x \text{ is nilpotent}\}.$$

This idea will be recovered in Engel's Theorem (see Theorem 2.1.14).

In nilpotent Lie algebras we can define the type of them:

Definition 2.1.26. Let $\mathfrak{n}$ be a non-abelian nilpotent Lie algebra, we call *type* of $\mathfrak{n}$ to the codimension of $\mathfrak{n}^2$ in $\mathfrak{n}$, i.e. the type is

$$d = \operatorname{codim} \mathfrak{n}^2 = \dim(\mathfrak{n}/\mathfrak{n}^2) = \dim \mathfrak{n} - \dim \mathfrak{n}^2.$$

In general, the t-tuple $(c_1, \dots, c_t)$ in which the i^{th} component is defined as $c_i = \dim \mathfrak{n}^i/\mathfrak{n}^{i+1}$, i.e. the dimension of the quotient of two consecutive terms in the DCS, will be called the *general type* of some $\mathfrak{n}$ $(t+1)$-step algebra; and $\dim \mathfrak{n} = \sum c_i$.

According to [Gauger, 1973, Corollary 1.3], a set $\mathfrak{m} = \{x_1, x_2, \dots, x_d\}$ generates a nilpotent Lie algebra $\mathfrak{n}$ (of type d) if and only if $\{x_1+\mathfrak{n}^2, \dots, x_d+\mathfrak{n}^2\}$ is a basis of $\mathfrak{n}/\mathfrak{n}^2$. So, the type of a Lie algebra is just the cardinality of any linear independent set $\mathfrak{m}$ such that $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{n}^2$, where $\mathfrak{u}$ is the linear span of $\mathfrak{m}$. The above condition implies that the set $\mathfrak{m}$ generates $\mathfrak{n}$ as an algebra. Therefore, the elements $x_i \in \mathfrak{m}$ can be viewed as a minimal set of generators (m.s.g.) of algebra $\mathfrak{n}$.

Finally, we will denote as $\mathfrak{n}_{d,t}$ the free t-step nilpotent Lie algebra on d generators (see [Benito and de-la-Concepción, 2013] for a formal specification and references therein). Just to give a brief definition (for the notion of free Lie algebras we follow [Jacobson, 1979, Chapter V, Section 4]):

Definition 2.1.27. Let $\mathfrak{FL}(d)$ be the free Lie algebra on a set of d generators over the field $\mathbb{F}$. The free t-nilpotent Lie algebra on d generators is denoted $\mathfrak{n}_{d,t}$ and defined as the quotient algebra

$$\mathfrak{n}_{d,t} = \mathfrak{FL}(d)/\mathfrak{FL}(d)^{t+1}.$$

The elements of $\mathfrak{FL}(d)$ are linear combinations of monomials

$$[x_{i_1}, \ldots, x_{i_s}] = [\ldots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_s}],$$

where $s \geq 1$ and $x_{i_j} \in \mathfrak{m}$. So, the free nilpotent algebra $\mathfrak{n}_{d,t}$ is generated as vector space by s-monomials $[x_{i_1}, \ldots, x_{i_s}]$, for $1 \leq s \leq t$. These monomials form the so-called Hall basis $\mathcal{H}_{d,t}$ for every $\mathfrak{n}_{d,t}$ (see Section 6.2.1 for details).

Any t-nilpotent Lie algebra $\mathfrak{n}$ of type d over $\mathbb{F}$ is an homomorphic image of $\mathfrak{n}_{d,t}$ according to Proposition 1.4 and Proposition 1.5 in [Gauger, 1973] or [Satô, 1971, Proposition 4]. This is called Universal Mapping Property (UMP), and can be seen in the next proposition.

Proposition 2.1.9. *For any k-step nilpotent Lie algebra $\mathfrak{n}$ with $k \leq t$ of type d, and any d-elements $y_1, \ldots, y_d$ of $\mathfrak{n}$, the correspondence $x_i \mapsto y_i$ extends to a unique algebra homomorphism $\mathfrak{n}_{d,t} \to \mathfrak{n}$. In the particular case that $\{y_1, \ldots, y_d\}$ is a m.s.g., the image contains a set of generators, so the map is surjective. Therefore, any t-step nilpotent Lie algebra of type d is an homomorphic image $\frac{\mathfrak{n}_{d,t}}{I}$ where I is an ideal such that $I \subseteq \mathfrak{n}_{d,t}^2$ and $\mathfrak{n}_{d,t}^t \not\subseteq I$.*

We can check [Grayson and Grossman, 1990] for a description of free nilpotent algebras by Hall basis generation and [Benito and de-la-Concepción, 2013] for main features. As main examples of $\mathfrak{n}_{d,2}$ and $\mathfrak{n}_{d,3}$, along the dissertation we will model them by using multilinear algebra as:

$$\begin{aligned}
&\mathfrak{n}_{d,2} = \mathfrak{u} \oplus \Lambda^2\mathfrak{u}, && [u, v] = u \wedge v, \\
&\mathfrak{n}_{d,3} = \mathfrak{u} \oplus \Lambda^2\mathfrak{u} \oplus \frac{\mathfrak{u} \otimes \Lambda^2\mathfrak{u}}{\Lambda^3\mathfrak{u}}, && [u, v \wedge w] = u \otimes v \wedge w \mod \Lambda^3\mathfrak{u}.
\end{aligned} \tag{2.7}$$

2.1.2.2 Semisimple and simple Lie algebras

The nullity of the maximal solvable ideal of a Lie algebra leads to semisimple algebras.

Definition 2.1.28. A *semisimple* Lie algebra is a Lie algebra with no solvable ideals, i.e. its solvable radical is zero.

Note that if L is semisimple, then $[L, L] = L$ and $Z(L) = 0$. This way $L = 0$ is semisimple. Moreover, for an arbitrary L, the quotient $L/\operatorname{Rad} L$ is also semisimple. But the main examples are simple Lie algebras.

Apart from semisimple algebras, there are some Lie algebras L such that $L^2 = L$, they are called *perfect*.

Example 2.1.3. Let L be a semisimple Lie algebra of dimension n, and let V be an irreducible module of L of dimension $m \geq 2$. We define over $L \times V$ the bracket

$$[(x,u),(y,v)]_{L\times V} = ([x,y]_L, x\cdot v - y\cdot u).$$

Then, $L \times V$ is a perfect Lie algebra of dimension $n+m$. Moreover, $\mathrm{Rad}(L \times V) = V$ so it is not semisimple. This split extension appears previously written as $L \ltimes V$.

Despite its definition in terms of its radical, a semisimple Lie algebra can be nicely described in characteristic zero using its decomposition as an ideal direct sum of simple algebras. These simple factors, as we have seen previously in the introductory chapter, were the first classified Lie algebras.

Definition 2.1.29. A Lie algebra is *simple* when it is not abelian and its unique ideals are the trivial ones (total and zero).

All these simple Lie algebras were classified by Élie Cartan in this PhD Thesis in 1894 using ideas previously introduced by Killing. Over an algebraically closed field, except five exceptional simple Lie algebras ($\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$), every simple Lie algebra is isomorphic to one of the following four families, known as classical algebras[4]:

- A_n for $n \geq 1$, known as the special linear series. This algebra is

  $$\mathfrak{sl}(V) = \{f \in \mathfrak{gl}(V) : \mathrm{Tr}\, f = 0\},$$

  where $\dim V = n+1$. Note the trace of an endomorphism is independent from the chosen basis and, under linear combinations and the commutator, that nullity is preserved.

- C_n for $n \geq 1$, known as the symplectic series. This algebra is

  $$\mathfrak{sp}(V,\varphi) = \{f \in \mathfrak{gl}(V) : \varphi(f(x),y) + \varphi(x,f(y)) = 0\},$$

  where φ is bilinear skew-symmetric nondegenerate, and $\dim V = 2n$.

[4]It is not unusual to relate simple Lie algebras to its Dynkin diagram. These four diagrams are the ones which define A_n, B_n, C_n and D_n types notation.

- B_n and D_n for $n \geq 3$, known as the orthogonal series. These algebras are

$$\mathfrak{so}(V,\varphi) = \{f \in \mathfrak{gl}(V) : \varphi(f(x),y) + \varphi(x,f(y)) = 0\}, \tag{2.8}$$

where φ is bilinear symmetric nondegenerate and $\dim V = 2n+1$ in the case B_n and $\dim V = 2n$ when the type is D_n. Some authors refers to them as $\mathfrak{o}$ instead of $\mathfrak{so}$.

Remark 2.1.10. For smaller dimensions, some types are equivalent. This way we have $A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, and $D_3 \cong A_3$. To avoid all these relations, some authors only consider $A_{n\geq 1}$, $B_{n\geq 2}$, $C_{n\geq 3}$ and $D_{n\geq 4}$.

Remark 2.1.11. If we extend D_n for $n \geq 1$ we obtain the semisimple but non-simple algebras $D_1 \cong \mathbb{F}$, $D_2 \cong A_1 \times A_1$.

Remark 2.1.12. Let $\varphi\colon V \times V \to \mathbb{F}$ bilinear non-degenerate, it induces an isomorphism from V to V^* which maps v to $\varphi(v,\cdot)$. This leads to the involution of the General Linear Lie algebra

$$\begin{aligned} \star_\varphi\colon \mathfrak{gl}(V) &\to \mathfrak{gl}(V) \\ f &\mapsto f^\star, \end{aligned}$$

where $f^\star$ is the only mapping such that

$$\varphi(f(x),y) = \varphi(x,f^\star(y)).$$

This way $\mathrm{Skew}(\mathfrak{gl}(V),\star_\varphi) = \{f \in \mathfrak{gl}(V) : f^\star = -f\}$ is equivalent to

- $\mathfrak{so}(V,\varphi)$ when φ is symmetric,
- $\mathfrak{sp}(V,\varphi)$ when φ is skew-symmetric.

All these families can be viewed taking basis in form of matrices. This way, the counterpart of $\mathfrak{sl}(V)$ appeared as $\mathfrak{sl}(n,\mathbb{F})$ in Subsection 2.1.1.1. For the rest of them, we can define the auxiliar subalgebra

$$\mathfrak{gl}_S(n,\mathbb{F}) := \{M \in \mathfrak{gl}(n,\mathbb{F}) : M^t S = -SM\}.$$

Now, if I_n is the identity matrix and 0_n the null matrix, both of dimension $n \times n$:

- The special orthogonal Lie algebra, $\mathfrak{so}(n,\mathbb{F}) = \mathfrak{gl}_S(n,\mathbb{F})$ for

$$S = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix} \qquad \text{or} \qquad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0_k & I_k \\ 0 & I_k & 0_k \end{pmatrix}$$

  according to the parity of n, i.e. $n = 2k$ or $n = 2k+1$.

- The symplectic Lie algebra (even dimension) $\mathfrak{sp}(2n,\mathbb{F}) = \mathfrak{gl}_S(2n,\mathbb{F})$ for

$$S = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$$

Applying this definitions we obtain:

$$\begin{aligned} \mathfrak{so}(2n,\mathbb{F}) &\cong \left\{ \left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix} \right) : M^t = -Q, N^t = -N, P^t = -P \right\} \\ \mathfrak{so}(2n+1,\mathbb{F}) &\cong \left\{ \begin{pmatrix} \begin{smallmatrix} 0 & p & q \\ r & M & N \\ s & P & Q \end{smallmatrix} \end{pmatrix} : \left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix} \right) \in \mathfrak{so}(2n,\mathbb{F}), r^t = -q, s^t = -p \right\} \\ \mathfrak{sp}(2n,\mathbb{F}) &\cong \left\{ \left(\begin{smallmatrix} M & N \\ P & Q \end{smallmatrix} \right) : M^t = -Q, N^t = N, P^t = P \right\} \end{aligned}$$

Analogous to Remark 2.1.10, we have $\mathfrak{so}(2,\mathbb{F}) \cong \mathbb{F}$, $\mathfrak{so}(3,\mathbb{F}) \cong \mathfrak{sl}(2,\mathbb{F}) \cong \mathfrak{sp}(2,\mathbb{F})$, $\mathfrak{so}(4,\mathbb{F}) \cong \mathfrak{sl}(2,\mathbb{F}) \times \mathfrak{sl}(2,\mathbb{F})$, $\mathfrak{so}(5,\mathbb{F}) \cong \mathfrak{sp}(4,\mathbb{F})$, and finally $\mathfrak{so}(6,\mathbb{F}) \cong \mathfrak{sl}(4,\mathbb{F})$.

In general, a Lie algebra L over a field $\mathbb{F}$ is said to be of classical type X ($X = A, B, C, D$) in case its extension by scalars (see [Jacobson, 1979, Chapter I, Section 8]) $L \otimes \overline{\mathbb{F}}$ is simple of type X. Here $\overline{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$. In this way, the number of Lie algebras over non-algebraically closed fields, as for instance $\mathbb{R}$, is way bigger than the closed case (check [Elduque, 2015]). This situation we can be observed in the following example.

Example 2.1.4. In $\mathbb{R}$ we can find other simple Lie algebras as

$$\mathfrak{su}(n,\mathbb{R}) = \{A \in \mathfrak{sl}(n,\mathbb{C}) : a_{ij} = -\overline{a_{ji}}\}$$

which has the same dimension of $\mathfrak{sl}(n,\mathbb{R})$ as a $\mathbb{R}$-vector space but it is not isomorphic. Although $\mathfrak{su}(n,\mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{sl}(n,\mathbb{C})$.

Simple Lie algebras whose extension by scalars to the algebraically closed field remains being simple are called *central simple*.

2.1.2.3 Graded Lie algebras

In contrast to previous types of algebras, another important family is graded algebras. This section is based on [Elduque and Kochetov, 2013, Section 1.1]. When restricting ourselves to just monoid grading on Lie algebras, we find the following definition:

Definition 2.1.30. Let G be a commutative semigroup, a G-*grading* on L is any decomposition of L into a direct sum of subspaces indexed by G

$$L = \bigoplus_{g\in G} L_g$$

which respects the Lie bracket, i.e. $[L_g, L_h] \subseteq L_{gh}$. In this case L is said to be *graded*.

On this memoir, the semigroup G can be $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}_n$ with the binary sum. When $G = \mathbb{N}$ we say the algebra is *naturally graded*, and when $G = \mathbb{N}^+$ it is *positive naturally graded*, depending on whether the zero in included or not.

Different types of Lie algebras can be seen as graded, including simple (see Example 2.1.5) or solvable (see Example 2.1.6).

Example 2.1.5. The smallest example of a gradation in a simple algebra is a $\mathbb{Z}_3$-gradation in $\mathfrak{sl}_2$. If we take the natural basis $\{e, f, h\}$ with products

$$[e, f] = h, \qquad [h, f] = -2f, \qquad [h, e] = 2e,$$

we have the decomposition

$$\mathfrak{sl}_2 = (\mathfrak{sl}_2)_{-1} \oplus (\mathfrak{sl}_2)_0 \oplus (\mathfrak{sl}_2)_1 = \mathrm{span}\langle e\rangle \oplus \mathrm{span}\langle h\rangle \oplus \mathrm{span}\langle f\rangle.$$

This same decomposition in three parts can be seen directly as a $\mathbb{Z}$-gradation, and, with slight changes, we can also define the $\mathbb{Z}_2$-gradation

$$\mathfrak{sl}_2 = (\mathfrak{sl}_2)_0 \oplus (\mathfrak{sl}_2)_1 = \mathrm{span}\langle h\rangle \oplus \mathrm{span}\langle e, f\rangle.$$

Example 2.1.6. All nilpotent 2-step Lie algebras L admit a $\mathbb{Z}_2$-gradation considering the decomposition

$$L = L_0 \oplus L_1$$

where $L_0 = L^2$ and L_1 is a complement of L_0.

Example 2.1.7. Let $\mathfrak{n}_{d,t}$ be the free nilpotent Lie algebra with $\mathfrak{m} = \{x_1, \ldots, x_d\}$ a m.s.g. If we set $\mathfrak{u} = \mathrm{span}\langle\mathfrak{m}\rangle$, the subspace $\mathfrak{u}^s = [\mathfrak{u}^{s-1}, \mathfrak{u}]$ is the linear span of the s-monomials. Thus $\mathfrak{n}_{d,t}$ is an $\mathbb{N}$-graded algebra whose s-th homogeneous component is $\mathfrak{u}^s$. The dimension of any subspace $\mathfrak{u}^s$, $1 \leq s \leq t$ is

$$\frac{1}{s}\sum_{a|s}\mu(a)d^{s/a},$$

where μ is the Möbius function.

Definition 2.1.31. Let $L = \bigoplus_{g\in G} L_g$ be a gradation. An ideal I of L is called *homogeneous* if

$$I = \bigoplus_{g\in G}(L_g \cap I).$$

Among natural graded Lie algebras, in this thesis we are interested in the so called quasi-cyclic

Definition 2.1.32. A finite-dimensional *quasi-cyclic* (also known as *homogeneous* or *Carnot*) Lie algebra is a positive naturally graded algebra $L = L_1 \oplus L_2 \oplus \cdots \oplus L_t$ generated as an algebra by L_1. This means $[L_i, L_j] \subseteq L_{i+j}$ (here $L_s = 0$ for $s > t$) and $L_i = [L_1, L_{i-1}] = [L_1, L_1^{i-1}]$.

As seen, this can only be applied for nilpotent Lie algebras.

Following [Cornulier, 2016, Definition 3.3], the terms quasi-cyclic, graded, naturally graded or homogeneous are better known in Lie algebras, while the word Carnot (graded) is more commonly used in sub-Riemannian and conformal geometry. These algebras were introduced at [Leger, 1963] by G. Leger in 1963. The variety of quasi-cyclic Lie algebras includes free nilpotent $\mathfrak{n}_{d,t}$, generalised Heisenberg (see Definition 2.2.9) and filiforms among others Lie algebras. It is not hard to see that a nilpotent Lie algebra $\mathfrak{n} \cong \frac{\mathfrak{n}_{d,t}}{I}$ is quasi-cyclic if and only if I is a homogeneous ideal of $\mathfrak{n}_{d,t}$. In fact, quasi-cyclic Lie algebras are the class of nilpotent Lie algebras that contain a m.s.g. $\{e_1, \ldots, e_d\}$ such that the correspondence $e_i \mapsto e_i$ extends to a derivation of $\mathfrak{n}$ according to [Johnson, 1975, Corollary 1]. Note that such a derivation is invertible.

Definition 2.1.33. An automorphism of a real Lie algebra is called *expanding automorphism* if it is semisimple[5] with eigenvalues greater than 1 in absolute value.

[5]Here semisimple refers to a group and means it has no non-trivial normal abelian subgroups.

According to [Dyer, 1970], quasi-cyclic Lie algebras admits expanding automorphisms, but the converse is false (see Example 5.1.3). In fact, real quasi-cyclic Lie algebras are those Lie algebras that admit *grading automorphisms* (see [Johnson, 1975]). And following [Deré, 2017, Theorem 3.1 and Theorem 3.3] (see also [Cornulier, 2016]), the class of real Lie algebras admitting expanding automorphisms is just the class of positive graded Lie algebras (positive naturally graded onwards) which is bigger than the quasi-cyclic class. In the realm of nilpotent Lie groups, the existence of an expanding map (respectively a non-trivial self-cover) in an infra-nilmanifold modeled on a Lie group G is equivalent to the fact that the real algebra Lie(G) admits a positive grading (respectively a naturally and non-trivial grading). Expanding automorphisms of real Lie algebras are hyperbolic (maps without eigenvalues ± 1), and Lie algebras admitting *hyperbolic automorphisms* are nilpotent (see [Smale, 1967, Proposition 3.6] and Definition 3.2.1). Some naturally graded parametric families of Lie algebras will be constructed in our Chapter 5.

2.1.3 Structure results

We will start explaining two main results that are equivalent to the nilpotency or solvability of a Lie algebra. The essence of both results is the existence of common eigenvectors for Lie algebra endomorphisms as it is established in the two lemmas preceding the main theorems. The second theorem requires the field to be algebraically closed in order to guarantee it contains all required eigenvalues.

Lemma 2.1.13. *Let L be a subalgebra of $\mathfrak{gl}(V)$. If L consists of nilpotent endomorphisms and $V \neq 0$, then there exists a nonzero $v \in V$ for which $x(v) = 0$ for all $x \in L$.*

Theorem 2.1.14 (Engel's Theorem)**.** *A Lie algebra L is nilpotent if and only if $\operatorname{ad} x$ is nilpotent for every element $x \in L$.*

In general, $\rho(x)$ is nilpotent for any element $x \in [L, \operatorname{Rad} L]$ for any representation $\rho\colon L \to \mathfrak{gl}(V)$. Thus, we can find the next result which relates radical and nilradical and appears in [Jacobson, 1979, Theorem 13, p. 51].

Theorem 2.1.15. *Let L be a finite dimensional Lie algebra with radical R and nilradical N, then $[L, R] \subseteq N$.*

Even more,

$$R^2 \subseteq [L,R] = L^2 \cap R = R(L^2) \subseteq N \subseteq R. \tag{2.9}$$

Ideal $[L,R]$ coincides with the *Jacobson radical* and will be denoted as $\mathcal{J}(L)$. We have that L is solvable if and only if $\mathcal{J}(L) = L^2$. This radical is originally defined in [Marshall, 1967], which also gives a short proof of expression (2.9), as

$$\mathcal{J}(L) = \bigcap \{I : I \text{ is a maximal ideal of L}\}.$$

Theorem 2.1.15 and equation (2.9) can also be seen as a consequence of Lie's Theorem, which applies in combination with the following lemma over algebraically closed fields of characteristic zero.

Lemma 2.1.16. *Let $\rho\colon L \to \mathfrak{gl}(V)$ be a representation of a solvable Lie algebra L. Then, there exists a nonzero element $v \in V$ such that $\rho(x)(v) \in \operatorname{span}\langle v\rangle$ for any $x \in L$.*

Remark 2.1.17. Previous lemma is valid even if the field is not an algebraically closed, as long as the minimum polynomial of $\rho(x)$ splits over it. In this case, ρ is said to be a *split representation*.

Theorem 2.1.18 (Lie's Theorem)**.** *Let $\rho\colon L \to \mathfrak{gl}(V)$ be a split representation of a solvable Lie algebra L. Then, there exists a basis of V such that the coordinate matrix of any $\rho(x)$ for $x \in L$ is upper triangular.*

Also as a consequence of Lie's Theorem we have the next lemma:

Lemma 2.1.19. *L is solvable if and only if L^2 is nilpotent.*

Although, solvable Lie algebras can also be characterized thanks to Cartan's criterion based on the traces of inner derivations.

Theorem 2.1.20 (Cartan's Criterion for solvability)**.** *A Lie algebra L in characteristic zero is solvable if and only if $\kappa(x,y) = 0$ for all $x \in L$ and $y \in L'$.*

Here $\kappa(x,y)$ is the *Killing form*, which is defined as $\kappa(x,y) = \operatorname{Tr}(\operatorname{ad}\, x \circ ad\, y)$ where Tr denotes the trace.

This same Killing form serves us to characterize semisimple algebras using another Cartan's criterion:

Theorem 2.1.21 (Cartan's Criterion for semisimplicity)**.** *A Lie algebra L in characteristic zero is semisimple if and only if the Killing form κ of L is non-degenerate.*

As a consequence of these criteria, we have the following properties:

- A Lie algebra is semisimple if and only if it is a direct sum of simple ideals.
- All ideals and quotients of semisimple Lie algebra are also semisimple
- All derivations of semisimple Lie algebras are inner derivations. So $\operatorname{Der} L = \operatorname{Inner} L = \{\operatorname{ad} x : x \in L\}$.

In view of the previous Cartan's criteria, the Killing form is extremely important. It is also symmetric, bilinear, and satisfies an associative property

$$\kappa([x, y], z) = \kappa(x, [y, z]). \tag{2.10}$$

The study of the representation theory of the semisimple Lie algebras is one of the most important areas of research in Lie algebras. An important result in this field is the next one:

Lemma 2.1.22 (Schur's lemma)**.** *Let $\rho\colon L \to \mathfrak{gl}(V)$ over an algebraically closed field of characteristic zero be an irreducible representation. Then, the only endomorphisms of V commuting with all $\rho(x)$ for $x \in L$ are the scalars.*

Using the notation introduced after equation (2.6), Schur's lemma can be rewritten as $\operatorname{Hom}_L(V, V) = \mathbb{F}\operatorname{Id}_V$. Moreover, note L itself is an L-module using the adjoint representation. Here an L-submodule is just an ideal, so it follows that a simple algebra L is irreducible as L-module, while a semisimple algebra is completely reducible as stated in the following theorem:

Theorem 2.1.23 (Weyl's Theorem)**.** *Let L be a semisimple Lie algebra. Then, each finite dimensional representation is completely reducible. Moreover, the number of irreducible submodules in every decomposition is invariant and, up to reorder, those submodules are isomorphic.*

Lemma 2.1.24. *Let $\rho\colon L \to \mathfrak{gl}(V)$ be a representation of a semisimple Lie algebra L, then $\rho(L) \subseteq \mathfrak{sl}(V)$. In particular, L acts trivially on 1-dimensional L-modules.*

Once explained these initial results, and known the two big families of Lie algebras (semisimple and solvable), we are ready to see how everything is connected. One of the important consequences of Weyl's Theorem is the Levi Theorem that provides the *Levi's decomposition* of any Lie algebra which is finite-dimensional.

Theorem 2.1.25 (Levi Theorem)**.** *Let L a Lie algebra of finite dimension over a field of characteristic zero, then there exists a subalgebra $S \subseteq L$ such that $L = S \oplus R$ where R is the radical of L.*

In this decomposition S is referred as *Levi's factor* or *Levi subalgebra*. But there are still some results to introduce before completely understanding this result. First, let us focus on the Levi's factor. Despite, not being unique, the following result from [Jacobson, 1979] indicates all possible factors are isomorphic.

Theorem 2.1.26 (Maltsev-Chandra's Theorem)**.** *Let L be a finite dimensional Lie algebra over a field of characteristic zero with Levi's decomposition $L = S \oplus R$. For each semisimple Lie algebra S_1 of L there exists $\varphi \in \operatorname{Int} L$ such that $\varphi(S_1) \subseteq S$.*

Here, $\operatorname{Int} L$ denotes the subgroup of $\operatorname{Aut} L$ generated by internal automorphisms, i.e. elements of the form

$$\exp(\operatorname{ad} x) = \sum_{n=0}^{\infty} \frac{(\operatorname{ad} x)^n}{n!}$$

for $x \in \operatorname{Nil} L$.

And, these Levi factors are semisimple Lie algebras as they are isomorphic to $L/\operatorname{Rad} L$ which is semisimple.

Finally, along this work we will use the following notion

Definition 2.1.34. We would say a Lie algebra L is *mixed* when in its Levi's decomposition $L = S \oplus R$ both S and R are not null.

Levi Theorem and previous notions on solvability and semisimplicity lead to the deconstruction of Lie algebras summarize in Figure 2.1. Here, we observe the deconstruction $L = R \rtimes_{\mathrm{ad}} S$ following Definition 2.1.20.

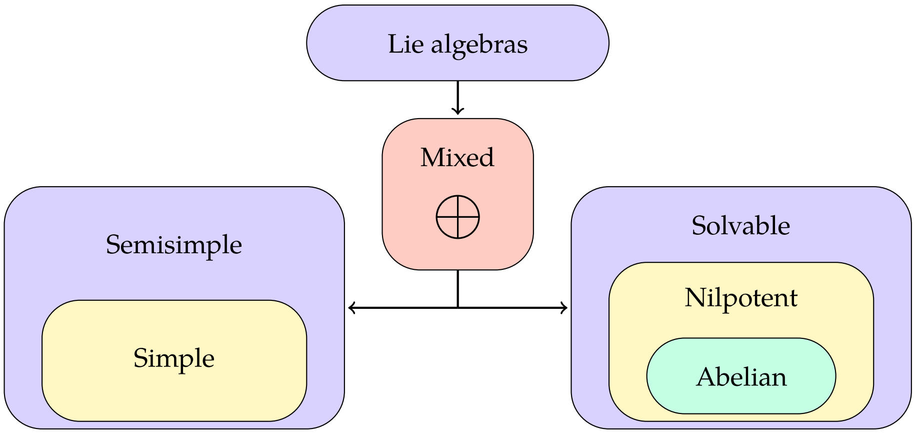

Figure 2.1: Schema of general Lie algebras types and decomposition.

2.2 Quadratic Lie algebras

The global structure of invariant Lie groups is encoded in the algebraic structure of their (real) metric Lie algebras. In 1985, Medina (see [Medina, 1985, Lemma 2.1 and Corollary 2.2]) provides the following equivalent conditions on the existence of bi-invariant metrics on Lie groups:

> For a given Lie group G and its Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, the following statements are equivalent:
>
> (a) The group G is endowed with a bi-invariant metric.
>
> (b) The algebra $\mathfrak{g}$ has a metric such that the adjoint action of G on $\mathfrak{g}$ is given by isometries.
>
> (c) The adjoint and coadjoint representations of $\mathfrak{g}$ are isomorphic by means of an isomorphism $\psi\colon \mathfrak{g} \to \mathfrak{g}^*$ that satisfies $\psi(a)(b) = \psi(b)(a)$.
>
> Moreover, if G is a connected group, condition (a) is also equivalent to,
>
> (d) The algebra $\mathfrak{g}$ has a quadratic form $q\colon \mathfrak{g} \to \mathbb{R}$ and for every $x \in \mathfrak{g}$, the linear transformations $\mathrm{ad}\, x$ is skew-adjoint with

respect to the bilinear φ form attached to q, i.e. $\varphi(x,y) = q(x+y) - q(x) - q(y)$.

These statements have been previously established in [Milnor, 1976, Lemmas 7.1 and 7.2]. Since $\operatorname{ad} x(y) = [x,y]$ is the left product on $\mathfrak{g}$, condition (d) is equivalent to

$$\varphi([x,y],z) + \varphi(y,[x,z]) = 0,$$

for all $x,y,z \in \mathfrak{g}$.

2.2.1 Basic facts

Definition 2.2.1. Let A be a non-associative[6] algebra A with product xy, and let $\varphi\colon A \times A \to \mathbb{F}$ be a bilinear form. The pair (A,φ) is named *pseudo-quadratic* whenever

- φ is invariant, i.e. $\varphi(xy,z) = \varphi(x,yz)$,
- and φ is *non-degenerated*, which means

$$\ker \varphi(\,\cdot\,,y) := \{x : \varphi(x,y) = 0\ \forall\, y\} = 0.$$

And, if φ is also symmetric (A,φ) is *metric* or *quadratic*.

In the literature they also appear named as metrized, metrizable (usual names for algebras over the real field), orthogonal, regular quadratic, quasi-classical or symmetric self-dual. Along this thesis we use mostly the term quadratic.

Remark 2.2.1. In [Hilgert and Neeb, 1996, Lemma 1] it is proved that for real Lie algebras pseudo-metric and metric notions are equivalent.

Given any Lie algebra L with product $[x,y]$, since it is skew-symmetric the invariant condition, also named associative as in equation (2.10), of the bilinear form φ can be rewritten as

$$\varphi([x,y],z) + \varphi(y,[x,z]) = 0,$$

[6]Non-associative includes associative. This remark will be useful when defining associative quadratic algebras combined with quadratic Lie algebras.

or using the adjoint representation

$$\varphi(\mathrm{ad}(x)(y), z) + \varphi(y, \mathrm{ad}(x)(z)) = 0.$$

This is equivalent to check if for some φ

$$\mathrm{Inner}\, L \subseteq \mathfrak{so}(L, \varphi). \tag{2.11}$$

As $\Lambda^2 V \cong \mathfrak{so}(V, \varphi)$ using the map $a \wedge b \to \varphi_{a,b}$, we can see $\mathfrak{so}(L, \varphi)$ is linearly spanned by the linear maps $\varphi_{a,b} = \varphi(a, \cdot)b - \varphi(b, \cdot)a$ for $a, b \in L$. This algebra is formed by the φ-skew endomorphisms. Also, at this point, we can define

$$\mathrm{Der}_\varphi(L) := \mathrm{Der}\, L \cap \mathfrak{so}(L, \varphi), \tag{2.12}$$

the φ-skew derivations, which will serve us later. This is a subalgebra of $\mathrm{Der}\, L$ that contains the inner derivations ideal.

Once we have a definition of being quadratic, we will see how it applies over semisimple Lie algebras.

Example 2.2.1. Any semisimple Lie algebra with its Killing form is quadratic.

In fact, the non-degeneration of this form characterizes (in characteristic zero) the class of semisimple Lie algebras according to Cartan's Criterion (see Theorem 2.1.21).

And which is more important, every bilinear symmetric nondegenerate invariant form associated to any semisimple Lie algebra is a linear combination of the Killing form of its simple ideals.

Proposition 2.2.2. *Let (L, φ) be a simple quadratic Lie algebra over a field of characteristic zero which is algebraically closed. Then φ is a scalar multiple of the Killing form of L.*

Proof. If L is simple, L is an irreducible L-module via the adjoint representation. Let $\varphi\colon L \times L \to \mathbb{F}$ be a bilinear from we can define the L-module isomorphism

$$\begin{aligned} \hat{\varphi}\colon L &\to L^* \\ x &\mapsto \varphi(x, \cdot). \end{aligned}$$

Thus, we can define $\hat{\kappa}$ and $\hat{\kappa}^{-1}$. This way, $\hat{\varphi} \circ \hat{\kappa}^{-1} \in \mathrm{Hom}_L(L, L) = \mathbb{F}\,\mathrm{Id}_L$ applying Schur's Lemma. Therefore, $\hat{\varphi} \circ \hat{\kappa}^{-1} = \lambda\,\mathrm{Id}_L$ for $\lambda \in \mathbb{F}$ so $\varphi = \lambda\kappa$. $\square$

As the decomposition in simple ideals of a semisimple algebra is orthogonal respect to the Killing form, we also have the following result:

Corollary 2.2.3. *Let (L, φ) be a semi-simple quadratic Lie algebra over a field $\mathbb{F}$ of characteristic zero which is algebraically closed. If L decomposes as a sum of simple ideals $L_1 \oplus \cdots \oplus L_n$, then for $a_i \in \mathbb{F}$*

$$\varphi = \bigoplus_{i=1}^{n} a_i \cdot \kappa_{L_i},$$

where κ_{L_i} denotes the Killing form of L_i.

On the other hand, for abelian Lie algebras the symmetric nondegenerate invariant bilinear form could be anyone which is symmetric and non-degenerated, as all bilinear forms are invariant. If we see an abelian Lie algebra simply as a vector space, this form will be any scalar product over it. And, as scalar products, this form introduces the orthogonality concept in Lie algebras.

Definition 2.2.2. Let S be a subspace of a quadratic Lie algebra (L, φ) we define naturally the *orthogonal* of S as

$$S^{\perp} = \{x \in L : \varphi(x, y) = 0 \,\forall\, y \in S\}.$$

Thanks to the φ-invariance if I is an ideal, then $I^{\perp}$ is another ideal, although this is not true for subalgebras. And using non-degeneration we have

$$\dim I + \dim I^{\perp} = \dim L \qquad \text{and} \qquad (I^{\perp})^{\perp} = I.$$

Definition 2.2.3. Let I be an ideal of a quadratic Lie algebra (L, φ). We say I is *non-degenerate* if $\operatorname{Rad} \varphi|_{I \times I} = 0$.

Here, in contrast to a Lie algebra radical, the *radical* of a bilinear form $\varphi \colon L \times L \to \mathbb{F}$ denotes

$$\operatorname{Rad}(\varphi) = \{x \in L : \varphi(x, y) = 0 \,\forall\, y \in L\} = L^{\perp}.$$

Proposition 2.2.4. *Let (L, φ) be a quadratic Lie algebra and I an ideal of L. Then $L = I \oplus I^{\perp}$ if and only if I is non-degenerate.*

This non-degeneration for ideals is not so common for invariant forms.

Proposition 2.2.5. *Let (L, φ) be a nilpotent Lie algebra and φ invariant. Then L^k are degenerate ideals for $k \geq 2$.*

Proof. If L is t-step nilpotent $\varphi(L^k, L^t) = \varphi(L^{k-1}, L^{t+1}) = 0$ as $L^{t+1} = 0$. □

Also, with orthogonality, other concepts appear:

Definition 2.2.4. A subspace V of some quadratic Lie algebra (L, φ) is called *isotropic* if $V \subseteq V^{\perp}$.

Remark 2.2.6. Some authors called these subspaces totally isotropic, while for them isotropic are simply those ones which contain an isotropic element ($x \neq 0$ such that $\varphi(x, x) = 0$).

Definition 2.2.5. An isotropic subspace is called *lagrangian* if it is a maximally isotropic subspace. And its dimension is called *Witt index*, which is an algebra invariant.

Remark 2.2.7. Let V be an isotropic subspace of a quadratic Lie algebra (L, φ) over an algebraically closed field of characteristic different from two. If L is finite dimensional, then V is a lagrangian subspace if and only if the dimension of V equals $\lfloor \frac{\dim L}{2} \rfloor$.

Later on, we will be studying algebras in which there exist a lagrangian subspace which is an ideal.

Coming back to the classification of quadratic Lie algebras, apart from semisimple and abelian Lie algebras, the structure is not so clear. In order to turn down the complexity, we can limit our study using the concept of reducibility which appears in the next definition. First, as stated in [Tsou and Walker, 1957], we need to see that

$$Z(L)^{\perp} = L^2 \tag{2.13}$$

or equivalently

$$(L^2)^{\perp} = Z(L) \tag{2.14}$$

Hence,

$$\dim L = \dim L^2 + \dim Z(L). \tag{2.15}$$

The dimension of $Z(L) \cap L^2$ is called the *isotropic index of* L. Moreover, this orthogonal relation can be extended through central series obtaining $(A^k)^{\perp} =$

$Z_{k-1}(A)$ (see [Bordemann, 1997, Proposition 2.1], [Medina and Revoy, 1985] or [Keith, 1984, Theorem 3.27]). This property is also covered in Proposition 5.3.6.

Now we are ready for the reducibility definition.

Definition 2.2.6. A Lie algebra L is said to be *reduced* in case $Z(L) \subseteq L^2$. And the pair (r, s), where $r = \dim L^2$ and $s = \dim Z(L)$, is called the *bi-type* of L.

Remark 2.2.8. If the Lie algebra is quadratic, by orthogonality, its dimension is just $r + s$. Also, this bi-type, defined in [Tsou and Walker, 1957], is quite similar to the general type introduced in Definition 2.1.26.

In the 2-step case, being reduced is equivalent to $Z(L) = L^2$ as the other inclusion comes from its nilpotency.

And, non-reduced algebras can be decomposed as expected following the next theorem (see [Tsou and Walker, 1957, Theorem 6.2]).

Theorem 2.2.9. *Any non-reduced and non-abelian quadratic Lie algebra (L, φ) decomposes as an orthogonal direct sum of proper ideals for $L = \mathfrak{g} \oplus \mathfrak{a}$, and $\varphi = \varphi_1 \perp \varphi_2$. Where $(\mathfrak{g}, \varphi_1)$ is a quadratic reduced Lie algebra and $(\mathfrak{a}, \varphi_2)$ is a quadratic abelian algebra. In particular, $L^2 = \mathfrak{g}^2$ and $Z(L) = Z(\mathfrak{g}) \oplus \mathfrak{a}$.*

Proof. Let $M = Z(L) \cap L^2$, let $\mathfrak{a}$ be a subspace complementary of M in $Z(L)$, let $\mathfrak{g}$ a complement to $\mathfrak{a}$ in L containing L^2. Note this is possible as $\mathfrak{a} \cap L^2 = 0$. Now, $\mathfrak{a}$ is an abelian ideal of L as $\mathfrak{a} \subseteq Z(L)$, and $\mathfrak{g}$ is an ideal of L as $L^2 \subseteq \mathfrak{g}$, and together $L = \mathfrak{g} \oplus \mathfrak{a}$. We just need to check $\mathfrak{g}$ is reduced. This happens as $Z(\mathfrak{g}) \subseteq Z(L)$, thus $Z(\mathfrak{g}) = \mathfrak{g} \cap Z(L)$. Also $\mathfrak{g}^2 = L^2$, so $Z(\mathfrak{g}) \subseteq \mathfrak{g}^2 = L^2$.

We now need to see if $\varphi_1 = \varphi|_{\mathfrak{g}}$ and $\varphi_2 = \varphi|_{\mathfrak{a}}$ are symmetric nondegenerate invariant bilinear forms. Symmetry and invariant are clearly transferred, so we just need to check they are nondegenerated. Let $V = \mathrm{Rad}(\varphi_1)$, then $V \subseteq Z(L) \cap \mathfrak{g} = Z(\mathfrak{g}) \subseteq L^2$, as $L^2 \subseteq \mathfrak{g}$. Let us consider $x_1 + x_2 \in L$ with $x_1 \in \mathfrak{g}$ and $x_2 \in \mathfrak{a} \subseteq Z(L)$. Then $\varphi(V, x_1 + x_2) = \varphi(V, x_1) + \varphi(V, x_2)$ but both terms are zero, the first because of V definition, and the second because $V \subseteq L^2 = Z(L)^{\perp}$. Therefore, as $\varphi(V, L) = 0$, V must be zero as φ is nondegenerate. $\square$

Remark 2.2.10. This proof is constructive as it also gives us a method of finding that decomposition.

This is the reason why we limit our classification to this type of algebras. It is important not to confuse reduced with indecomposable.

Definition 2.2.7. A quadratic algebra (A, φ) is called *decomposable* if it contains a proper ideal I that is non-degenerated (i.e. $\varphi\ |_{I\times I}$ is non-degenerate), and *indecomposable* otherwise.

Remark 2.2.11. Equivalently, (A, φ) is decomposable if and only A can be express as a direct sum of two orthogonal non-degenerate proper ideals, i.e. $A = I \oplus I^{\perp}$.

So, every abelian algebra of dimension greater than one or non-reduced algebra is decomposable. And also, any quadratic Lie algebra is the orthogonal direct sum of indecomposable quadratic Lie algebras. This assertion, which we can see in [Astrakhantsev, 1978], follows easily from the fact that I is an ideal of A if and only if its orthogonal space $I^{\perp}$ also is.

Other examples of decompositions we can find come from semisimple Lie algebras as they can be viewed as a sum of its simple ideals. These algebras are a subfamily of a bigger class formed by the orthogonal sum of both an abelian and a semisimple algebra, known as reductive.

Definition 2.2.8. A Lie algebra L is called *reductive* if $\operatorname{Rad} L = Z(L)$. More concretely, a Lie algebra is reductive if it is a direct sum as ideals of a semisimple Lie algebra and an abelian Lie algebra.

Example 2.2.2. Lie algebra $\mathfrak{gl}(n, \mathbb{F})$ is reductive. We can decompose

$$\mathfrak{gl}(n, \mathbb{F}) = \mathfrak{sl}(n, \mathbb{F}) \oplus \mathbb{F}I_n = \mathfrak{gl}(n, \mathbb{F})^2 \perp Z(\mathfrak{gl}(n, \mathbb{F})).$$

Here $\mathfrak{gl}(n, \mathbb{F})^2 = \mathfrak{sl}(n, \mathbb{F})$ is a simple Lie algebra, and

$$\operatorname{Rad}(\mathfrak{gl}(n, \mathbb{F})) = Z(\mathfrak{gl}(n, \mathbb{F})) = \mathbb{F}I_n,$$

the scalar matrices, form an abelian Lie algebra of dimension 1.

Reduced quadratic Lie algebras are a big family, in fact, in [Tsou, 1962, Theorem 5.1] we find the following result:

Theorem 2.2.12 (Tsou, 1962)**.** *There exist reduced quadratic Lie algebras of arbitrary bi-type* (r, s) $r \geq 3$ *except for* $(5, 0)$, $(7, 0)$ *and* $(4, s)$, $0 \leq s \leq 4$.

The proof of this theorem, based on multilinear arguments and tools, does not make it clear how to build them. However, these techniques will inspire the quadratic family method and its constructions in Chapter 3.

Some general properties all quadratic Lie algebras share are gather in the following proposition:

Proposition 2.2.13. *Let (L, φ) a quadratic Lie algebra, U any subspace and I an ideal. Then:*

(a) $\dim L = \dim U + \dim U^{\perp}$ *and, if U is non-degenerate, $\mathfrak{g} = U \oplus U^{\perp}$.*

(b) *U is an ideal of L if and only if $[U^{\perp}, U] = 0$.*

(c) *Any minimal and non-degenerate ideal of L is simple or one-dimensional. Minimal degenerated ideals are isotropic and abelian.*

(d) *The algebra L decomposes as the orthogonal direct sum, as ideals, of a reduced quadratic Lie algebra and an abelian Lie algebra.*

(e) *If L is indecomposable and I is proper, then $I \cap I^{\perp} \neq 0$ and $Z(L) \subseteq L^2$. In particular, L is a reduced algebra and has no simple ideals.*

(f) *The centre of a nonzero solvable quadratic algebra is nonzero. Even more, $Z(L) \cap L^2 \neq 0$ and, in particular, $N^{\perp} \subseteq N$.*

(g) *If L is indecomposable, then either L is one-dimensional or simple or can be obtained up to isometrically isomorphisms using a technique called double extension of some quadratic Lie algebra by another one-dimensional or simple.*

(h) *For any subalgebra S, $[S, S^{\perp}] \subseteq S$, in particular, $S \cap S^{\perp}$ is an ideal of S.*

Proof. A detailed proof of items (b), (c) and (g) can be found in [Figueroa-O'Farrill and Stanciu, 1996]. Item (d) is just Theorem 2.2.9 and (e) follows easily from previous items. For item (f), apply equation (2.15) and $L^2 \neq L$ because of L is nonzero and solvable. Now, observe that if $Z(L) \cap L^2 = 0$ then $L = L^2 \oplus Z(L)$ and in that case $L' = L^2 = [L^2, L^2] = L^{(2)}$ which contradict the solvability of L. Lastly, as $Z(L) + L^2 \subseteq N$ so by orthogonality

$$N^{\perp} \subseteq (Z(L) + L^2)^{\perp} = L^2 \cap Z(L) \subseteq N.$$

The final case (h) is also true as

$$\varphi([S, S^{\perp}], S) = \varphi(S^{\perp}, [S, S]) = \varphi(S^{\perp}, S) = 0,$$

and $[S \cap S^{\perp}, S] \subseteq [S, S] \cap [S^{\perp}, S] \subseteq S \cap S^{\perp}$. $\square$

Remark 2.2.14. The double extension procedure mentioned in item (g) will be described in Section 2.2.2.1.

Now, we can question ourselves which are the smallest quadratic Lie algebras we can have:

Lemma 2.2.15. *The smallest non-zero-dimensional indecomposable Lie algebra is:*

- *of dimension 1, when it is abelian;*
- *of dimension 3, when it is simple;*
- *of dimension 4, and isomorphic to $\mathbb{F}a \ltimes \mathfrak{h}_3$ described in equation* (2.16)*, when it is non-simple and non-abelian, in particular when it is solvable; and*
- *of dimension 5 and isomorphic to $\mathfrak{n}_{2,3}$, when it is nilpotent non-abelian.*

So, there are no nilpotent quadratic indecomposable Lie algebras of dimension two, three or four.

Proof. All abelian and simple Lie algebras are quadratic. So, using [Jacobson, 1979, Chapter 1], the smallest simple Lie algebra is 3-dimensional proving the first two items.

For L solvable non-abelian, we see $\dim L \geq 4$. This happens because we need a reduced Lie algebra in order not to be decomposable. There are three options when considering $0 \neq Z(L) \subseteq L^2 \subsetneq L$ and $\dim L = 3$.

- $Z(L) = L^2$ both 1-dimensional. This is the Heisenberg algebra which is not quadratic as $\dim Z(L) + \dim L^2 \neq \dim L$ as equation (2.15) imposes.
- $Z(L) = L^2$ both 2-dimensional. It is contradictory as $\dim L/Z(L) = 1$ so L is abelian.
- $Z(L) \subsetneq L^2$. Again impossible as $L = \operatorname{span}\langle x, y, z\rangle$ and $Z(L) = \operatorname{span}\langle z\rangle$ implies $L^2 = \operatorname{span}\langle [x, y]\rangle$ is 1-dimensional.

Now that we know $\dim L \geq 4$, we can distinguish again two cases:

- $Z(L) = L^2$: Using equation (2.15), $\dim L = 2d$ with $d = \dim Z(L)$. Case $d = 2$ does not work as $L = \operatorname{span}\langle x, y\rangle \oplus Z(L)$, so L^2 should be 1-dimensional. When $d = 3$ we have the free nilpotent Lie algebra $\mathfrak{n}_{3,2}$ which is quadratic as we will see in Example 2.2.5.
- $Z(L) \subsetneq L^2$: In this case, $\dim L = n + 2d$, where $n = \dim L^2/Z(L)$ and $d = \dim Z(L)$ with $d, n \geq 1$ in order not to be abelian.
  - When $n = 1$ the smallest valid d is 2 producing the quadratic nilpotent algebra $\mathfrak{n}_{2,3}$ from Example 2.2.4.
  - When $n = 2$ and $d = 1$ the algebra is of the form $L = \mathbb{F}a \oplus L^2$, $Z(L) = \mathbb{F}z$ and

    $$L^2 = \operatorname{span}\langle x, y, z\rangle = \operatorname{span}\langle [a, x], [a, y], [x, y]\rangle.$$

    Observe $0 \neq [x, y] \in Z(L)$ because $L^2 \cong \mathfrak{h}_3$ using Lemma 2.1.19. This way we have products

    $$[a, x] = \alpha_1 x + \alpha_2 y + \alpha_3 z, \qquad [a, y] = \beta_1 x + \beta_2 y + \beta_3 z,$$

    with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ in order L^2 to have dimension 3. Without loss of generality, we can rescale the basis to have products

    $$[a, x] = y, \qquad [a, y] = \alpha x + \beta y, \qquad [x, y] = z. \tag{2.16}$$

    Applying the Jacobi identity

    $$0 = [a, z] = [a, [x, y]] = -[x, [y, a]] - [y, [a, x]] = \beta z,$$

    so $\beta = 0$. The resulting algebra admit a symmetric invariant bilinear given by the matrix

    $$\begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \frac{-\lambda_2}{\alpha} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ \lambda_2 & 0 & 0 & 0 \end{pmatrix}$$

    in basis $\{a, x, y, z\}$. When $\lambda_1 = 0$ and $\lambda_2 = -\alpha = 1$ we obtain the oscillator algebra from next Example 2.2.3. □

Example 2.2.3. The *oscillator algebra* $(\mathfrak{d}_4, \varphi)$, mentioned in previous lemma, is the smallest solvable non-abelian quadratic algebra. We can define it taking basis $\{d, x, y, z\}$ with nonzero products

$$[d, x] = y, \qquad [d, y] = -x, \qquad [x, y] = z,$$

and bilinear form $\varphi(d, z) = 1 = \varphi(x, x) = \varphi(y, y) = 1$. Its derived algebra $\mathfrak{d}_4^2 = [\mathfrak{d}_4, \mathfrak{d}_4] = \mathfrak{h}_3$ is the Heisenberg Lie algebra (see Definition 2.2.9). An easy computation yields $\mathrm{Der}_\varphi \, \mathfrak{d}_4 = \mathrm{Inner} \, \mathfrak{d}_4$. This algebra arises over the reals in the quantum mechanical description of a harmonic oscillator and φ is a Lorentzian[7] form. More information is available at [Ovando, 2006] and in Section 5.2.2.

In contrast to these examples, we can also find algebras which are not quadratic. For instance, the *Heisenberg algebra* $\mathfrak{h}_3$ which have just appeared as the square of $\mathfrak{d}_4$ in Example 2.2.3. This is a 3-dimensional algebra with basis $\{x, y, z\}$ and Lie product $[x, y] = z = Z(\mathfrak{h})$. From equality (2.15), it is immediate that $\mathfrak{h}_3$ is not quadratic. But this algebra does not appear alone, we can define Generalized Heisenberg Algebras (GHA) in a similar way. This family will take an important role later in Section 5.2.2.

Definition 2.2.9. The *generalized Heisenberg algebra* series, following [Dixmier, 1996], is determined through the property $\mathfrak{h}^2 = Z(L) = \mathrm{span}\langle z \rangle$. For such algebra, the Lie bracket $[x, y] = b_z(x, y)z$ provides an alternating bilinear form $b_z \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{F}$ and $(\mathfrak{h})^\perp = \mathfrak{h}^2 = \mathbb{F} \cdot z$. Note that b_z is non-degenerate on any complement summand W of the centre $Z(\mathfrak{h})$ in $\mathfrak{h}$. Hence W is a vector space of dimension even and $b_z|_{W \times W}$ has a canonical basis $\{x_1, \ldots, x_n, y_1, , \ldots, y_n\}$ such that $b_z(x_i, y_i) = 1 = -b_z(y_i, x_i)$. Then, Heisenberg algebras have odd dimension and, for any natural $n \geq 1$, there is a unique Lie algebra of Heisenberg type of dimension $2n + 1$ that we call $\mathfrak{h}_{2n+1}$. Therefore, the algebra $\mathfrak{h}_{2n+1}$ has a *standard basis* $\{x_1, \ldots, x_n, y_1, \ldots y_n, z\}$ with nonzero brackets $[x_i, y_i] = z$.

Note that equation (2.15) fails, as for $\mathfrak{h} = \mathfrak{h}_3$, so $\mathfrak{h}_{2n+1}$ are not quadratic.

According to [del Barco and Ovando, 2012], the 5-dimensional algebra $\mathfrak{n}_{2,3}$ and the 6-dimensional $\mathfrak{n}_{3,2}$ are the unique quadratic free nilpotent Lie algebras. Any algebra in the free nilpotent series $\mathfrak{n}_{d,t}$ ($d \geq 2$ and $t \geq 2$) satisfies

[7] A Lorentzian form is a non-degenerate bilinear form in a n-dimensional vector space such that its signature is $n - 2$.

$d = \operatorname{codim} \mathfrak{n}_{d,t}^2$ and the centre, $Z(\mathfrak{n}_{d,t}) = (\mathfrak{n}_{d,t})^t$ has dimension $\frac{1}{t}\sum_{a|t}\mu(a)d^{t/a}$, where μ the is Möbius function (see Example 2.1.7 or [Gauger, 1973]). Then, the assertion is a corollary of item (a) in Proposition 2.2.13.

Example 2.2.4. The quadratic structure of $\mathfrak{n}_{2,3}$ is given in (Hall) basis $\{a_i\}_{i=1}^5$ by nonzero products $[a_2, a_1] = a_3$, $[a_3, a_1] = a_4$, $[a_3, a_2] = a_5$ and the symmetric form $\varphi(a_i, a_j) = (-1)^{i-1}$ for $i \leq j$ and $i + j = 6$ and zero otherwise.

Example 2.2.5. For $\mathfrak{n}_{3,2}$, we can take (Hall) basis $\{a_i\}_{i=1}^6$ and nonzero products $[a_2, a_1] = a_4$, $[a_3, a_1] = a_5$, $[a_3, a_2] = a_6$ with symmetric form $\phi(a_i, a_j) = (-1)^{i-1}$ for $i \leq j$ and $i + j = 7$ and zero otherwise.

Remark 2.2.16. The structure mentioned in Example 2.2.4 is unique over algebraically closed fields, while the one from Example 2.2.5 is unique over any field of characteristic zero (see [Benito et al., 2017]).

We come back now to items (c) and (d) of the multiple characterisation result of Lie groups with bi-invariant metrics and their Lie algebras (see [Medina, 1985]) that we have mentioned at the beginning of this section. Both items highlight the natural relationship between invariant bilinear forms on Lie algebras and homomorphisms of their adjoint and coadjoint representations. As vector spaces, the set of invariant bilinear forms, $\mathrm{Bi}_{\mathrm{inv}}(L)$, and $\mathrm{Hom}_L(L, L^*)$, are isomorphic:

$$\Delta\colon \mathrm{Bi}_{\mathrm{inv}}(L) \to \mathrm{Hom}_L(L, L^*), \quad \Delta(f)(x) = f(x, \cdot\,) = \psi_f(x), \tag{2.17}$$

and $\Delta^{-1}(\psi) = f_\psi$, with $f_\psi(x, y) = \psi(x)(y)$. Even more, Δ sends a nondegenerate invariant form into a L-module isomorphism and conversely.

On the other hand, for any bilinear form φ of L, we can set the bilinear form $\varphi^t(x, y) := \varphi(y, x)$. From the anticommutativity of L, it is easily checked that φ^t is invariant if and only if φ so is. This way, over fields of characteristic not 2, the usual decomposition of φ as sum of its symmetric part and its skew-symmetric part

$$\varphi = \frac{1}{2}(\varphi + \varphi^t) + \frac{1}{2}(\varphi - \varphi^t)$$

preserves the invariance. Therefore, $\mathrm{Bi}_{\mathrm{inv}}(L)$ decomposes as the direct sum of the vector spaces of symmetric invariant forms, $\mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L)$, and that of the skew-symmetric forms $\mathrm{Bi}^{\mathrm{as}}_{\mathrm{inv}}(L)$:

$$\mathrm{Bi}_{\mathrm{inv}}(L) = \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L) \oplus \mathrm{Bi}^{\mathrm{as}}_{\mathrm{inv}}(L).$$

So for invariant forms, at matrix level, we recover the natural decomposition of a matrix as sum of a symmetric matrix and a skew-symmetric matrix.

Our next result restates and expands Lemma 1 in [Hilgert and Neeb, 1996] and clarifies the equivalent assertions for Lie algebras attached to Lie gropus with bi-invariant metrics given by Medina and Milnor.

Proposition 2.2.17. *Let L be a Lie algebra over a field of characteristic different from 2. Then it is equivalent:*

(a) *There exists a nondegenerate form $\varphi \in \mathrm{Bi}_{\mathrm{inv}}(L)$.*

(b) *There exists a nondegenerate form $\varphi \in \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L)$.*

(c) *The adjoint and coadjoint representations of L are isomorphic.*

Moreover, over fields that have more than $\dim L$ elements, the vector space $\mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L)$ is linearly generated by the set of invariant and non-degenerate symmetric bilinear forms and previous assertions are also equivalent to:

(d) $\dim \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L)$ *is greater or equal than one.*

Proof. Let us assume (a) and, using [Bordemann, 1997, Proposition 2.4], (b) follows. From (b) we get (c) taking into account that the isomorphism Δ in expression (2.17) sends any non-degenerate invariant form f into the isomorphism $\varphi_f\colon L \to L^*$, $\varphi_f(x) = f(x, \cdot)$. This map is one-to-one because f is nondegenerate and, since L and L^* are equidimensional, it is bijective. Moreover as f is invariant, for all $x, y \in L$:

$$\varphi_f([x,y]) = f([x,y], \cdot) = -f(y, (\mathrm{ad}\, x)(\cdot)) = -f(y, \cdot) \circ \mathrm{ad}\, x = -\varphi_f(y) \circ \mathrm{ad}\, x.$$

This is just equation (2.6), so $\varphi_f \in \mathrm{Hom}_L(L, L^*)$ and adjoint and coadjoint representations are isomorphic. Finally, for any bijective map $\psi \in \mathrm{Hom}_L(L, L^*)$, $f_\psi(x,y) = \psi(x)(y)$ is a bilinear form and $f_\psi(x, L) = 0$ implies that $\psi(x)$ is a null map, so $x = 0$ because ψ is one-to-one. Thus f_ψ is nondegenerate. Moreover $f_\psi([x,y], z) = \psi([x,y])(z)$ and from equation (2.6), $\psi([x,y]) = -\psi(y) \circ \mathrm{ad}\, x$ and $\psi([x,y])(z) = -\psi(y)([x,z]) = -f_\psi(y, [x,z])$. Hence $f_\psi \in \mathrm{Bi}_{\mathrm{inv}}(L)$, and item (a) follows. According to Lemma2.1 in [Bajo and Benayadi, 1997], $\mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L) = \mathrm{span}\langle \varphi \in \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L) : \varphi \text{ is non-degenerate}\rangle$ over the real field. The arguments in the proof of this lemma are also valid for fields that have more than $\dim L$ elements. Therefore, the equivalence of the four statements is proved. $\square$

In Figure 2.2 we can find a diagram representing how Δ mapping from equation (2.17) works.

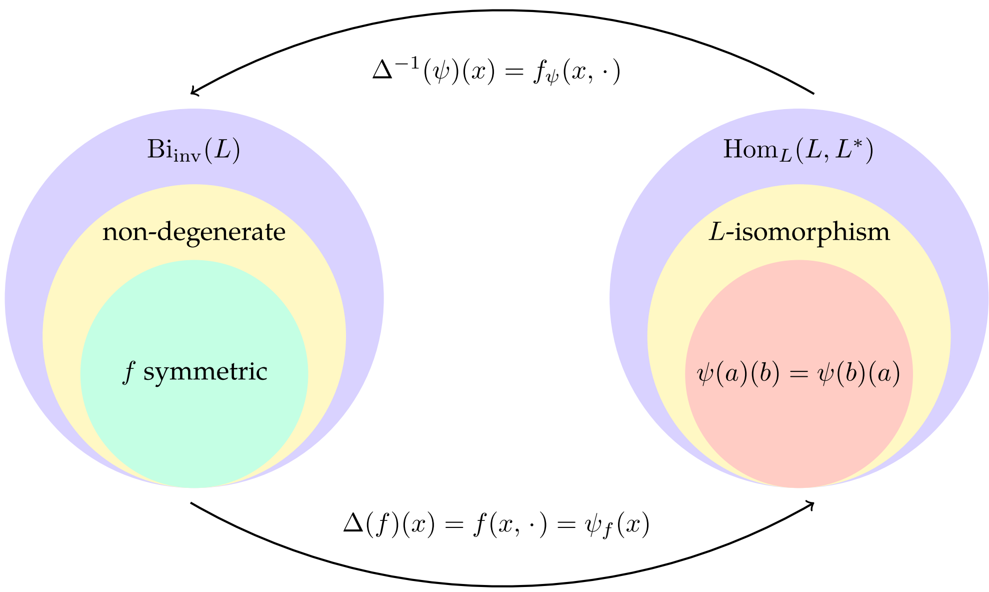

Figure 2.2: Δ isomorphism between $\mathrm{Bi}_{\mathrm{inv}}(L)$ and $\mathrm{Hom}_L(L, L^*)$.

At the end of previous proposition appeared the dimension of the vector space $\mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L)$. This plays an important role in quadratic Lie algebras as it tells us how many appear.

Definition 2.2.10 (See [Tsou and Walker, 1957]). We call *quadratic dimension* or metric index to the number of linearly independent symmetric nondegenerate invariant bilinear forms some algebra admits. For a given Lie algbra L we write

$$m(L) = \dim \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L).$$

If the algebra is not quadratic its quadratic dimension is zero. If the algebra is abelian of dimension n, then its quadratic dimension is $\frac{n(n+1)}{2}$. But there is an important lemma when we look for dimension one. Although, for good basis and low dimensions we will be able to compute these quadratic dimensions using algorithms from Chapter 6.

Lemma 2.2.18. *Every non-abelian algebra with quadratic dimension one is simple. Over algebraically closed fields the reverse also holds. And, over the reals, if an algebra is simple its quadratic dimension is*

- *one if its* complex extension[8] *is simple,*
- *or two if its complex extension is not simple.*

This lemma appears first in [Tsou and Walker, 1957] for $\mathbb{R}$ and $\mathbb{C}$ and for more general fields in [Bajo and Benayadi, 1997].

The reason why for some real simple Lie algebras the quadratic dimension is two is related to the fact that over the complex field they are semisimple and decompose as two simple conjugate Lie algebras. This lemma also serves as a proof that only the Killing form or scalar multiples of it work as a quadratic form for complex simple Lie algebras (see Proposition 2.2.2).

Also, in [Tsou and Walker, 1957, 9.2], the authors state that for an ideal decomposition $L_1 \oplus L_2$ we have

$$m(L_1 \oplus L_2) = m(L_1) + m(L_2) + \dim Z(L_1) \cdot \dim Z(L_2).$$

This formula is compatible with the semisimple case. In those algebras the decomposition in simple ideals has trivial centres, and $m(S_1 \oplus \cdots \oplus S_n) = m(S_1) + \cdots + m(S_n) \geq n$. This last inequality turns into an equal sign when we are over an algebraically closed field or, in the real case, when every simple ideal has a simple complex extension. If $k \leq n$ ideals have no simple complex extension, are central simple, then $m(S_1 \oplus \cdots \oplus S_n) = n + k$.

Apart from abelian or semisimple Lie algebras, and following [Hofmann and Keith, 1986, Proposition A], we can generate quadratic Lie algebras with arbitrary quadratic dimension. First, we consider the tensor product $(\mathfrak{g} \otimes \mathfrak{a}, \varphi \otimes q)$ where $(\mathfrak{g}, \varphi)$ is quadratic Lie with bracket $[x, y]$ and $(\mathfrak{a}, q)$ is an associative and commutative algebra with invariant symmetric and nondegenerate form q and product ab produces the quadratic algebra

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab,$$

[8] An algebra over the complex field which has the same structural constants.

under the form

$$(\varphi \otimes q)(x \otimes a, y \otimes b) = \varphi(x, y)q(a, b).$$

Example 2.2.6. As a remarkable example we point out the algebra $\mathfrak{g} = \mathfrak{s} \otimes \frac{\mathbb{F}[t]}{\langle t^n \rangle}$ for any $n \geq 1$, with $\mathfrak{s}$ simple. This algebra as seen in [Bajo and Benayadi, 1997, Proposition 2.2] is non-semisimple, irreducible, perfect and $m(\mathfrak{g}) \geq n$.

Also, for generic non-abelian quadratic algebras we can get bounds for its quadratic dimension:

Lemma 2.2.19. *See [Tsou and Walker, 1957, Section 9.2] If L is non-abelian and quadratic, and* $\dim Z(L) = c$*, then in* $\mathbb{C}$

$$m(L) \geq 1 + \frac{c(c+1)}{2}.$$

With all these results, and using computational software from Chapter 6, we can easily find the quadratic multiplicity in all previous examples:

- $m(\mathfrak{d}_4) = 2$,
- $m(\mathfrak{n}_{2,3}) = 4$,
- and $m(\mathfrak{n}_{3,2}) = 7$.

2.2.2 Classical constructions

The early work of Tsou and Walker makes use of multilinear algebra to establish the existence of quadratic algebras. Although the description of these algebras is not explicit, multilinear tools point the way to reduce the classification of 2-step quadratic Lie algebras to trivectors (see [Noui and Revoy, 1997] and Chapter 4). In the 1980s and 1990s, methods of double extensions and T^*-extensions appeared. They allow us to construct finite dimensional quadratic algebras in a more explicit way. The first method is used in the realm of Lie algebras, mainly characteristic zero. According to [Rodríguez-Vallarte and Salgado, 2018], the double extension process also lets us produce Lie algebras with the same type of geometric (quadratic, symplectic or contact) structures. The T^*-extension technique can be applied to the variety of non-associative algebras (characteristic not two), but only generates quadratic algebras of even dimension. Since the end of the '90s and up to now, related to quadratic constructions, there have appeared bi-extensions and inflactions in [Keith, 1984] and [Hofmann and Keith, 1986], amalgamated products in [Favre and Santharoubane, 1987] and two-fold or quadratic extensions in [Kath and Olbrich, 2004] among other notions and procedures. In the case of quadratic nilpotent Lie algebras a classification scheme has been proposed in [Benito et al., 2017] based on free nilpotent Lie algebras and their invariant forms. Other techniques to produce examples of quadratic Lie algebras are given by tensor products, the so called *current Lie algebras* (see [Zusmanovich, 2014]), or representations of simple Lie algebras. This technique will appear in our last chapter and it was also used in [Benayadi and Elduque, 2014].

Both classical methods, double extensions and T^*-extensions, present difficulties when dealing with the classification problem, but we will tackle this task later. Despite this problem, as we will show along the thesis, they are good at producing examples.

2.2.2.1 Double extension

Chronologically, the double extension process appears first (see [Figueroa-O'Farrill and Stanciu, 1996] for a nice presentation). It was developed during the 1980s and was introduced in several independent works from Keith and Hofmann (see [Keith, 1984]), Favre and Santharoubane (see [Favre and Santharoubane, 1987]), and Medina and Revoy (see [Medina and Revoy, 1985]). This procedure, according to [Favre and Santharoubane, 1987], follows the main ideas that V. G. Kac had written in several exercises for his students (see [Kac, 1983, Exercise 2.10 and 2.11]). In Exercise 2.10, the double one-dimensional extension is defined, and the fact that every indecomposable solvable quadratic Lie algebra of dimension $n + 2$ can be obtained from a quadratic Lie algebra of dimension n is established in Exercise 2.11 (see [Favre and Santharoubane, 1987, Proposition 2.9] for a complete proof).

The double extension method consists in an iterative process that allows us to find new quadratic Lie algebras starting from a smaller dimensional one. The formal description we present here follows from [Bordemann, 1997, Theorem 2.2]), but slightly changed to match our notation. From now on, B^* will denote the dual space of B.

Theorem 2.2.20. *Let (A, f) be a finite-dimensional quadratic Lie algebra over a field $\mathbb{F}$. Let B be another finite-dimensional Lie algebra over $\mathbb{F}$ and suppose there is a Lie homomorphism $\phi\colon B \to \mathrm{Der}_f(A)$ from B onto the space of all f-skew-symmetric* *derivations of A. Denote by $w\colon A \times A \to B^*$ the bilinear skew-symmetric map $(a, a') \mapsto (b \mapsto f(\phi(b)(a), a'))$. Take the vector space direct sum $A_B := B \oplus A \oplus B^*$ and define the following multiplication for b, $b' \in B$, a, $a' \in A$, and $\beta, \beta \in B^*$:*

$$[b + a + \beta, b' + a' + \beta'] := [b, b']_B + \phi(b)(a') - \phi(b')(a) + [a, a']_A \\ + w(a, a') + \mathrm{ad}^*(b)(\beta') - \mathrm{ad}^*(b')(\beta). \quad (2.18)$$

Moreover, define the following symmetric bilinear form f_B on A_B:

$$f_B(b + a + \beta, b' + a' + \beta') := \beta(b') + \beta'(b) + f(a, a'). \quad (2.19)$$

Then the pair (A_B, f_B) is a quadratic Lie algebra over $\mathbb{F}$ and is called the double extension *of (A, f) by (B, ϕ).*

Now that once the method is explained we can understand where the name comes from. As it suggests, it consists of two extensions. First, over (A, f) we find B such that $\phi\colon B \to \mathrm{Der}_f(A)$ exists. From here, we define the 2-cocycle w and obtain the vector space $A \oplus B^*$ endowed with product

$$[a + \beta, a' + \beta'] = [a, a']_A + w(a, a').$$

This is the central extension of A by means of w. Now, we can define the Lie algebra homomorphism

$$\begin{aligned} \hat{\phi}\colon B &\to \mathrm{Der}(A \oplus B^*) \\ b &\mapsto \quad \hat{\phi}\colon A \oplus B^* \to A \oplus B^* \\ &\qquad\qquad a + \beta \mapsto \hat{\phi}(b)(a + \beta) = \phi(b)(a) + \mathrm{ad}^*(b)(\beta), \end{aligned}$$

to obtain the semidirect product $B \ltimes_{\hat{\phi}} (A \oplus B^*)$, which is the double extension.

It is worth mentioning that this construction, when B is abelian, also appear in the literature as a particular case of bi-extensions (see [Keith, 1984]).

One crucial property of this method is that every quadratic Lie algebra in characteristic zero can be obtained this way. So, it covers every possible case and allows us to deconstruct Lie algebras in Chapter 3. Note that, as it preserves dimension parity (it adds an even dimension to the extended algebra), we need to start the process in an algebra of the same parity as the one we want to end up.

To illustrate this method we are going to show how to obtain the oscillator algebra from Example 2.2.3 as a double extension by a 1-dimensional algebra.

Example 2.2.7 (Oscillator as one-dimensional double extension). Starting with the $\mathbb{R}$-vector space W_2 of dimension 2 with a bilinear symmetric and nondegenerate form φ, and being $\{x, y\}$ an orthonormal basis. The set $\mathrm{End}_\varphi(W_2)$ of linear maps that satisfy expression (2.12) is a one-dimensional subspace generated by the linear isomorphism $\phi(x) = y$ and $\phi(y) = -x$. Then, the Lie algebra obtained as double extension of the abelian quadratic algebra (W_2, φ) by $(\mathrm{span}\langle d\rangle, \hat{\phi})$ where $\hat{\phi}(d) = \phi$ has the same product and bilinear form as in Example 2.2.3. Note, $\mathrm{span}\langle d, d^*\rangle$ is an hyperbolic subspace and we obtain the real algebra $\mathfrak{d}_4(\mathbb{R})$, the harmonic oscillator algebra, with metric signature $(3, 1)$. This algebra is also known as diamond or Nappi-Witten Lie algebra (see [Casati et al., 2010] and references therein) and it can also be obtained as the central extension of the Poincaré Lie algebra in two dimensions.

This construction, in Section 5.2.2, will let us define generalized oscillator algebras. For other examples of double extensions, we can find them distributed through the following chapters.

2.2.2.2 T*-extension

In 1997, Bordemann (see [Bordemann, 1997]) introduced another method for contructing these algebras: the T^*-extension. This technique can be applied to all known classes of non-associative algebras over fields of characteristic different from 2. The method produces quadratic algebras of dimension $2n$ (even) and Witt index n (half of its dimension). This is proved in [Bordemann, 1997, Theorem 3.2], where we can see T^*-extensions are just quadratic non-associative algebras of dimension $2n$ that contain an isotropic subspace U of dimension n such that $U^2 = 0$.

This is a one-step method, in contrast to the multistep double extension. We are focused on the study of Lie algebras, so we will only see its definition applied on these algebras (for a general definition we can see the original method in [Bordemann, 1997]).

Along this subsection, $(B, [x, y]_B)$ will be a Lie algebra. Let V a B-module given by the representation $\rho\colon B \to \mathfrak{gl}(V)$ (indeed a Lie algebra homomorphism as we saw $\mathfrak{gl}(V)$ denotes the general Lie algebra of endomorphisms over the vector space V). In order to reach the definition of T^*-extension we need the following basic cohomology notions.

Definition 2.2.11. Let $w\colon B \times B \to V$ be a bilinear map, V a B-module given by the representation ρ, and a, b, c arbitrary elements of B. Then we say:

- w is *non-degenerate* if its radical is zero, i.e.
$$\operatorname{Rad} w = \{b \in B : w(b, \cdot) = 0\} = 0;$$
- w is *cyclic* if
$$w(a,b)(c) = w(c,a)(b) = w(b,c)(a); \tag{2.20}$$
- and w is a 2-*cocycle* if w is skew-symmetric and
$$\sum_{\substack{\circlearrowleft \\ a,b,c}} w([a,b],c) = \sum_{\substack{\circlearrowleft \\ a,b,c}} \rho(a)w(b,c).$$

The vector space of 2-cocycles with values in V is denoted by $Z^2(B,V)$.

Remark 2.2.21. In our construction we will work with a 2-cocycle w with values in B^* using the coadjoint[9] representation, i.e. $w \in Z^2(B,B^*)$. Despite this structure, we can also see it like a 3-cocycle with values in $\mathbb{F}$. Given $w\colon B\times B \to B^*$, we can define the trilinear map $\phi_w\colon B \times B \times B \to \mathbb{F}$ as $\phi_w(a,b,c) = w(a,b)(c)$. It is straightforward to infer w is cyclic, $w \in Z^2(B,B^*)$ where $V = B^*$ by the coadjoint representation if and only if ϕ_w is a 3-cocycle. This means it is a 3-alternating form such that

$$\begin{aligned}\phi_w([b_0,b_1],b_2,b_3) + \phi_w([b_1,[b_0,b_2],b_3) + \phi_w([b_1,b_2,[b_0,b_3]) = \\ \phi_w(b_0,[b_1,b_2],b_3) + \phi_w(b_0,b_2,[b_1,b_3]) - \phi_w(b_0,b_1,[b_2,b_3]).\end{aligned}$$

The vector space of scalar 3-cocycles is denoted as $Z^3(B,\mathbb{F})$. Here, $V = \mathbb{F}$ comes from the trivial representation.

Consider now an arbitrary bilinear form $w\colon B \times B \to B^*$, and define the multiplication on the vector space $B \oplus B^*$ for $b, b' \in B$ and for $\beta, \beta' \in B^*$ as

$$[b+\beta,\ b'+\beta'] := [b,b']_B + w(b,b') + \operatorname{ad}^*(b)(\beta') - \operatorname{ad}^*(b')(\beta), \tag{2.21}$$

where ad^* is the *coadjoint representation*. So for any $\beta\colon B \to \mathbb{F}$, $\beta \in B^*$ and $b, b' \in B$

$$\operatorname{ad}^*(b)(\beta)(b') = -\beta([b,b']) = -\beta \circ \operatorname{ad} b(b'). \tag{2.22}$$

Moreover, we construct the symmetric bilinear form q_B as:

$$q_B(b+\beta, b'+\beta') := \beta(b') + \beta'(b). \tag{2.23}$$

[9]See definition in equation (2.22).

Proposition 2.2.22. *Let B, B^*, w, and q_B be as above. Then:*

(a) *The vector space $B \oplus B^*$ with the binary product given in equation* (2.21) *is a Lie algebra if and only if B is a Lie algebra and $w \in Z^2(B, B^*)$.*

(b) *The form q_B defined in equation* (2.23) *is an invariant bilinear form of the Lie algebra $B \oplus B^*$ if and only if w is cyclic.*

So, $(B \oplus B^*, q_B)$ is a quadratic Lie algebra if and only if the bilinear form w is a cyclic 2-cocycle and $(B, [x, y]_B)$ is a Lie algebra.

Proof. Assertion (a) follows from [Bordemann, 1997, p. 177]. Here we can see Jacobi identity is satisfied if and only if B is a Lie algebra and w is a 2-cocycle. On the other hand, product in equation (2.21) is skew-symmetric if and only if $[x, y]_B$ and w are skew, which also comes from being a 2-cocyle. Finally, to prove assertion (b) about the T^* construction, we use [Bordemann, 1997, Lemma 3.1], which adds us the cyclic condition in order to be quadratic. □

Definition 2.2.12. Let $w\colon B \times B \to B^*$ be cyclic 2-cocycle and B Lie algebra. The quadratic algebra $(B \oplus B^*, q_B)$, with product and quadratic form defined in equations (2.21) and (2.23) respectively, is called the *T^*-extension* of B by w, and we denote it as $(T^*_w B, q_B)$.

Finally, we have the following theorem (check [Bordemann, 1997, Theorem 3.2]) which gives us conditions about when we can build a quadratic Lie algebra using T^*-extensions.

Theorem 2.2.23. *Let (A, f) be a quadratic Lie algebra of finite dimension n over a field $\mathbb{F}$ of characteristic not equal to two. Then (A, f) will be isometric to a T^*-extension $(T^*_w B, q_B)$ if and only if n is even and A contains an isotropic ideal I of dimension $n/2$. In this case, as a Lie algebra, B is isomorphic to the quotient A/I.*

Remark 2.2.24. As seen in the proof, any isotropic ideal of dimension $n/2$ will work. We also note that, any isotropic subspace V of A whose dimension is $\dim A/2$, also named lagrangian, is an ideal of A if and only if it is abelian, in other words, $V^2 = 0$. This assertion follows from $V = V^\perp$ and item (d) in Proposition 2.2.13.

As a consequence, we can summarize and say:

Corollary 2.2.25. *The class of T^*-extensions is just the class of quadratic Lie algebras of dimension $2n$ with a lagrangian n-dimensional ideal.*

Although at first glance T^*-extensions only work for even dimensional algebras, [Bordemann, 1997] says we can use them to end up obtaining odd dimensional Lie algebras applying 1-dimensional quotients afterwards.

Some examples of Lie algebras we can obtain through this technique are the following:

Example 2.2.8. T^*-extensions of Lie algebras by the null 2-cocycle, T_0^*B are just split extensions of that Lie algebra B by means of its coadjoint representation. And even more, any invariant bilinear form $f\colon B \times B \to \mathbb{F}$ let us define another invariant quadratic form $Q_{q_B,f}$ on (T_0^*B, q_B):

$$Q_{q_B,f}(a+\alpha, b+\beta) = q_B(a+\alpha, b+\beta) + f(a,b) = \alpha(b) + \beta(a) + f(a,b).$$

The resulting quadratic Lie algebra $(T_0^*B, Q_{q_B,f})$ was introduced in [Hofmann and Keith, 1986] as the *inflaction of* B with respect to the forms q_B and f. According to [Hofmann and Keith, 1986, Lemma 2.9], inflactions occur prominently in the structure of quadratic mixed Lie algebras.

Remark 2.2.26. Let B a Lie algebra, T_0^*B coincides with the double extension of the trivial Lie algebra of dimension zero by B.

Our next example follows ideas in [Bordemann, 1997, Example 4.2].

Example 2.2.9. Let $\mathfrak{h} = \operatorname{span}\langle x, y, z\rangle$ be the Heisenberg 3-dimensional Lie algebra given by the non-zero products $[x,y] = z$. Apart from the 6-dimensional 2-step Lie algebra $T_0^*\mathfrak{h}$, we can construct a 6-dimensional 3-step Lie algebra taking the cyclic 2-cocyle w defined as

$$w(v_1, v_2)(v_3) = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix},$$

where $v_i = \lambda_i x + \beta_i y + \gamma_i z$. This produces a Lie algebra $T_w^*\mathfrak{h}$ with basis $\{x, y, z, x^*, y^*, z^*\}$ and the following non-null multiplication table

$$[x,y] = z + z^*, \qquad [x,z] = [x,z^*] = -y^*, \qquad [y,z] = [y,z^*] = x^*.$$

This way, $(T_w^*\mathfrak{h})^2 = \operatorname{span}\langle z+z^*, x^*, y^*\rangle$ and $(T_w^*\mathfrak{h})^3 = \operatorname{span}\langle x^*, y^*\rangle$. This algebra does not appear directly in quadratic nilpotent classifications like the one in [Benito et al., 2017] as it is non-reduced. It decomposes in ideals as

$$T_w^*\mathfrak{h} = \operatorname{span}\langle x, y, x^*, y^*, z + z^*\rangle \oplus \operatorname{span}\langle z - z^*\rangle,$$

where the second term in abelian and the first one is isomorphic to $\mathfrak{n}_{2,3}$.

2.3 Chronology

Since quadratic Lie algebras appeared on 1955, there have been several advances. Hereunder, we list chronologically which have been the major events in the timeline until this memoir have been written. These articles have somehow inspired this work.

Note not all the articles in this chronology appear in the final bibliography. Instead, only the most used ones are referenced. Additionally, in this list, we have marked with symbol ■ those articles key to the development of this thesis, and with symbol ⧫ our own articles included, and sometimes expanded, to form this memoir.

1955 ● Tsou PhD: *On Metrisable Lie groups and algebras*.

1957 ■ Tsou-Walker: *Metrisable Lie groups and algebras*. Definitions of real metrizable algebra (quadratic), reduced, type and metric multiplicity (quadratic dimension). Techniques based on metric tensors and structure constants. Proof of the existence of arbitrary type (except for 7). Proof of $Z(L) = (L^2)^\perp$. Real and complex algebras of quadratic dimension 1.

1957 Ruse: *On the geometry of metrisable Lie algebras*. Properties of quadratic Lie algebras in terms of projective spaces geometry. Low dimensional examples.

1962 Tsou: *On the Construction of Metrisable Lie Algebras*. Proved the existence theorem of specific types of quadratic real Lie algebras announced in the previous article without proof.

1962 Walker: *Note on metrisable Lie groups and algebras*. Example of metrisable Lie groups and Lie algebras of quadratic dimension two..

1976 Milnor: *Curvatures of left invariant metrics on Lie groups*. Outline of classical theory on bi-invariant metrics. Definition of bi-invariant metric on real Lie as inner derivations being skew-adjoint. Metric Lie decomposition as orthogonal sum of ideals. Ricci curvatures.

1978 Astrakhantcev: *On the decomposability of Lie algebras*. Uniqueness of orthogonal ideal decomposition of quadratic Lie.

1984 Keith PhD: *On invariant bilinear forms of f-d Lie algebras*. Structure and selfduality of Lie algebras with invariant metrics. Definition of bi-extensión as tool to build metric algebras. Self-duality of ideal lattices. Orthogonality among the terms of upper and lower central series. Structure of metric mixed.

1984 Kac: *Infinite-dimensional Lie algebras, Birkhauser*. Definition of one-dimensional double extension (two exercices).

1985 Hilgert-Hofmann: *Lorentzian cones in real Lie algebras*. Discovery of a whole countable series of solvable real Lie algebras that support invariant Lorentzian forms (oscillator algebras).

1985 — Medina: *Groupes de Lie munis de métriques bi-invariantes*. One-dimensional double extension construction. Orthogonality among the terms of upper and lower central series. Properties of ideals in quadratic Lie.

1985 — Medina-Revoy: *Algèbres de Lie et produit scalaire invariant*. Indecomposable quadratic Lie as double extensions of either simple quadratic or 1-dimensional quadratic (characteristic zero).

1986 — Keith-Hofmann: *Invariant quadratic forms on f.d. Lie algebras*. Tensor products of Lie quadratics by commutative to enrich the examples of quadratic nilpotent (arbitrary nilindex) and mixed. Inflaction construction. Indecomposable mixed quadratic: $\mathrm{Rad}(L)^{\perp} \subset \mathrm{Rad}(L)$ and $\mathrm{Rad}(L)^{\perp} \oplus S$, being S a Levi factor, is isomorphic to the inflaction of the Levi factor.

1987 — Favre-Santharoubane: *On invariant bilinear forms on f.d. Lie algebras*. Indecomposable solvable quadratic: centre non-null and totally isotropic. One-dimensional double construction (solvable case; Kac's ideas). Quadratic nilpotent as amalgamated products.

1996 — Figueroa O'Farrill-Stanciu: *On the structure of symmetric self-dual Lie algebras*. Pretty outline of double extension method. Applications to Conformal field Theory.

1996 — Hilgert-Neeb: *Orthogonal Lie algebras with cone potential*. Root decomposition for some solvable quadratic indecomposable Lie algebras.

1997 — Noui-Revoy: *Algébres de Lie algebras orthogonales et formes trilineaires alaternées*. Reduced quadratic 2-step nilpotent Lie and trivectors. Equivalent classifications.

1997 ······ Bajo-Benayadi: *Lie algebras admitting a unique quadratic structure*. Examples of mixed quadratic of arbitrary quadratic dimension.

2002 ······ Eberlein: *Riemannian submersions and lattices in 2-step nilpotent Lie groups*. Introduction of standard real metrics (inner product, not necessarily invariant) Lie algebras. Relations to metric 2-step nilpotent algebras.

2003 ······ Baum-Kath: *Doubly extended Lie groups-curvature, holonomy and parallel spinors*. Geometry of doubly extended Lie groups with their natural bi-invariant metric.

2004 ······ Kath-Olbrich: *Metric Lie algebras with maximal isotropic center*. Real quadratic algebras through orthogonal representations. Two-fold quadratic extensions.

2006 ······ Ovando: *Small oscillations on $\mathbb{R}^2$ and Lie theory*. The real Lie oscillator 4-dim algebra as a model for the simple harmonic oscillator.

2006 ······ Kath-Olbrich: *Metric Lie algebras and quadratic extensions*. Building real quadratic without simple ideals. Radical descending and socle ascending series of ideals as tool (Bergery's idea).

2007 ······ Ovando: *Small oscillations and the Heisenberg Lie algebra*. Classical mechanical systems with a quadratic hamiltonian on $\mathbb{R}^{2n}$. Real oscillator algebras and $(2n+2)$-dimensional Lie motion of n Harmonic oscillators.

2007 ······ Kath: *Nilpotent quadratic Lie algebras of small dimension*. Classification of real nilpotent quadratic up to dimension 10.

2007 ······ Ovando: *Two-step nilpotent Lie algebras with ad-invariant metrics and a special kind of skew-symmetric maps*.

2007 — Bajo-Benayadi: *Lie algebras with quadratic dimension* 2. This type of algebras are local: admits a unique maximal ideal and a unique minimal.

2007 — Bajo-Benayadi-Medina: *Symplectic structures on quadratic Lie algebras*. Structure of quadratic algebras admitting non-degenerate skew-forms. Examples of Manin algebras.

2008 — Andrada-Barberis-Ovando: *Lie bialgebras of complex type and associated Poisson Lie groups*. Lie bialgebras arising from Hermitian structures on Lie algebras with ad-invariant metrics. Double procedure to build Lie algebras.

2009 — Kath-Olbrich: *Structure of pseudo-Riemannian symmetric spaces*.

2012 — Dat-Duong-Vu: *Solvable quadratic Lie algebras in low dimensions*. Quadratic classification up to dimension 6..

2012 — Duong-Pinczon-Ushirobira: *A new invariant of quadratic Lie algebras*.

2012 — del Barco-Ovando: *Free nilpotent Lie algebras admitting ad-invariant metrics*. (Real) Quadratic free nilpotent Lie algebras: $\mathfrak{n}_{2,3}$, 3-step 5-dimensional and $\mathfrak{n}_{3,2}$ 2-step 6-dimensional.

2013 — Duong: *Two-Step Nilpotent Quadratic Lie Algebras and 8-Dimensional Non-commutative Symmetric Novikov Algebras*.

2014 — Benayadi-Elduque: *Classification of quadratic Lie algebras of low dimension*. Mixed quadratic up to dimension 13.

2014 — Zusmanovich: *A Compendium of Lie Structures on Tensor Products*. Invariant bilinear forms on tensor product Lie algebras, also named current Lie algebras. Existence of Poisson structures.

2016 Cornullier: *On the Koszul map of Lie algebras*. A 12-dimensional quadratic nilpotent with non-zero Koszul map. Current Lie algebras 2-cohomology group description.

2016 Ovando: *Lie Algebras with ad-invariant metrics: a survery-guide*.

2017 Benito-de la Concepción-Laliena: *Free nilpotent and nilpotent quadratic Lie algebras*. General classification scheme of nilpotent quadratic. Scheme based on invariant forms of free nilpotent Lie algebras.

2017 Duong-Ushirobira: *Solvable quadratic Lie algebras of dimensions* ≤ 8. arXiv preprint, no journal publication reference found.

2018 Autenried-Furutani et al: *Pseudo-metric 2-step nilpotent Lie algebras*. Generalizations of definitions and results of Eberlein (2002) to pseudo-metric Lie algebras.

2018 Rodriguez-Salgado: *Geometric structures on Lie algebras and double extensions*. Quadratic, symplectic and contact structures on one-dimensional extensions of central extensions of real and complex Lie algebras.

2019 Benito-Roldán López et al: *Quadratic 2-step Lie algebra: Computational algorithms and classification*.

2019 Camacho-Kaminjarov-Ladra-Omirov: *Leibniz algebras constructed by representations of general diamond Lie algebras*. Diamond (also named as oscillator) algebras: solvable quadratic attached to Lorentzian cones.

2020 García Delgado-Salgado-Sánchez: *Invariant metrics on central extensions of quadratic Lie algebras*. Structural properties.

2020 Benito-Roldán López: *Derivations and Automorphisms of Free Nilpotent Lie Algebras and Their Quotients*.

2021 — Albuquerque-Barreiro-Benayadi et al.: *Poisson algebras and symmetric Leibniz bialgebra structures on oscillator Lie algebras*.

2021 — Zusmanovich: *On regular Lie algebras*. Quadratic regular algebras and a question formulated by Amitsur and Rowen in 1994.

2021 — Conti-del Barco-Rossi: *Uniqueness of ad-invariant metrics* and *Ad-invariant metrics on non-nice nilpotent Lie algebras*.

2022 — Benito-Roldán López: *Lie algebras with a finite number of ideals*.

2022 — Benayadi-Lebzioui: *Flat left-invariant pseudo-Riemannian metrics on quadratic Lie groups*.

2022 — García Delgado: *Invariant metrics on current Lie algebras*. Quadratic structures on tensor products of generalized oscillator algebras.

2023 — Benito-Roldán López: *Equivalent constructions of nilpotent quadratic Lie algebras*.

2023 — Bachaou-Bajo-Louzari: *On pseudo-Hermitian quadratic nilpotent Lie algebras*. Inductive description of Lie algebras with complex and quadratic structures.

2023 — Benito-Roldán López: *Examples and patterns on quadratic Lie algebras*. In press.

202x — Benito-Roldán López: *Lie structures and chain ideal lattices*. Accepted for publication.

202x — Benito-Roldán López: *Metrics related to oscillator Lie algebras*. Under review, preprint available in arXiv.

3 Deconstructing quadratic Lie algebras

CHAPTER

Along previous chapter, we have seen some basic families of Lie algebras which are always quadratic: abelian, simple and semisimple. Non-degenerate invariant bilinear forms in the abelian case are simply non-degenerate bilinear forms, or scalar products over the vector space. When considering simple and semisimple Lie algebras. The possibilities for their forms are greatly reduced as indicated by their quadratic dimension. In fact, over algebraically closed fields, they are just linear combinations of the Killing forms of their simple ideals in algebraically closed fields. So all these three families are full studied.

In this chapter, we will be focused on studying quadratic mixed Lie algebras. We will see we can reduce their study, first, to the solvable algebras and, later, to the nilpotent ones. This deconstruction will end up in the study of the two-step nilpotent case. It is precisely, on this variety of algebras, where a new approach to obtain them will arise. This leads to some computational algorithms to build quadratic algebras. All these last sections are based on the published paper [Benito et al., 2019].

3.1 From mixed quadratic to nilpotent

According to Levi's Theorem, any Lie algebra appears as the direct sum of a semisimple subalgebra and a solvable ideal. Thanks to previous works from Lie, Killing and Engel; at 1894, Cartan achieved the classification of complex (semi)sim-ple Lie algebras. But the analogous problem for the solvable type is wild. In 1945, Anatoly I. Maltsev reduced this problem to the classification of nilpotent Lie algebras, their derivation algebras, automorphism groups and some other invariants. This idea is the starting point of this section changing Lie for quadratic Lie.

3.1.1 Mixed Lie algebras

First, we need to define some important ideals in our algebra:

Definition 3.1.1. Given a Lie algebra L we define

- the *socle* of L, $\mathrm{soc}(L)$, as the sum of the minimal ideals,
- the *simple socle* of L, $\mathrm{ssoc}(L)$ as the sum of the minimal simple ideals,
- the *abelian socle* of L, $\mathrm{asoc}(L)$ as the sum of the minimal abelian ideals.

Notice, using Proposition 2.2.13, we have

$$\mathrm{soc}(L) = \mathrm{ssoc}(L) \oplus \mathrm{asoc}(L).$$

And some general structure results:

Proposition 3.1.1. *Let $L = S \oplus R$ a Levi decomposition, then*

(a) $L^2 = S \oplus \mathcal{J}(L)$.

And, when (L, φ) is quadratic, we also have for $N = N(L)$

(b) $\mathcal{J}(L)^{\perp} = R^{\perp} \oplus Z(L) = \mathrm{soc}(L)$,

(c) $Z(R) = \mathrm{asoc}(L) \subseteq N$,

(d) $R^\perp = \mathrm{ssoc}(L) \oplus (\mathrm{asoc}(L) \cap R^\perp)$ *with* $R \cap R^\perp = \mathrm{asoc}(L) \cap R^\perp$ *and* $R + R^\perp = \mathrm{ssoc}(L) \oplus R$,

(e) $N^\perp = \mathrm{ssoc}(L) \oplus (\mathrm{asoc}(L) \cap N^\perp)$ *with* $N \cap N^\perp = \mathrm{asoc}(L) \cap N^\perp$ *and* $N + N^\perp = \mathrm{ssoc}(L) \oplus N$.

Proof. First item can be proved directly by computation and applying Jacobson radical definition introduced in Theorem 2.1.15. This way

$$L^2 = [S \oplus R, S \oplus R] = [S, S] + [L, R] = S^2 \oplus \mathcal{J}(L) = S \oplus \mathcal{J}(L).$$

For the second, we need to use the last equality from Jacobson radical definition in equation (2.9) which says $\mathcal{J}(L) = L^2 \cap R = [L, R]$. Then, by orthogonality,

$$\mathcal{J}(L)^\perp = (L^2)^\perp + R^\perp = Z(L) + R^\perp.$$

But this sum is direct, as if $x \in Z(L) \cap R^\perp$ then $x \in \mathrm{Rad}\,\varphi$ because

$$\varphi(x, L) = \varphi(x, S + R) = \varphi(x, S) = \varphi(x, [S, S]) = \varphi([x, S], S) = 0.$$

For the final equality in item (b) we can observe that $\mathcal{J}(L)$ is the intersection of the maximal ideals, so its orthogonal must be $\mathrm{soc}(L)$.

Now, in order to prove item (c) first we notice $\mathrm{asoc}(L) \subseteq N$ trivially as $\mathrm{asoc}(L)$ is abelian, so nilpotent. Next, we are going to prove the equality by double inclusion. First, $Z(R)$, which is an abelian ideal, can be decompose as a direct sum of irreducible $\mathrm{ad}\, S$-modules using Theorem 2.1.23. Each summand is a minimal abelian ideal as $[Z(R), R] = 0$, thus $Z(R) \subseteq \mathrm{asoc}(L)$. On the other hand, let I be a minimal abelian ideal, applying item (b) and $\mathrm{asoc}(L) \subseteq N \subseteq R$, we obtain

$$I \subseteq \mathrm{asoc}(L) \subseteq \mathrm{soc}(L) \cap R = \mathcal{J}(L)^\perp \cap R. \tag{3.1}$$

Moreover, $[I, L] = 0$ or $[I, L] = I$ by minimality, in that second case

$$\varphi(I, R) = \varphi([I, L], R) = \varphi(I, [L, R]) = \varphi(I, \mathcal{J}(L)) = 0,$$

by the inclusion chain in expression (3.1). So, in this case $I \subseteq R^\perp$ and, using Proposition 2.2.13 item (b), we conclude $[I, R] = 0$ in both cases. As $I \subseteq R$ then $I \subseteq Z(R)$ proving $\mathrm{asoc}(L) \subseteq Z(R)$.

Lastly, items (d) and (e) can be determined together. We start with the relation $J(L) \subseteq N \subseteq R$ given in Theorem 2.1.15. By orthogonality, and in combination with previous item, $R^\perp \subseteq N^\perp \subseteq J(L)^\perp = \mathrm{soc}(L) = \mathrm{asoc}(L) \oplus \mathrm{ssoc}(L)$. Notice $\mathrm{ssoc}(L) \subseteq R^\perp \subseteq N^\perp$ as

$$\varphi(\mathrm{ssoc}(L), R) = \varphi([\mathrm{ssoc}(L), \mathrm{ssoc}(L)], R) = \varphi(\mathrm{ssoc}(L), [\mathrm{ssoc}(L), R]) = 0,$$

using $\mathrm{ssoc}(L) = \mathrm{ssoc}(L)^2$ as it is semisimple and both $\mathrm{ssoc}(L)$ and R are ideals. Therefore,

$$R^\perp = \mathrm{ssoc}(L) \oplus (\mathrm{asoc}(L) \cap R^\perp)$$
$$N^\perp = \mathrm{ssoc}(L) \oplus (\mathrm{asoc}(L) \cap N^\perp)$$

Finally, employing $\mathrm{asoc}(L) \subseteq N \subseteq R$ we obtain all the values of $R + R^\perp$, $N + N^\perp$, $R \cap R^\perp$, $N \cap N^\perp$. Alternatively, we can see $R^\perp \subseteq \mathrm{soc}(L)$ using Theorem 2.1.23 to express $R^\perp = \sum_{i=1}^t U_i$ with U_i irreducible $\mathrm{ad}\, S$-modules. Observe $[S, U_i] \subseteq U_i$ and $[R, U_i] = 0$, as R is an ideal so $[R, R^\perp] = 0$ by Proposition 2.2.13. Therefore, U_i are minimal ideals of L. □

Now, let (L, φ) be a finite dimensional mixed quadratic Lie algebra over a field of characteristic zero. Then L can be decomposed in an orthogonal[1] sum of ideals

$$L = L_0 \oplus L_1 \tag{3.2}$$

with $L_0 = \mathrm{ssoc}(L)$. The resulting algebra L_1 is quadratic and has no simple ideals.

Using now Theorem 2.1.26 we obtain $L_0 \subseteq S$ for any S Levi factor of L, in particular, $S = L_0 \oplus S_1$ with S_1 Levi factor of L_1 such that $L_1 = S_1 \oplus R$. Moreover, $(L_1, \varphi|_{L_1})$ as any quadratic Lie algebra can be decomposed using Theorem 2.2.9 as the orthogonal sum of ideals

$$L_1 = \mathfrak{a} \oplus L_2,$$

where $\mathfrak{a}$ is abelian and L_2 is quadratic reduced.

At this point we can list some properties of the algebras obtained in this decomposition:

Lemma 3.1.2. *Let $L = L_0 \oplus L_1 = L_0 \oplus \mathfrak{a} \oplus L_2$ be as above, then:*

(a) $R(L) = R(L_1) = R = \mathfrak{a} \oplus R(L_2)$,

[1]Not confuse this notation with gradations.

(b) $N(L) = N(L_1) = N = \mathfrak{a} \oplus N(L_2)$,

(c) $\mathcal{J}(L) = \mathcal{J}(L_1) = \mathcal{J}(L_2)$,

(d) $Z(L) = Z(L_1) = \mathfrak{a} \oplus Z(L_2) \subseteq \mathfrak{a} \oplus L_2^2$,

(e) $L_1^2 = L_2^2$ *and* $L^2 = L_0 \oplus L_1^2$.

Proof. Most items can be obtained directly by applying L and L_i definitions, by using $\mathfrak{a} \subseteq Z(L) \subseteq R(L) \subseteq L_1$, that L_0 and L_1 are ideals with null bracket between them, or that L_2 is reduced. Just as an example,

$$\mathcal{J}(L) = [L, R] = [L_1, R] = \mathcal{J}(L_1) = [L_2, R(L_2)] = \mathcal{J}(L_2)$$

is the proof of the third item, where we have also applied equation (2.9) which describes properties of the Jacobson radical. □

This way, all these ideals follow the content relationships among ideals shown in Figure 3.1. The study of ideals will take a leading role in Section 5.3.

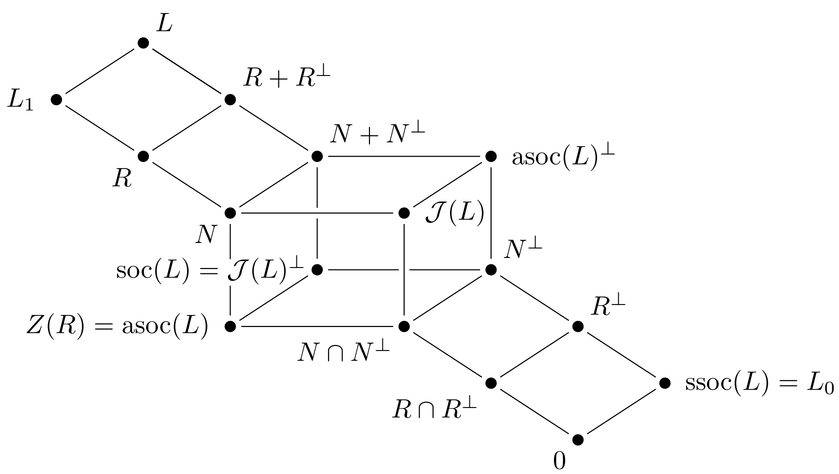

Figure 3.1: Structure of ideals in a mixed Lie algebra. Here $L/(R + R^\perp)$ is isomorphic to S_1 Levi factor of L_1 as observed in equation (3.2). Every ideal is contained in those that appear connected above or to their left.

So, from the mathematical development above including Proposition 3.1.1 and Lemma 3.1.2, we get our desired result:

Theorem 3.1.3. *Any quadratic mixed Lie algebra decomposes as an orthogonal direct sum of ideals in a reductive quadratic Lie algebra and a reduced quadratic Lie algebra with no simple ideals.*

This theorem goes beyond Theorem 2.2.9 and its decomposition, in combination with properties mentioned above, results in a much simpler ideal structure. At this point we can suppose without loss of generality that $L = S \oplus R$ is mixed quadratic without simple ideals and reduced. In this case

$$\operatorname{asoc}(L) = \operatorname{soc}(L) \subseteq N \subseteq R \subseteq L,$$

and the other way around,

$$R^{\perp} \subseteq N^{\perp} \subseteq \mathcal{J}(L)^{\perp} = \operatorname{soc}(L) = \operatorname{asoc}(L).$$

This way, we have the ideal structure shown in Figure 3.2.

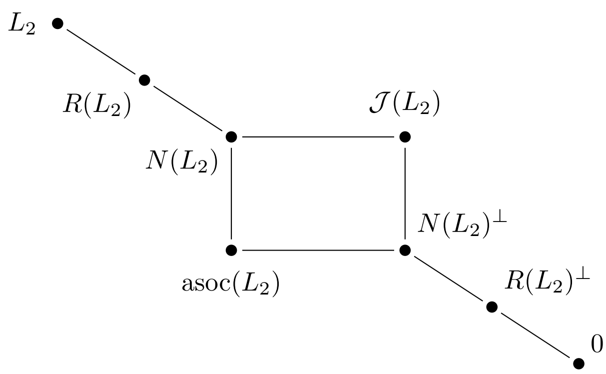

Figure 3.2: Structure of ideals in the reduced Lie algebra $L = L_2$. Every ideal is contained in those that appear connected above or to their left.

Before starting the decomposition, we need to see first a couple results in order to be able to determine when we can obtain a Lie algebra through a double extension process.

Proposition 3.1.4. *Let* (L, φ) *be a quadratic Lie algebra,* I *an ideal and* M *a subalgebra such that* $L = M \oplus I$*, then*

$$\begin{aligned} \Omega \colon I &\to M^* \\ x &\mapsto \varphi_x \colon M \to \mathbb{F} \\ &\qquad\qquad y \mapsto \varphi(x, y) \end{aligned}$$

is an M-module homomorphism, via the adjoint and coadjoint representations respectively, whose $\ker\Omega = M^\perp \cap I$.

Proof. First, Ω is linear as φ is bilinear. Now,

$$\ker\Omega = \{x \in I : \varphi(x, M) = 0\} = I \cap M^\perp.$$

Let see it respects the module actions. Let $x \in M$, $y \in I$ and $z \in M$ then

$$\begin{aligned}\Omega(x \cdot y)(z) &= \Omega(\mathrm{ad}(x)(y))(z) = \Omega([x,y])(z) = \varphi_{[x,y]}(z)\\ &= \varphi([x,y],z) = -\varphi(y,[x,z]) = -\varphi_y([x,z]) = (-\varphi_y \circ \mathrm{ad}(x))(z)\\ &= (\mathrm{ad}^*(x)(\varphi_y))(z) = (\mathrm{ad}^*(x)(\Omega(y)))(z) = (x \odot \Omega(y))(z).\end{aligned}$$

Here $\cdot$ and $\odot$ represent respectively the adjoint and coadjoint actions. □

Remark 3.1.5. From now on, extending this notation, $\varphi_x = \varphi(x, \cdot)$ for $x \in L$.

Analogously, we can see $I^\perp \cong (L/I)^*$ as L-modules (also M-modules) using

$$\begin{aligned}\hat{\Omega}\colon I^\perp &\to (L/I)^*\\ x &\mapsto \hat{\varphi}_x\colon L/I \to \mathbb{F}\\ &\qquad y+I \mapsto \varphi(x,y)\end{aligned}$$

Lemma 3.1.6. *Let (L,φ) be a quadratic Lie algebra, and I and ideal of L such that $I^\perp \subseteq I$. In that case, $(I/I^\perp, \tilde{\varphi})$ is a quadratic Lie algebra with*

$$\tilde{\varphi}(x + I^\perp, y + I^\perp) = \varphi(x,y).$$

In particular, for any ideal J we can take $I = J + J^\perp$ to obtain a quadratic Lie algebra.

Proof. First, observe $\tilde{\varphi}$ is bilinear and it is well defined as $\varphi(I, I^\perp) = 0$. So, we just need to check is non-degenerate

$$\mathrm{Rad}\,\tilde{\varphi} = \{x + I^\perp : \varphi(x, I) = 0\} = \{x + I^\perp : x \in I^\perp\} = 0.$$ □

Let (L,φ) be a quadratic Lie algebra, I an ideal such that $I^\perp \subseteq I$, and M subalgebra. We can define the Lie algebra homomorphism

$$\begin{aligned}\Phi_M\colon M &\to \mathrm{Der}_{\tilde{\varphi}}(I/I^\perp)\\ x &\mapsto \Phi_M(x)\colon I/I^\perp \to I/I^\perp\\ &\qquad y + I^\perp \mapsto [x,y] + I^\perp\end{aligned}$$

where $\tilde{\varphi}$ is the bilinear form induced over the quadratic algebra $(I/I^{\perp}, \tilde{\varphi})$ using Lemma 3.1.6.

And, finally we need to see double extensions admit other invariant forms apart from the one defined in the method itself.

Proposition 3.1.7. *Let (A_B, f_B) be the double extension of (A, f) by (B, ϕ). Then $(A_B, f_B + \varphi)$ is a quadratic Lie algebra for any φ bilinear symmetric such that $\varphi|_{B\times B} \in \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(B)$ and $\varphi|_{(A+B^*)\times A_B} = 0$.*

Proof. The proof is a straightforward computation and consists on checking $f_B + \varphi$ remains being non-degenerate and $f_B + \varphi \in \mathrm{Bi}^{\mathrm{s}}_{\mathrm{inv}}(A_B)$. Observe the matrix structure of this new bilinear form is

$$\left(\begin{array}{c|c|c} Q & 0 & I \\ \hline 0 & P & 0 \\ \hline I & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c|c} 0 & 0 & I \\ \hline 0 & P & 0 \\ \hline I & 0 & 0 \end{array}\right) + \left(\begin{array}{c|c|c} Q & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right),$$

where 0 represent null submatrices, I the identity, P is the matrix associated to f and Q is the matrix associated to $\varphi|_{B\times B}$. $\square$

Now, with all previous results proved, we are ready to introduce our main deconstruction theorem.

Theorem 3.1.8. *Let (L, φ) a quadratic Lie algebra, I and ideal which contains its orthogonal, i.e. $I^{\perp} \subseteq I$, and M a subalgebra of L such that $L = M \oplus I$. In that case L is isomorphic to $((I/I^{\perp})_M = M \oplus I/I^{\perp} \oplus M^*, \hat{\varphi}_M)$ the double extension of $(I/I^{\perp}, \hat{\varphi})$ by (M, Φ_M). And isometrically isomorphic to $((I/I^{\perp})_M, \hat{\varphi}_M + \phi)$ where ϕ is bilinear symmetric such that $\phi|_{M\times M} = \varphi|_{M\times M}$ and $\phi|_{(I/I^{\perp}\oplus M^*)\times(I/I^{\perp})_M} = 0$.*

Proof. Denote E the Lie algebra obtained in that double extension, i.e. $E = M \oplus I/I^{\perp} \oplus M^*$. Now, as $L = M \oplus I$ we have $L^{\perp} = M^{\perp} \cap I^{\perp} = 0$. So, using $I^{\perp} \cap M^{\perp} = 0$ and $I \cap M = 0$, we have

$$(I^{\perp} \oplus M) \cap (I^{\perp} \oplus M)^{\perp} = (I^{\perp} \oplus M) \cap I \cap M^{\perp} = 0.$$

In addition, $I^{\perp} \oplus M$ is a non-degenerate subalgebra which splits L as

$$L = (I^{\perp} \oplus M) \oplus (I^{\perp} \oplus M)^{\perp} = M \oplus (I \cap M^{\perp}) \oplus I^{\perp}$$

This decomposition of L allows us to define $\Psi\colon L \to E$ as

$$\begin{aligned}\Psi|_M &= \mathrm{Id}_M,\\ \Psi|_{I\cap M^\perp}(x) &= x + I^\perp,\\ \Psi|_{I^\perp} &= \Omega|_{I^\perp},\end{aligned}$$

where Ω is the one defined in Proposition 3.1.4. This Ψ is a Lie algebra homomorphism as for $m, m' \in M$, $x, x' \in I \cap M^\perp$ and $y, y' \in I^\perp$ we have

- as M is a subalgebra $\Psi([m,m']_L) = [m,m']_M = [\Psi(m), \Psi(m')]_E$,
- The trickiest case appears when combining two elements x, x'. Its product can have parts both in the intersection or in $I^\perp$. Therefore, we decompose $[x,x']_L = a + b$ where

  $$\begin{aligned}\pi_{I\cap M^\perp}([x,x']_L) &= a \in I \cap M^\perp\\ \pi_{I^\perp}([x,x']_L) &= b \in I^\perp\end{aligned}$$

  First,

  $$\begin{aligned}\Psi([x,x']_L) = \Psi(a+b) = (a + I^\perp) + \varphi_b &= (a + b + I^\perp) + (\varphi_a + \varphi_b)\\ &= ([x,x'] + I^\perp) + \varphi_{[x,x']},\end{aligned}$$

  as $\varphi_a(m) = \varphi(a,m) = 0$ for all $a \in M^\perp$. Next

  $$\begin{aligned}[\Psi(x), \Psi(x')]_E = [x+I^\perp, x'+I^\perp]_E &= ([x,x']+I^\perp) + \omega(x+I^\perp, x'+I^\perp)\\ &= ([x,x'] + I^\perp) + \varphi_{[x,x']},\end{aligned}$$

  since

  $$\begin{aligned}&\omega(x + I^\perp, x' + I^\perp)(m) = \tilde{\varphi}(\Phi_M(m)(x + I^\perp), x' + I^\perp)\\ &= \tilde{\varphi}([m,x] + I^\perp, x' + I^\perp) = \varphi([m,x], x') = \varphi([x,x'], m) = \varphi_{[x,x']}(m).\end{aligned}$$

- On one hand, $[\Psi(m), \Psi(x)]_E = [m, x+I^\perp]_E = \Phi_M(m)(x+I^\perp) = [m,x] + I^\perp$. On the other hand, $\Psi([m,x]_L) = [m,x] + I^\perp$ as $[m,x] \in I \cap M^\perp$ because $x \in I$ and I is an ideal, and $\varphi([m,x], m') = -\varphi(x, [m,m']) = 0$ since $x \in M^\perp$ and M is subalgebra.

- For M against $I^{\perp}$, it also works as

$$[\Psi(m), \Psi(y)]_E = \mathrm{ad}^*(m)(\varphi_y) = -\varphi_y \circ \mathrm{ad}(m) = -\varphi(y, [m, \cdot\,]) \\ = \varphi([m, y], \cdot\,) = \varphi_{[m,y]} = \Psi([m, y]_L).$$

- Finally, both $[\Psi(y), \Psi(y')]_E$ and $[\Psi(x), \Psi(y)]_E$ equal zero by definition of the double extension product. In the first case we have

$$\Psi([y, y']_L)(m) = \varphi_{[y,y']}(m) = \varphi([y, y'], m) = \varphi(y, [y', m]) = 0,$$

  using $I^{\perp}$ is an ideal and $I^{\perp} \subseteq I$. While in the second, analogously,

$$\Psi([x, y]_L)(m) = \varphi_{[x,y]}(m) = \varphi([x, y], m) = -\varphi(y, [x, m]) = 0.$$

And, to see it is bijective we can observe both dimensions coincide and

$$\ker \Psi = \{x \in I \cap M^{\perp} : x \in I^{\perp}\} = 0.$$

Finally, to check the isometry we must check it preserve the bilinear forms for the elements in each part. Let $f = \hat{\varphi}_M + \phi$ be the bilinear form of the double extension, $m, m' \in M$, $x, x' \in I \cap M^{\perp}$ and $y, y' \in I^{\perp}$.

$$\begin{aligned} f(\Psi(m), \Psi(m')) &= f(m, m') = \phi(m, m') = \varphi(m, m'), \\ f(\Psi(m), \Psi(x)) &= f(m, x + I^{\perp}) = \hat{\varphi}_M(m, x + I^{\perp}) = 0 = \varphi(m, x), \\ f(\Psi(m), \Psi(y)) &= f(m, \varphi_y) = \hat{\varphi}_M(m, \varphi_y) = \varphi_y(m) = \varphi(y, m) = \varphi(m, y), \\ f(\Psi(x), \Psi(x')) &= f(x + I^{\perp}, x' + I^{\perp}) = \hat{\varphi}_M(x + I^{\perp}, x' + I^{\perp}) \\ &= \hat{\varphi}(x + I^{\perp}, x' + I^{\perp}) = \varphi(x, x'), \\ f(\Psi(x), \Psi(y)) &= f(x + I^{\perp}, \varphi_y) = \hat{\varphi}_M(x + I^{\perp}, \varphi_y) = 0 = \varphi(x, y), \\ f(\Psi(y), \Psi(y')) &= f(\varphi_y, \varphi_{y'}) = \hat{\varphi}_M(\varphi_y, \varphi_{y'}) = 0 = \varphi(y, y'). \end{aligned}$$ □

Remark 3.1.9. This theorem allows us to obtain some quadratic Lie algebras as double extensions. After reducing the algebra to a quotient of ideals and double extending it again by a subalgebra, we obtain the original Lie algebra, but its bilinear form can differ unless $M \subseteq M^{\perp}$. In those cases, we can obtain an isometric double extension using Proposition 3.1.7 by adding a summand to the bilinear form to control it over $M \times M$.

Remark 3.1.10. Any algebra obtained from a double extension is of this form. In $B \ltimes (A \oplus B^*)$ we can distinguish an ideal $A \oplus B^*$, containing its orthogonal B^*, and a subalgebra B.

All these decompositions work not only for ideals I such that $I \subseteq I^{\perp}$, but for any ideal J considering $I = J + J^{\perp}$ with $I^{\perp} = J \cap J^{\perp} \subseteq I$ (see Figure 3.3).

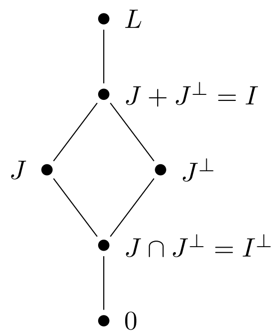

Figure 3.3: Diagram showing how to obtain an ideal containing its orthogonal. Every ideal is contained in those that appear above.

We can observe the Levi decomposition $L = S \oplus R$ satisfies, in our reduced algebras with no simple ideals, that $R^{\perp} \subseteq R$. Therefore, we can apply Theorem 3.1.8.

Corollary 3.1.11. *Let (L, φ) be a quadratic reduced Lie algebra with no simple ideals. Then L is isomorphic to the double extension $((R/R^{\perp})_S = S \oplus R/R^{\perp} \oplus S^*, \hat{\varphi}_S)$ of $(R/R^{\perp}, \hat{\varphi})$ by (S, Φ_S). And it is isometrically isomorphic to $((R/R^{\perp})_S, \hat{\varphi}_S + \phi)$ where ϕ is bilinear symmetric with $\phi|_{S\times S} = \varphi|_{S\times S}$ and $\phi|_{(R/R^{\perp}\oplus S^*)\times(R/R^{\perp})_S} = 0$.*

This result appears as Theorem 2.2 (iii) in [Bordemann, 1997] without proof and avoiding isometric isomorphisms.

Thanks to the result from Corollary 3.1.11, we can reduce the study of generic quadratic Lie algebras to just the semisimple, abelian and solvable-reduced ones separately. In conclusion,

$$\underbrace{(L,\varphi)}_{\text{Any mixed}} = \underbrace{(L_0,\varphi_0)}_{\text{Semisimple ideal}} \perp \underbrace{(\mathfrak{a},\varphi_1)}_{\text{Abelian ideal}} \perp \underbrace{(L_1,\varphi_2)}_{\text{Double extension of a solvable}}$$

As semisimple and abelian cases are completely known, as mentioned before, we are now going to focus just on the solvable one.

3.1.2 Solvable Lie algebras

At this point, we are working with solvable reduced Lie algebras. This way, $Z(L) \cap L^2 = Z(L) \neq 0$ seeing Proposition 2.2.13 item (f). So we can take an isotropic element $z \in Z(L)$, which can define a one-dimensional ideal $I^\perp = \mathbb{F}z$ contained in its orthogonal I. Using dimensions, this I has codimension one, so we can decompose

$$L = M \oplus I,$$

where $M = \mathbb{F}x$ is a subalgebra, I is an ideal and $I^\perp \subseteq I$. These are exactly the conditions required by Theorem 3.1.8.

Corollary 3.1.12. *Any solvable, quadratic and reduced Lie algebra (L, φ) is isometrically isomorphic to the double extension of $(I/I^\perp, \hat{\varphi})$ by means $(X = \mathbb{F}x, \Phi_X)$ where $I^\perp = \mathbb{F}z$ for some $z \in Z(L)$ and X is a complement (as a vector space) of I such that $\varphi(x, x) = 0$ and $\varphi(x, z) = 1$.*

Proof. Observe that x exists as for some y complement of I, if $\varphi(y, y) \neq 0$ we can consider $x = z - \frac{2\varphi(y,z)}{\varphi(y,y)} y$ up to scalars. □

Remark 3.1.13. This corollary is just 2.7 Lemma in [Favre and Santharoubane, 1987]. In addition, this same result can be extended to non-solvable Lie algebras when $Z(L) \cap L^2 \neq 0$. Indeed, this is similar to the form of the result which appears in [Bordemann, 1997], but specifying the double extension process we must follow to obtain our algebra.

This result allows us to obtain any solvable quadratic Lie algebra throughout a chain of double extensions. Each algebra in this chain has dimension two less than its predecessor. An idea like the one in this corollary appears in [Favre and Santharoubane, 1987, 2.8 Lemma], which acts as a source of motivation along this section. Thanks to this result, we could obtain any solvable Lie algebra from successive double extensions starting at the zero or 1-dimensional Lie algebra, depending or whether its dimension is even or odd. This chain of double extensions by 1-dimensional algebras serves as an inspiration for our chain in Figure 4.1 found later in Chapter 4. But this result does not reduce our classification problem to some other smaller family, they are all still solvable. To fix this problem, we can take the following approach.

Let $L = R$ be a reduced solvable quadratic Lie algebra. In this case we have the chain of ideals

$$0 \subsetneq Z(R) \subseteq R^2 \subseteq N \subseteq R = L.$$

By orthogonality

$$0 \subseteq N^\perp \subseteq Z(R) \subseteq R^2 \subsetneq R.$$

So we have the chain

$$0 \subseteq N^\perp \subseteq Z(R) \subseteq R^2 \subseteq N \subseteq R = L,$$

which results in a simplified version of Figure 3.2 shown in Figure 3.4.

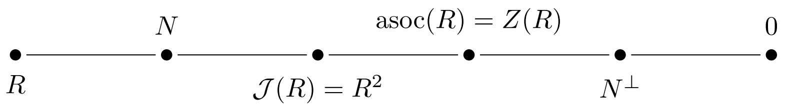

Figure 3.4: Ideals in a solvable reduced Lie algebra, where an ideal is contained within those to its left. Here we have used $\mathcal{J}(R) = [L, R] = [R, R]$ and $\mathcal{J}(R)^\perp = \mathrm{asoc}(R) = (R^2)^\perp = Z(R)$.

Proposition 3.1.14. *Let (R, φ) be a solvable non-nilpotent quadratic reduced Lie algebra, we can obtain a chain of quadratic Lie algebras*

$$R = R_0, R_1, \ldots, R_n,$$

such that R_i is isometrically isomorphic to the double extension of R_{i+1} by a 1-dimensional Lie algebra, R_n is nilpotent and $n = \mathrm{codim}_R N$.

Proof. Let us proceed by induction on $n = \mathrm{codim}_R N$. If $n = 1$, then there exists $x \notin N$ so $R = N \oplus \mathbb{F}x$. Here $\mathbb{F}x$ is a subalgebra, and N an ideal containing its orthogonal. So, effectively, we can apply Corollary 3.1.12 obtaining the chain with two algebras $R_0 = R$, $R_1 = N/N^\perp$.

Let us assume this is satisfied for every n and see what happens for $n + 1$. Since R is not nilpotent, $N \neq R$ and $N^\perp \neq 0$. In this case, every $z \in N^\perp \subseteq Z(R) \subseteq R^2 = Z(R)^\perp$ is isotropic. If we consider $I^\perp = \mathbb{F}z \subseteq I$, I would have codimension one, and we can decompose $R = I \oplus \mathbb{F}x$ for some $x \notin I$. Again, employing Corollary 3.1.12, R is the double extension of $R_1 = I/I^\perp$.

Moreover, $N(R_1) = T/I^\perp$ for some T, thus $T^k \subseteq I^\perp \subseteq Z(R)$. This way, $T^{k+1} = 0$, so T is nilpotent.

Note that $\mathcal{J}(R) = R^2 \subseteq I$ because I is a maximal ideal. Then, $R^2/I^\perp$ is a nilpotent ideal of $I/I^\perp$ and, by definition of nilpotent radical, $R^2/I^\perp \subseteq T/I^\perp$ and $R^2 \subseteq T$. Therefore, T is an ideal of R ($[T,R] \subseteq R^2 \subseteq T$) and, as it is nilpotent, $T \subseteq N$. But $I^\perp = \mathbb{F}z \subseteq N^\perp$, so $T \subseteq N \subseteq I$ and $T = N$ by maximality of the nilradical $T/I^\perp$ of R_1.

Moreover, as

$$\begin{aligned}\operatorname{codim}_{R_1} N(R_1) = \dim R_1 - \dim N(R_1) &= \dim R - 2 - \dim N + 1 \\ &= \dim R - \dim N - 1 = \operatorname{codim}_R N - 1,\end{aligned}$$

we can apply our induction hypothesis over R_1. In order to apply induction, we must check $R_1 = I/I^\perp$ is reduced. As I^2 is an ideal of R, $(I^2)^\perp \subseteq I^2$ because R is reduced. Then, from $R^2 \subseteq I$, we obtain

$$I^\perp \subseteq (R^2)^\perp = Z(R) \subseteq (I^2)^\perp \subseteq I^2 \subseteq R^2 \subseteq I.$$

On the other hand,

$$R_1^2 = \left(\frac{I}{I^\perp}\right)^2 = \frac{I^2 + I^\perp}{I^\perp} = \frac{I^2}{I^\perp}$$

and

$$Z(R_1) = (R_1^2)^\perp = \{x + I^\perp : x \in I \cap (I^2)^\perp = (I^2)^\perp\} = \frac{(I^2)^\perp}{I^\perp} \subseteq \frac{I^2}{I^\perp} = R_1^2. \quad \square$$

This result gives us a shorter path of double extensions when looking for a nilpotent algebra at the end of our chain.

When writing about reducing the study to just nilpotent Lie algebras, there is another result worth mentioning from [Keith, 1984, Proposition 5.61]. We recall that from Proposition 2.2.13 item (f) $N^\perp \subseteq N$ and then we have.

Proposition 3.1.15. *If (L, φ) is a quadratic solvable Lie algebra over a field $\mathbb{F}$ of characteristic zero, then (L, φ) is a central bi-extension of $(N/N^\perp, \varphi|_{N\times N})$ where $N = N(L)$.*

Remark 3.1.16. Bi-extension is the name given in [Keith, 1984, Definition 5.60] to those algebras which contain an ideal I such that $I^\perp \subseteq I$. In that same thesis, the author obtained some of those bi-extensions using a procedure equivalent to the double extension by an abelian Lie algebra, but the question of how to obtain the rest remain opened. Moreover, the term central refers to $N^{\perp} \subseteq Z(L)$.

Anyway, thanks to Proposition 3.1.14, we are led directly into the following section: the study of those nilpotent quadratic algebras.

3.1.3 Nilpotent Lie algebras

Thanks to all the previous deconstruction, nilpotent Lie algebras remain being one of the main fields of study of quadratic Lie algebras. And its complete classification is still an open problem. In this section we will see different results for these algebras.

To begin with, as nilpotent algebras are solvable, Corollary 3.1.12, which we have previously seen, can be applied over any nilpotent Lie algebra to obtain it from successive double extensions with one-dimensional algebras (vector spaces) starting from the zero or the abelian one-dimensional Lie algebra.

At this point, where we cannot reduce our nilpotent family of study to a smaller one, we should start studying these algebras. As this is a wild problem, research in this line has focused on more restrictive subvarieties, for instance: low dimensional in [Kath, 2007], 2-step in [Ovando, 2007b] or [Duong, 2013], free nilpotent in [del Barco and Ovando, 2012] or even solvable with maximal isotropic centre in [Kath and Olbrich, 2004]. However, these are just specific approaches with no technical generalization.

In [Benito et al., 2017], a classification scheme based on free nilpotent algebras and their invariant forms was introduced. Here, a new technique for constructing quadratic nilpotent Lie algebras out of invariant symmetric bilinear forms on free nilpotent Lie algebras appears. In the sequel, we summarize briefly some of the results we will use along the next section. The following lemma comes from [Benito et al., 2017, Proposition 4.1].

Lemma 3.1.17. *Let $\mathfrak{n}$ be the factor Lie algebra $\mathfrak{n}_{d,t}/I$ where I is an ideal of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq I \subseteq \mathfrak{n}^2_{d,t}$. Then, there exists a symmetric, invariant and non-degenerate bilinear form φ on $\mathfrak{n}$ if and only if there exists a symmetric and invariant bilinear form ψ on $\mathfrak{n}_{d,t}$ such that $I = \mathfrak{n}^{\perp}_{d,t}$. The relation between φ and ψ is given by $\psi(a, b) = \varphi(a + I, b + I)$ for all $a, b \in \mathfrak{n}_{d,t}$.*

Previous lemma reduces the classification of quadratic nilpotent Lie algebras to that of invariant bilinear forms on free nilpotent Lie algebras. Moreover, we can establish a categorical approach. We distinguish two categories:

- $\mathrm{NilpQuad}_{d,t}$ stands for the category whose objects are the quadratic t-nilpotent Lie algebras $(\mathfrak{n}, \varphi)$ of type d. Morphisms $f\colon (\mathfrak{n}, \varphi) \to (\mathfrak{n}', \varphi')$ are *isometric Lie homomorphisms*, that means $f([x, y]) = [f(x), f(y)]$ and $\varphi(x, y) = \varphi'(f(x), f(y))$.

- $\mathrm{Sym}_0(d, t)$ is the category whose objects are the symmetric invariant bilinear forms ψ on $\mathfrak{n}_{d,t}$ for which $\ker \psi \subseteq \mathfrak{n}^2_{d,t}$ and $\mathfrak{n}^t_{d,t} \nsubseteq \ker \psi$. The morphisms are isometric Lie homomorphisms of $\mathfrak{n}_{d,t}$ moduli the relation of equivalence
$$f_1 \sim f_2 \iff (f_1 - f_2)(\mathfrak{n}_{d,t}) \subseteq \ker(\psi_2),$$
where $f_i\colon (\mathfrak{n}_{d,t}, \psi_1) \to (\mathfrak{n}_{d,t}, \psi_2)$ for $i = 1, 2$.

- $\mathrm{Q}_{d,t}\colon \mathrm{Sym}_0(d, t) \to \mathrm{NilpQuad}_{d,t}$ is the functor that associates to each object ψ in the category $\mathrm{Sym}_0(d, t)$, the object in $\mathrm{NilpQuad}_{d,t}$:
$$\mathrm{Q}_{d,t}(\psi) = (\mathfrak{n}_{d,t}/\ker \psi, \varphi), \quad \varphi(a + \ker \psi, b + \ker \psi) = \psi(a, b).$$

In [Benito et al., 2017] it is proved that the functor $\mathrm{Q}_{d,t}$ provides an equivalence between the categories $\mathrm{Sym}_0(d, t)$ and $\mathrm{NilpQuad}_{d,t}$. Moreover, there is a natural action of $\mathrm{Aut}\,\mathfrak{n}_{d,t}$, the group of automorphisms of $\mathfrak{n}_{d,t}$, on the set of objects of the category $\mathrm{Sym}_0(d, t)$,

$$\mathrm{Aut}\,\mathfrak{n}_{d,t} \times \mathrm{Obj}(\mathrm{Sym}_0(d, t)) \to \mathrm{Obj}(\mathrm{Sym}_0(d, t)) \quad \text{given as} \quad (\theta, \psi) \mapsto \psi_\theta \tag{3.3}$$

where $\psi_\theta(x, y) = \psi(\theta(x), \theta(y))$. Using the functor $\mathrm{Q}_{d,t}$ and the action $\theta \cdot \psi = \psi_\theta$ introduced in expression (3.3), from [Benito et al., 2017, Corollary 4.3 and Lemma 4.4] we get:

Lemma 3.1.18. *For every $\psi_1, \psi_2 \in \mathrm{Obj}(\mathrm{Sym}_0(d, t))$, the following assertions are equivalent:*

- *ψ_1 and ψ_2 are isomorphic objects in the category $\mathrm{Sym}_0(d, t)$.*

- *$\mathrm{Q}_{d,t}(\psi_1)$ and $\mathrm{Q}_{d,t}(\psi_2)$ are isometrically isomorphic Lie algebras.*

- *There exists an isometric automorphism $\theta\colon (\mathfrak{n}_{d,t}, \psi_1) \to (\mathfrak{n}_{d,t}, \psi_2)$.*

Lemma 3.1.19. *The orbit set* $\mathrm{Orb}_{\mathrm{Aut}\,\mathfrak{n}_{d,t}}(\psi) = \{\psi_\theta : \theta \in \mathrm{Aut}\,\mathfrak{n}_{d,t}\}$*, for all* $\psi \in \mathrm{Obj}(\mathrm{Sym}_0(d,t))$*, equals to the set of bilinear invariant symmetric forms that are isomorphic to* ψ *in the category* $\mathrm{Sym}_0(d,t)$*. Therefore, the number of orbits of the action* $\theta \cdot \psi = \psi_\theta$ *given in equation* (3.3) *is exactly the number of quadratic t-nilpotent Lie algebras of type d up to isometric isomorphisms.*

Thanks to all these results, the authors were able to study up to isometric isomorphisms, the indecomposable quadratic nilpotent Lie algebras over any algebraically closed filed $\mathbb{F}$ of characteristic zero whose type is 1, or 2 when the nilindex is less than 6; or type 3 for nilindex less than 4. They end up with 7 algebras all of different dimension. For the real case they add more algebras. All these can be found summarized in Table 3.1, and in an extended way in the original paper (see [Benito et al., 2017]).

Type	**Nilindex**	**Dimension**
1	1	$1, 1^-$
2	3	$5, 5^-$
	5	$7, 7^-, 8, 8^-, 8^*$
3	2	6
	3	$8, 8^-, 8^*, 9, 9^-, 9^*$

Table 3.1: Quadratic Lie algebras of small type classified in [Benito et al., 2017]. Here, n^* denotes an algebra of dimension n which only appears in the real case, n^- refers to an algebra $(\mathfrak{n}, -\varphi)$ of dimension n which belongs also to $\mathbb{R}$ for some algebra $(\mathfrak{n}, \varphi)$ which exists in both $\mathbb{R}$ and $\mathbb{C}$ and appears in this list simply as n.

Moreover, we observe the real algebra $\mathfrak{n}_{2,3}(\mathbb{R})$ admits two non-isometric invariant metrics, the metric in Example 2.2.4 and its opposite. In fact, the number of non-isometrically isomorphic quadratic structures on $\mathfrak{n}_{2,3}(\mathbb{F})$ is equal to the cardinality of the quotient group $\mathbb{F}^\times/(\mathbb{F}^\times)^2$, where $\mathbb{F}^\times$ is the multiplicative group of the field $\mathbb{F}$ as it is established in [Benito et al., 2017, Theorem 5.2 and Corollary 5.3]. So, there is only one quadratic structure on the complex algebra $\mathfrak{n}_{2,3}(\mathbb{C})$ and infinite in case $\mathfrak{n}_{2,3}(\mathbb{Q})$, where $\mathbb{Q}$ is the rational field. In contrast, up to isometric isomorphisms, there is only one indecomposable and quadratic 2-step Lie algebra of type 3 over any field of characteristic zero. This algebra is the 6-dimensional free nilpotent $\mathfrak{n}_{3,2}(\mathbb{F})$ according to [Benito et al., 2017, Corollary 5.7], see Example 2.2.5. In fact, [del Barco and Ovando, 2012] proved that $\mathfrak{n}_{3,2}(\mathbb{F})$ and $\mathfrak{n}_{2,3}(\mathbb{F})$ are the only free nilpotent Lie algebras that admit a quadratic structure.

Recall at the end of Section 2.2.1, just before Section 2.2.2, we said $m(\mathfrak{n}_{2,3}) = 4$ and $m(\mathfrak{n}_{3,2}) = 7$ despite there are all isometrically isomorphic in $\mathbb{C}$.

3.2 Quadratic 2-step nilpotent Lie algebras

Nilpotent Lie algebras are a huge family to be analyzed thoroughly. This is why we are going to focus on the 2-step case. A Lie algebra $\mathfrak{n}$, as seen in Definition 2.1.25, is said to be 2-step when $\mathfrak{n}^3 = 0$ but $\mathfrak{n}^2 \neq 0$. These are the first nilpotent algebras we can study that are not trivial (abelian).

Along this section, we are recombining ideas and results in [Benito et al., 2017] and [Ovando, 2007b] to get an explicit classification of quadratic 2-step nilpotent Lie algebras.

The real Lie groups associated to quadratic 2-step real Lie algebras provide examples of compact pseudo-Riemannian nilmanifolds. According to [Noui and Revoy, 1997, Corollary 3.6], there is a finite number (up to isometric isomorphisms) of quadratic and reduced 2-step nilpotent Lie algebras of dimension up to 17. Following that same reference, the classification of those algebras is equivalent to the classification of alternating trilinear forms, which is an open problem. Ovando (see [Ovando, 2007b]) shows the existence of real quadratic 2-step Lie algebras of arbitrary type $d \geq 3$ with $d = 4$ as the only exception. In [Ovando, 2007b], it is proved that the algebras in this class can be achieved from any injective homomorphism $\rho\colon \mathfrak{v} \to \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ of a real vector space $\mathfrak{v}$ equipped with an inner product $\langle \cdot, \cdot \rangle$. In addition, the map ρ must satisfy the relation $\rho(v)(v) = 0$ for all $v \in \mathfrak{v}$. This last condition can be rewritten as,

$$\rho(v)(u) + \rho(u)(v) = 0. \tag{3.4}$$

From now on, the rest of the chapter, based on the published article [Benito et al., 2019], is aimed to introduce a new method to construct any 2-step nilpotent quadratic algebra of d generators or type d. We are going to establish some alternative results on the existence and isomorphisms of quadratic

2-step nilpotent Lie algebras over arbitrary fields of characteristic 0. Along it, we will show that the key of the structure and classification of this class of metric algebras relies on certain families of skew-symmetric matrices. Moreover, we will give some computational examples for d up to 8.

3.2.1 Basic examples

The smallest example of a reduced 2-step nilpotent Lie algebra is the free nilpotent $\mathfrak{n}_{3,2}$ of dimension 6. Its metric was defined in Example 2.2.5. We note that $Z(\mathfrak{n}_{3,2})$ is an isotropic subspace of dimension half of the algebra, 3 in this case. So this space is a lagrangian (see Definition 2.2.5) and φ is a metabolic metric (equivalently hyperbolic symmetric). This motivates the following general definition, included in [Elman et al., 2008, Chapter I, Section 1 and 1C], which is remarkable for the explicit description of quadratic 2-step Lie algebras on the real field.

Definition 3.2.1. Let $\mathfrak{v}$ denotes a vector space over an arbitrary field $\mathbb{F}$ and $\lambda = \pm 1$. We define the hyperbolic λ-bilinear form on $\mathfrak{v}$ to be the form $\varphi_{\mathfrak{v}}^{\lambda}$ on $\mathbb{H}(\mathfrak{v}) = \mathfrak{v} \oplus \mathfrak{v}^*$, where

$$\varphi_{\mathfrak{v}}^{\lambda}(v_1 + f_1, v_2 + f_2) = f_1(v_2) + \lambda f_2(v_1),$$

for all $v_1, v_2 \in \mathfrak{v}$ and $f_1, f_2 \in \mathfrak{v}^*$. If $\lambda = 1$, the form $\varphi_{\mathfrak{v}}^{1}$ is symmetric, and if $\lambda = -1$, it is alternating (skew-symmetric). A bilinear form ψ is called a *hyperbolic bilinear form* if it is isometric to $\varphi_{\mathfrak{v}}^{\lambda}$ for some vector space $\mathfrak{v}$ and some $\lambda = \pm 1$.

Definition 3.2.2. A non-degenerate bilinear form $\varphi\colon V \times V \to \mathbb{F}$ is called *metabolic* if there is an isotropic subspace W of V of half its dimension, i.e. there is a lagrangian subspace.

In characteristic different from two, following [Elman et al., 2008, Corollary 1.26], hyperbolic symmetric and metabolic forms are equivalent. This metrics are the basement for the following general example.

Example 3.2.1. Let $(\mathfrak{v}, \langle\cdot,\cdot\rangle, \rho)$ be the a triple where $(\mathfrak{v}, \langle\cdot,\cdot\rangle)$ is a metric vector space over $\mathbb{R}$ (so $\langle\cdot,\cdot\rangle$ is an inner product on $\mathfrak{v}$) and $\rho\colon \mathfrak{v} \to \mathfrak{so}(\mathfrak{v})$ denotes an injective linear map that satisfies condition (3.4). For every $u, v \in \mathfrak{v}$, let $f_{u,v} : \mathfrak{v} \to \mathbb{R}$ be defined as

$$f_{u,v}(w) = \langle \rho(w)(u), v\rangle \underset{(3.4)}{=} -\langle \rho(u)(w), v\rangle$$

for all $w \in \mathfrak{v}$. Consider now the vector space $\mathfrak{n}(\mathfrak{v}, \rho) = \mathfrak{v} \oplus \mathfrak{v}^*$ endowed with the canonical hyperbolic metric

$$\varphi_{\mathfrak{v}}^1(v_1 + f_1, v_2 + f_2) = f_1(v_2) + f_2(v_1).$$

In $\mathfrak{n}(\mathfrak{v}, \rho)$ we define the bracket product

$$[u + g, v + h] = [u, v] = f_{u,v},$$

for $u, v \in \mathfrak{v}$ and $g, h \in \mathfrak{v}^*$. Since $\rho(a) \in \mathfrak{so}(\mathfrak{v})$, the product $[\cdot, \cdot]$ is skewsymmetric. It is also clear that $[[u, v], w] = 0$. Hence, $\mathfrak{n}(\mathfrak{v}, \rho)$ is a 2-step Lie algebra. From condition (3.4) of ρ, we get that $\varphi_{\mathfrak{v}}^1$ is an invariant bilinear form. Therefore $(\mathfrak{n}(\mathfrak{v}, \rho), \varphi_{\mathfrak{v}}^1)$ is a quadratic 2-step Lie algebra named as the *modified cotangent* of $\mathfrak{v}$. In fact $\mathfrak{n}(\mathfrak{v}, \rho)^2 = \mathfrak{v}^* = Z(\mathfrak{n}(\mathfrak{v}, \rho))$ because of ρ is injective. So the Lie algebra $\mathfrak{n}(\mathfrak{v}, \rho)$ is reduced (equivalently, its *corank*, defined as $\dim Z(\mathfrak{n}) - \dim \mathfrak{n}^2$, is zero).

Theorem 3.2 in [Ovando, 2007b] establishes that for every integer $m \geq 1$, if we provide $\mathbb{R}^m$ (consider as an abelian Lie algebra) with a metric Φ_m, the real quadratic 2-step nilpotent Lie algebras (up to isometries) are of the form $(\mathbb{R}^m \oplus \mathfrak{n}(\mathfrak{v}, \rho), \Phi_m \perp \varphi_{\mathfrak{v}}^1)$. In [Ovando, 2007b, Theorem 3.6], she describes the isomorphisms between 2-step nilpotent real quadratic algebras. We will see that these results are also valid for arbitrary fields of characteristic zero.

3.2.2 Theoretical support

The existence of a quadratic structure on a Lie algebra imposes strong conditions on its algebraic structure. For real 2-step nilpotent Lie algebras this structure has been completely described by [Ovando, 2007b] by means of the construction of the *modified cotangent of a real vector space* $\mathfrak{v}$. According to Example 3.2.1, three ingredients are needed for this construction:

- a metric vector space $(\mathfrak{v}, \langle \cdot, \cdot \rangle)$,
- an injective linear map $\rho : \mathfrak{v} \to \mathfrak{so}(\mathfrak{v}, \langle \cdot, \cdot \rangle)$ satisfying $\rho(a)(a) = 0$ and
- the canonical hyperbolic metric φ on the vector space $\mathfrak{v} \oplus \mathfrak{v}^\star$.

In this section, we will rewrite this construction throughout the notion of *d-quadratic family of matrices* which let us to present the modified cotangent in terms of basis, structure constants and canonical metabolic bilinear forms. In this way, we will escape from the initial metric $\langle \cdot, \cdot \rangle$ and the map ρ (both are integrated in the notion of a d-quadratic family of matrices) and we will open the construction to any field of characteristic zero, not only the reals. Although the notion of modified cotangent is deeper from an algebraic and geometric point of view, this new approach is available for generic fields and more useful for our computational purposes. We will recover and extend results on structure and existence established in [Ovando, 2007b]. And, by using techniques introduced in [Benito et al., 2017], we will give a condition of isometric isomorphisms between quadratic 2-step Lie algebras.

For a given quadratic and reduced 2-step nilpotent Lie algebra $(\mathfrak{n}, \varphi)$ we have that $Z(\mathfrak{n}) = \mathfrak{n}^2 = (\mathfrak{n}^2)^\perp$ because of equality (2.13). Therefore $\mathfrak{n}^2$ is an isotropic subspace of $\mathfrak{n}$ of dimension $\frac{1}{2}\dim \mathfrak{n}$. It follows that any non-degenerate invariant bilinear form attached to a quadratic and reduced 2-step nilpotent Lie algebra is metabolic.

Let $(\mathfrak{n}, \varphi)$ be a quadratic and reduced 2-step nilpotent Lie algebra of type d over the field $\mathbb{F}$. Since φ is metabolic, for a given basis $\{z_1, z_2, \dots, z_d\}$ of $\mathfrak{n}^2$, there exists a set $\{v_1, v_2, \dots, v_d\}$ of orthogonal vectors such that $\varphi(z_i, v_j) = \delta_{ij}$. Therefore, $\mathcal{B} = \{v_1, v_2, \dots, v_d, z_1, z_2, \dots, z_d\}$ is an ordered basis of $\mathfrak{n}$ and φ, in the basis $\mathcal{B}$, is determined by the matrix

$$B(d) = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ I_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{3.5}$$

Here $0_{d\times d}$ denotes the null matrix of dimension $d \times d$ and $I_{d\times d}$ refers to the identity matrix of order d. In the sequel we will also use the following notations and facts. The decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}^2$, $\mathfrak{v} = \operatorname{span}\langle v_1, v_2, \dots, v_d\rangle$ induces a natural $\mathbb{Z}_2$-graduation in the algebra $\mathfrak{so}(\mathfrak{n}, \varphi)$ by declaring as even part $\mathfrak{so}_0(\mathfrak{n}, \varphi) = \{h \in \mathfrak{so}(\mathfrak{n}, \varphi) : h(\mathfrak{v}) \subseteq \mathfrak{v}, h(\mathfrak{n}^2) \subseteq \mathfrak{n}^2\}$ and $\mathfrak{so}_1(\mathfrak{n}, \varphi) = \{h \in \mathfrak{so}(\mathfrak{n}, \varphi) : h(\mathfrak{v}) \subseteq \mathfrak{n}^2, h(\mathfrak{n}^2) \subseteq \mathfrak{v}\}$ as odd part. From the description of $\mathfrak{so}(\mathfrak{n}, \varphi)$ given in expression (2.8) and the previous decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}^2$, we have

$$\begin{aligned} \mathfrak{so}(\mathfrak{n}, \varphi) &= \operatorname{span}\langle \varphi_{x,y} = \varphi(x, \cdot)y - \varphi(y, \cdot)x : x, y \in \mathfrak{n}\rangle \\ &= \operatorname{span}\langle \varphi_{x,y} : x, y \in \mathfrak{v} \cup \mathfrak{n}^2\rangle. \end{aligned}$$

Since $\varphi_{x,y} = -\varphi_{y,x}$ and $\mathfrak{v}$ and $\mathfrak{n}^2$ are isotropic, we get

$$\mathfrak{so}(\mathfrak{n}, \varphi) = \varphi_{\mathfrak{v},\mathfrak{v}} \oplus \varphi_{\mathfrak{v},\mathfrak{n}^2} \oplus \varphi_{\mathfrak{n}^2,\mathfrak{n}^2} = \mathfrak{so}_0(\mathfrak{n}, \varphi) \oplus \mathfrak{so}_1(\mathfrak{n}, \varphi), \tag{3.6}$$

with

$$\mathfrak{so}_0(\mathfrak{n},\varphi)=\varphi_{\mathfrak{v},\mathfrak{n}^2} \quad \text{and} \quad \mathfrak{so}_1(\mathfrak{n},\varphi)=\varphi_{\mathfrak{v},\mathfrak{v}}\oplus\varphi_{\mathfrak{n}^2,\mathfrak{n}^2},$$

where $\varphi_{\mathfrak{p},\mathfrak{q}} = \operatorname{span}\langle\varphi_{x,y} : x \in \mathfrak{p}, y \in \mathfrak{q}\rangle$. Hence, any $h \in \mathfrak{so}(\mathfrak{n},\varphi)$ can be decomposed as $h = h_0 + h_1$ where $h_i \in \mathfrak{so}_i(\mathfrak{n},\varphi)$. This lets us describe and manage the orthogonal map h and any Lie product $[h,h'] = hh' - h'h$ in an easier way.

On the other hand, φ is related to some invariant form of the 2-step free nilpotent algebra $\mathfrak{n}_{d,2}$ because of Lemma 3.1.17. We will use as model of $\mathfrak{n}_{d,2}$:

$$\mathfrak{n}_{d,2}(\mathbb{F})=\mathbb{F}^d\oplus\Lambda^2\mathbb{F}^d, \quad [a+b\wedge c, a'+b'\wedge c']=a\wedge a',$$

which is given in [Gauger, 1973]. If we take a basis $\{v_1,\dots,v_d\}$ of $\mathbb{F}^d$, the set $\{v_i,[v_i,v_j]=v_i\wedge v_j : 1\le i\le d, i<j\}$ is a basis of $\mathfrak{n}_{d,2}(\mathbb{F})$ called *Hall basis* (see Section 6.2.1 for a definition, algorithm and examples).

Definition 3.2.3. For any $d \geq 2$, a family $\{M_1,\dots,M_d\}$ of square matrices of order d with entries in $\mathbb{F}$ is called *d-quadratic* if the following properties are satisfied:

(1) Every matrix M_i is skewsymmetric ($M_i^t = -M_i$, where M_i^t is the transpose matrix of M_i).

(2) The j^{th} column of M_i is the additive inverse of the i^{th} column of M_j.

Let $M_{i<j}$ denote the submatrix of M_i given by the set of all j^{th} columns of M_i such that $i<j$. In case the matrix

$$\mathcal{F}(M_1,\dots,M_d)=[M_{1<j}M_{2<j}\dots M_{d-1<j}], \tag{3.7}$$

of order $d\times\frac{d(d-1)}{2}$, has rank d we say this is a *non-degenerate d-quadratic family*.

Remark 3.2.1. From item (2) we can observe the i^{th} column of every M_i is null.

The next theorem gives us an alternative formulation (and proof) of the structure result established in Theorem 3.2 in [Ovando, 2007b] for quadratic and reduced 2-step algebras. The current formulation given here is the key of the computational algorithms we will develop in the next Section. The algorithms are based on a description of any quadratic and reduced 2-step algebra through basis and structure constants.

Theorem 3.2.2. *For any $d \geq 2$, the following assertions are equivalent:*

(a) $(\mathfrak{n}, \varphi)$ *is a quadratic and reduced 2-step nilpotent Lie algebra of type d.*

(b) *There exists a basis $\mathcal{B} = \{v_1, \ldots, v_d, z_1, \ldots, z_d\}$ of $\mathfrak{n}$ for which the structure constants are determined by a family of non-degenerate d-quadratic matrices $\{M_i = (m_{ijk}) : 1 \leq i \leq d\}$ in the following way: $[z_i, \mathfrak{n}] = 0$ and $[v_i, v_j] = \sum_{k=1}^{d} m_{ijk} z_k$.*

In this case, the bilinear form φ is metabolic and, in the ordered basis $\mathcal{B}$, φ is given by the canonical matrix $B(d)$ described in equation (3.5).

Proof. First, assume that $(\mathfrak{n}, \varphi)$ is quadratic and reduced. From equality (2.13) we get $Z(\mathfrak{n}) = \mathfrak{n}^2 = (\mathfrak{n}^2)^{\perp}$. Hence, we can take a minimal set of generators $\{v_1, \ldots, v_d\}$ of $\mathfrak{n}$ and a basis $\{z_1, \ldots, z_d\}$ of $\mathfrak{n}^2$ such that $\varphi(v_i, z_j) = \delta_{ij}$. Therefore, $\mathfrak{n}$ decomposes as the direct sum,

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{n}^2,$$

where $\mathfrak{v}$ denotes the linear span of $\{v_1, \ldots, v_d\}$ and the matrix of φ attached to the basis $\mathcal{B} = \{v_1, \ldots, v_d, z_1, \ldots, z_d\}$ is the matrix $B(d)$ described in equation (3.5). Now, the inner derivation algebra of $\mathfrak{n}$, $\operatorname{Inner} \mathfrak{n} \cong \mathfrak{n}/Z(\mathfrak{n})$ according to equation (2.5), is generated by the set $\{\operatorname{ad} v_i : 1 \leq i \leq n\}$. Since $\mathfrak{n}^2 \subseteq Z(\mathfrak{n})$, this algebra is abelian and its dimension is d because of $\mathfrak{v} \cap Z(\mathfrak{n}) = 0$ (the map $\mathfrak{v} \to \operatorname{Inner} \mathfrak{n}$, $v \mapsto \operatorname{ad} v$ is one-one).

From the inclusion given in expression (2.11), the form φ is invariant if and only if $\operatorname{Inner} \mathfrak{n}$ is a subalgebra of the orthogonal Lie algebra $\mathfrak{so}(\mathfrak{n}, \varphi)$. As it is explained in the decomposition (3.6), the algebra $\mathfrak{so}(\mathfrak{n}, \varphi)$ is $\mathbb{Z}_2$-graded with even and odd parts, $\mathfrak{so}_0(\mathfrak{n}, \varphi) = \varphi_{\mathfrak{v}, \mathfrak{n}^2}$ and $\mathfrak{so}_1(\mathfrak{n}, \varphi) = \varphi_{\mathfrak{v}, \mathfrak{v}} \oplus \varphi_{\mathfrak{n}^2, \mathfrak{n}^2}$ respectively. Therefore, any $h \in \operatorname{Inner} \mathfrak{n}$ decomposes as $h = h_0 + h_1$ where $h_i \in \mathfrak{so}_i(\mathfrak{n}, \varphi)$. Since h is an inner derivation and $\mathfrak{n}$ is 2-nilpotent we have

$$h(\mathfrak{v}) \subseteq \mathfrak{n}^2, \text{ and } h(\mathfrak{n}^2) = 0.$$

Hence $h_0 = 0$ and the matrix representation of $h = h_1$ with respect to the basis $\mathcal{B}$ is

$$\hat{M}(h) = \begin{pmatrix} 0_{d \times d} & 0_{d \times d} \\ M & 0_{d \times d} \end{pmatrix}. \tag{3.8}$$

Now, $h \in \mathfrak{so}(\mathfrak{n}, \varphi)$ if and only if

$$\varphi(h(x), y) + \varphi(x, h(y)) = 0 \Longleftrightarrow \hat{M}(h)^t B(d) + B(d)\hat{A}(h) = 0. \tag{3.9}$$

Note that the assertion (3.9) is equivalent to $M^t = -M$. Let M_i denote the skewsymmetric matrix of order $d \times d$ attached to $\hat{M}(\operatorname{ad} v_i)$ for $i = 1, \dots, d$. We claim that $\{M_1, \dots, M_d\}$ is a family of non-degenerate d-quadratic matrices. Conditions (1) and (2) in Definition 3.2.3 follow from assertion (3.9) and the anticommutativity of the Lie product of $\mathfrak{n}$. Non-degenerate condition in Definition 3.2.3 follows from the basic fact that $\operatorname{span}\langle h(v_i) : i = 1, \dots, d, h \in \operatorname{Inner} \mathfrak{n}\rangle = \mathfrak{n}^2$. This implies that the rank of $\mathcal{F}(M_1, \dots, M_d)$ is exactly d.

For the converse, we note that the Lie bracket given in item (b) from Theorem 3.2.2 is anticommutative by using property (2) in Definition 3.2.3 of d-quadratic matrices. The triple product $[[x, y], z]$ is null in $\mathfrak{n}$, so, the Jacobi identity is trivial. Therefore, $(\mathfrak{n}, [\cdot, \cdot])$ is a Lie algebra. From the non-degenerate condition of a d-quadratic family, we have $0 \neq \mathfrak{n}^2 = \operatorname{span}\langle z_1, \dots, z_d\rangle$. Hence, $\mathfrak{n}$ is a 2-step nilpotent Lie algebra of type d. The structure constants m_{ijk} of $\mathfrak{n}$ show us that the matrix representation of $\hat{M}(\operatorname{ad} v_i)$ is determined by the matrix M_i. According to condition (1) in Definition 3.2.3, every matrix M_i is skewsymmetric. Hence, the non-degenerate bilinear form $\varphi(v_i, z_j) = \delta_{ij}$ and $\varphi(v_i, v_j) = \varphi(z_i, z_j) = 0$ is invariant because of the equivalence established in condition (3.9). In order to prove that $\mathfrak{n}$ is reduced, we will test that $Z(\mathfrak{n}) \cap \mathfrak{v} = 0$, where $\mathfrak{v} = \operatorname{span}\langle v_1, \dots, v_d\rangle$. Let $u = x_1 v_1 + \cdots + x_d v_d$ (x_i scalars) and assume $u \in Z(\mathfrak{n})$. Denote $u = (x_1, \dots, x_n)$ and note that $M_i u = 0$ for $i = 1, \dots, d$. Since $M_i^t = -M_i$, we have $[M_1, \dots, M_d]^t u = 0$. From the non-degenerate condition of a d-quadratic family, the rank of $[M_1, \dots, M_d]$ is d. Therefore $u = 0$ which proves that $Z(\mathfrak{n}) \cap \mathfrak{v} = 0$. □

From now on, we denote as $(\mathfrak{n}(M_1, \dots, M_d), \varphi)$ the quadratic Lie algebra attached to the family of d-quadratic matrices $\{M_1, \dots, M_d\}$ as it is described in item (b) of Theorem 3.2.2. For any $n \geq 1$, we also denote $(\mathbb{F}^n, \Phi_n)$ the quadratic abelian Lie algebra of type n where Φ_n is the non-degenerate bilinear form given by $\Phi_n(u, v) = \sum_{i=1}^n u_i v_i$ (standard inner product).

Remark 3.2.3. Summarizing Theorem 3.2.2, the Lie algebra $(\mathfrak{n}(M_1, \dots, M_d), \varphi)$ satisfies:

- $\mathfrak{n}(M_1, \dots, M_d) = \operatorname{span}\langle v_1, \dots, v_d, z_1, \dots z_d\rangle$ has type d and dimension $2d$.

- The invariant bilinear form φ is given by the formulas $\varphi(v_i, v_j) = 0 = \varphi(z_i, z_j)$ and $\varphi(v_i, z_j) = \delta_{ij}$. The matrix of φ is $B(d)$ as described in equation (3.5).

- $\mathcal{F}(M_1, \dots, M_d)$ determines completely the Lie product of $\mathfrak{n}(M_1, \dots, M_d)$ from the matrix equation:

$$\begin{aligned}([v_1, v_2], \dots, [v_1, v_d], [v_2, v_3], \dots, [v_2, v_d], [v_3, v_4], \dots, [v_{d-1}, v_d])\\ = (z_1, \dots, z_d) \cdot \mathcal{F}(M_1, \dots, M_d).\end{aligned}$$

In the following examples we give the whole set of non-degenerate d-quadratic families of matrices for $2 \leq d \leq 4$. When $d = 2, 4$ there is none d-quadratic family, while for $d = 3$ there is a one-parametric family. For other arbitrary values of d, it is possible to get examples of this type of families using the computational methods found next in Section 3.3. The quadratic 2-step Lie algebras of type 3 attached to 3-quadratic families of Example 3.2.3 are given in [Ovando, 2007b], [del Barco and Ovando, 2012] and [Benito et al., 2017].

Example 3.2.2. The 2×2 skewsymmetric matrices are of the form

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad a \in \mathbb{F}.$$

So, there are no non-degenerate 2-quadratic families of matrices. Hence, in characteristic zero, there are no 2-step quadratic Lie algebras of type 2.

Example 3.2.3. The non-degenerate 3-quadratic families of matrices are of the form $\{M_1, M_2, M_3\}$ where

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & -a \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and $0 \neq a \in \mathbb{F}$. As there are no 2-step quadratic nilpotent Lie algebras of type 2, any quadratic 2-step of type 3 is reduced. Therefore the class of quadratic 2-step Lie algebras of type 3 are of the form $(\mathfrak{n}(M_1, M_2, M_3), \varphi)$. From Remark 3.2.3, we obtain the one-parametric family of Lie algebras $(\mathfrak{n}_a, \varphi)$ described by the basis $\{v_1, v_2, v_3, z_1, z_2, z_3\}$, the bilinear form φ given by the matrix $B(3)$ and non-zero products $[v_1, v_2] = -az_3 = -[v_2, v_1]$, $[v_1, v_3] = az_2 =$

$-[v_3, v_1]$ and $[v_2, v_3] = -az_1 = -[v_3, v_2]$ obtained from

$$([v_1, v_2], [v_1, v_3], [v_2, v_3]) = (z_1, z_2, z_3) \cdot \begin{pmatrix} 0 & 0 & -a \\ 0 & a & 0 \\ -a & 0 & 0 \end{pmatrix}.$$

All of them are isometrically isomorphic. So, up to isometries, we get that $(\mathfrak{n}_1, \varphi)$ is the unique quadratic 2-step nilpotent Lie algebra of type 3. Following [Benito et al., 2017], the algebra $(\mathfrak{n}_1, \varphi)$ is just the quadratic Lie algebra $(\mathfrak{n}_{3,2}(\mathbb{F}), \Psi)$, where $\mathfrak{n}_{3,2}(\mathbb{F}) = \mathbb{F}^3 \oplus \Lambda^2\mathbb{F}^3$ and the matrix of Ψ with respect to the basis $\{e_i, e_i \wedge e_j\}$ is (here $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ *is the classic canonical basis of* $\mathbb{F}^3$):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example 3.2.4. For $d = 4$, the existence of quadratic and reduced 2-step Lie algebras is equivalent to the existence of a 4×6 matrix of rank 4 with the following shape:

$$\mathcal{F}(M_1, M_2, M_3, M_4) = \begin{pmatrix} 0 & 0 & 0 & -a & -b & -c \\ 0 & a & b & 0 & 0 & -d \\ -a & 0 & c & 0 & d & 0 \\ -b & -c & 0 & -d & 0 & 0 \end{pmatrix}.$$

Any minor of order 4 of this type of matrices is null. So,

$$\operatorname{rank} \mathcal{F}(M_1, M_2, M_3, M_4) \leq 3,$$

and, therefore, in characteristic zero there are no 2-step reduced quadratic Lie algebras of type 4. Up to isometric isomorphisms the non-reduced are of the form $(\mathfrak{n}_{3,2}(\mathbb{F}) \oplus \mathbb{F}, \Psi \perp \Phi_1)$.

From now on, our goal is to investigate, the existence problem of quadratic Lie algebras $(\mathfrak{n}(M_1, \ldots, M_d), \varphi)$ for any arbitrary $d \geq 5$. We will also study the problem of isometric isomorphisms. The first problem is a simple exercise of linear algebra. For the second one, we will use the functorial relation introduced in Section 3.1.3. Our next result ensures the existence of quadratic and reduced 2-step algebras of arbitrary type d different from $1, 2$ and 4, and over any field $\mathbb{F}$ of characteristic zero. This result has also been established in [Ovando, 2007b, Proposition 3.3]. We provide here an alternative proof.

Proposition 3.2.4. *For any $d \neq 1, 2, 4$, there exist non-degenerate d-quadratic families of matrices.*

Proof. Assume first d is odd and $d > 3$ and choose a $d \times d$ skewsymmetric matrix M_1 of rank $d-1$, with its first row and its first column being null. Then, $U_1 = \{u \in \mathbb{F}^d : uM_1 = M_1 u^t = 0\} = \mathbb{F} \cdot e_1$ where $e_1 = (1, 0, \cdots, 0)$. As $d > 1$, we can take a second skewsymmetric matrix M_2 such that its first row is the additive inverse of the second row of M_1. Clearly, $e_1 \notin \{u \in \mathbb{F}^d : uM_2 = 0\}$. Therefore, $\{u \in \mathbb{F}^d : u[M_1, M_2] = 0\} = 0$. This implies that $[M_1, M_2]$ is a matrix of maximal rank d. We can now extend the set $\{M_1, M_2\}$ to a non-degenerate d-quadratic family $\{M_1, M_2, \ldots, M_d\}$ in an easy (and not unique) way.

If d is even and $d > 4$, take a $d \times d$ skewsymmetric matrix M_1 with rows $r_1 = r_d = (0, \ldots, 0)$ and rows $r_2, \ldots, r_{d-1}$ being a set of linearly independent row vectors of $\mathbb{F}^d$. Therefore, the rank of M_1 is $d - 2$. It is clear that $U_1 = \{u \in \mathbb{F}^d : uM_1 = 0\} = \mathbb{F} \cdot e_1 \oplus \mathbb{F} \cdot e_d$. Now, choose a second matrix M_2 with first row $-r_2 = -(0, 0, -s_3, \ldots, -s_d) \neq 0$ and its second row being null. Since $d > 4$, we can take the 3^{rd} and 4^{th} rows of M_2 as $p_3 = (s_3, 0, 0, 0, \ldots, 1)$ and $p_4 = (s_4, 0, 0, 0, 0, \ldots, a)$ where $a \neq s_4/s_3$ if $s_3 \neq 0$ or $a = 0$ otherwise. Finally we add j^{th} rows for $j = 5, \ldots, d$ just to get M_2 as a skewsymmetric matrix. For the pair M_1, M_2, we claim that $U_2 = \{u \in \mathbb{F}^d : u[M_1\, M_2] = 0\} = 0$.

Suppose there is $0 \neq u \in U_2$. Since $u \in U_1$ we can write $u = t \cdot e_1 + s \cdot e_d$ for some $(0, 0) \neq (t, s) \in \mathbb{F} \times \mathbb{F}$. We note that $u \neq e_1$ because of the first row of M_2 is not null. Rescaling if necessary, we can assume $u = t_0 \cdot e_1 + e_d$. In particular, we have that

$$u \cdot p_3^t = t_0 s_3 + 1 = 0 = u \cdot p_4^t = t_0 s_4 + a.$$

Hence, $s_3 \neq 0$ and $t_0 = -\dfrac{1}{s_3}$ which implies $a = \dfrac{s_4}{s_3}$, a contradiction. This proves our claim. Now, $U_2 = 0$ implies that $[M_1\, M_2]$ is a matrix of maximal rank d and, reasoning as in the odd case, we get our result. $\square$

An element $\psi \in \mathrm{Sym}_0(d,2)$ is called reduced if and only if $Q_{d,2}(\psi) = \dfrac{\mathfrak{n}_{d,2}}{\ker \psi}$ is a reduced Lie algebra. This assertion is equivalent to

$$\mathfrak{n}_{d,2}^2 = \{x \in \mathfrak{n}_{d,2} : [x, \mathfrak{n}_{d,2}] \subseteq \ker \psi\}.$$

For any d-quadratic family $\{M_1, \ldots, M_d\}$, we denote

$$\mathcal{Q}(M_1, \ldots, M_d) = \begin{pmatrix} 0_{d\times d} & \mathcal{F}(M_1, \ldots, M_d) \\ \mathcal{F}(M_1, \ldots, M_d)^t & 0_{\frac{d(d-1)}{2}\times\frac{d(d-1)}{2}} \end{pmatrix}, \tag{3.10}$$

where $\mathcal{F}(M_1, \ldots, M_d)$ is defined as in equation (3.7) from Definition 3.2.3).

Following Proposition 3 in [Satô, 1971], for a given $\{v_1, \ldots, v_d\}$ basis of a d-dimensional vector space $\mathfrak{v}$, any linear map $f \colon \mathfrak{v} \to \mathfrak{n}_{d,2}(\mathfrak{v}) = \mathfrak{v} \oplus \Lambda^2\mathfrak{v}$, for which the vectors $f(v_1), \ldots, f(v_d)$ are linearly independent, extends to the automorphism τ_f of $\mathfrak{n}_{d,2}$ by defining

$$\tau_f([v_i, v_j]) = \tau_f(v_i \wedge v_j) = f(v_i) \wedge f(v_j) = [f(v_i), f(v_j)]. \tag{3.11}$$

Even more, any automorphism of $\mathfrak{n}_{d,2}$ is of this form. Hence, in the Hall basis $\mathcal{H}_{\mathfrak{v}} = \{v_i, v_i \wedge v_j : i = 1, \ldots, d, i < j\}$, the automorphisms of $\mathfrak{n}_{d,2}(\mathfrak{v})$ are represented by matrices of the form

$$\tau(Q, X) = \begin{pmatrix} Q & 0_{d\times\frac{d(d-1)}{2}} \\ X & \hat{Q} \end{pmatrix},$$

where X is a any matrix of order $\frac{d(d-1)}{2} \times d$, Q is a regular matrix of order $d \times d$, and $\hat{Q}$ is a matrix completely determined from Q by the rule (3.11). In case $Q = (b_{ij})$, from a straightforward computation we get that

$$\tau_f(v_i \wedge v_j) = \sum_{1\le r<s\le n} \det \begin{pmatrix} b_{ri} & b_{rj} \\ b_{si} & b_{sj} \end{pmatrix} v_r \wedge v_s. \tag{3.12}$$

The formula in equation (3.12) provides the entries of the matrix $\hat{Q}$ in terms of the entries of the matrix Q.

Theorem 3.2.5. *Let $\{M_1, \ldots, M_d\}$ and $\{N_1, \ldots, N_d\}$ be two families of non-degenerate d-quadratic matrices and let $(\mathfrak{n}(M_1, \ldots, M_d), \varphi)$ and $(\mathfrak{n}(N_1, \ldots, N_d), \psi)$ be the quadratic Lie algebras attached to them as it is described in Theorem 3.2.2. Then, the Lie algebras $(\mathfrak{n}(M_1, \ldots, M_d), \varphi)$ and $(\mathfrak{n}(N_1, \ldots, N_d), \psi)$ are isometrically isomorphic if and only if there exists a regular $d \times d$ matrix Q such that*

$$B(N_1, \ldots, N_d) = Q^t \mathcal{F}(M_1, \ldots, M_d)\hat{Q},$$

where $\hat{Q}$ is given in terms of $Q = (b_{ij})$ through the formula in equation (3.12) *and* $\mathcal{F}(N_1, \dots, N_d)$*,* $\mathcal{F}(M_1, \dots, M_d)$ *are matrices of order* $d \times \frac{d(d-1)}{2}$ *which are described in the general form* $(R = M$ *or* $N)$

$$\mathcal{F}(R_1, \dots, R_d) = [R_{1<j} R_{2<j} \dots R_{d-1<j}],$$

where $R_{i<j}$ *denotes the submatrix of* R_i *given by the set of all* j^{th} *columns of* R_i *such that* $i < j$*.*

Proof. Along the proof we will denote $\mathfrak{n}(M_1, \dots, M_d)$ and $\mathfrak{n}(N_1, \dots, N_d)$ as $\mathfrak{n}_M$ and $\mathfrak{n}_N$ respectively. According to Lemma 3.1.17, $(\mathfrak{n}_M, \varphi)$ is isometrically isomorphic to $(\mathfrak{n}_{d,2}/\ker\hat{\varphi}, \hat{\varphi})$ for some bilinear form $\hat{\varphi} \in \mathrm{Sym}_0(d, 2)$ which is reduced and invariant. Then, from Theorem 3.2.2, there exists a minimal generator set $\mathfrak{u} = \{u_1, \dots, u_d\}$ of $\mathfrak{n}_{d,2}$ of isotropic vectors and an isotropic central ideal $\mathfrak{z} = \mathrm{span}\langle z_1, \dots, z_d\rangle$ such that

$$\mathfrak{n}_{d,2} = \mathfrak{u} \oplus \wedge^2\mathfrak{u} = \mathrm{span}\langle u_1, \dots, u_d\rangle \oplus (\mathfrak{z} \oplus \ker\hat{\varphi}),$$

$$[u_i, u_j] = u_i \wedge u_j \equiv \sum_{k=1}^{d} m_{ijk} z_k \pmod{\ker\hat{\varphi}},$$

$$\varphi(u_i, z_j) = \delta_{ij},$$

and the set $\{m_{ijk}\}$ of structure constants is determined by the non-degenerate d-quadratic family $\{M_1, \dots, M_d\}$. So, if we fixed a basis $\mathcal{B}$ of $\ker\hat{\varphi} = \mathfrak{n}_{d,2}^{\perp}$, the matrix of $\hat{\varphi}$ attached to $\mathcal{B}' = \{u_1, \dots, u_d, z_1, \dots, z_d\} \cup \mathcal{B}$ is

$$\mathcal{M} = \begin{pmatrix} 0_{d\times d} & I_{d\times d} & 0_{d\times \frac{d(d-3)}{2}} \\ I_{d\times d} & 0_{d\times d} & 0_{d\times \frac{d(d-3)}{2}} \\ 0_{\frac{d(d-3)}{2}\times d} & 0_{\frac{d(d-3)}{2}\times d} & 0_{\frac{d(d-3)}{2}\times \frac{d(d-3)}{2}} \end{pmatrix}.$$

For every $i = 1, \dots, d$, the inner derivation $\mathrm{ad}\, u_i$ is represented by a matrix (respect to the basis $\mathcal{B}'$) of the form

$$\begin{pmatrix} 0_{d\times d} & 0_{d\times d} & 0 \\ M_i & 0_{d\times d} & 0 \\ C_i & 0 & 0 \end{pmatrix}.$$

Let $C_{i<j}$ be the submatrix of C_i given by the set of all j^{th} columns of C_i such that $i < j$. Denote as D the matrix $\mathcal{F}(C_1, \dots, C_d) = [C_{1<j} C_{2<j} \dots C_{d-1<j}]$,

and let P be the matrix

$$P = \begin{pmatrix} I_{d\times d} & 0_{d\times \frac{d(d-1)}{2}} \\ 0_{d\times d} & \mathcal{F}(M_1,\dots,M_d) \\ 0_{\frac{d(d-3)}{2}\times d} & D \end{pmatrix}$$

Clearly P is a regular matrix because of $\mathfrak{n}_{d,2}^2 = \operatorname{span}\langle h(u_i) : h \in \operatorname{Inner} \mathfrak{n}_{d,2}\rangle$. In fact, P provides the change of basis from the Hall basis $\mathcal{H}_\mathfrak{u} = \{u_i, u_i \wedge u_j\}$ to $\mathcal{B}'$. Therefore, the matrix of φ attached to $\mathcal{H}_\mathfrak{u}$ is $P^t \mathcal{M} P = \mathcal{Q}(M_1,\dots,M_d)$ just as defined in (3.10). In an analogous way, the Lie algebra $(\mathfrak{n}_N, \psi)$ is isometrically isomorphic to $(\mathfrak{n}_{d,2}/\ker \hat{\psi}, \hat{\psi})$ for some reduced bilinear form $\hat{\psi} \in \operatorname{Sym}_0(d,2)$. From this isomorphism we get a Hall basis $\mathcal{H}_\mathfrak{v} = \{v_i, v_i \wedge v_j\}$ and a regular matrix R such that $R^t \mathcal{M} R = \mathcal{Q}(N_1,\dots,N_d)$ is the matrix of ψ attached to $\mathcal{H}_\mathfrak{v}$. The change of basis from $\mathcal{H}_\mathfrak{u}$ to $\mathcal{H}_\mathfrak{v}$ provides the automorphism $\tau(Q,X)$ of $\mathfrak{n}_{d,2}$ and the matrix that represents φ in the basis $\mathcal{H}_\mathfrak{v}$ is

$$\tau(Q,X)^t \mathcal{Q}(M_1,\dots,M_d)\tau(Q,X) = \begin{pmatrix} X^t \mathcal{F}_M^t Q + Q^t \mathcal{F}_M X & Q^t \mathcal{F}_M \hat{Q} \\ \hat{Q}^t \mathcal{F}_M^t Q & 0_{\frac{d(d-1)}{2}\times\frac{d(d-1)}{2}} \end{pmatrix},$$

where $\mathcal{F}_M = \mathcal{F}(M_1,\dots,M_d)$. Now, $(\mathfrak{n}_N,\psi)$ and $(\mathfrak{n}_M,\varphi)$ are isometrically isomorphic if and only if $\hat{\varphi}$ and $\hat{\psi}$ are isomorphic in the category $\operatorname{Sym}_0(d,2)$. From Lemma 3.1.19, the latter assertion is equivalent to the existence of an isometric automorphism $\theta\colon (\mathfrak{n}_{d,2},\hat{\psi}) \to (\mathfrak{n}_{d,2},\hat{\varphi})$. Hence, $\hat{\psi} = \hat{\varphi}_\theta$. In the Hall basis $\mathcal{H}_\mathfrak{v}$, the automorphism θ is of the form $\tau(S,Y)$. Then,

$$\tau(S,Y)^t \tau(Q,X)^t \mathcal{Q}(M_1,\dots,M_d)\tau(Q,X)\tau(S,Y) = \mathcal{Q}(N_1,\dots,N_d).$$

and

$$\tau(QS, XS + \hat{Q}Y)^t \mathcal{Q}(M_1,\dots,M_d)\tau(QS, XS+\hat{Q}Y) = \mathcal{Q}(N_1,\dots,N_d).$$

This implies $(QS)^t \mathcal{F}_M \widehat{QS} = \mathcal{F}_N$ which proves the result. □

Finally, the next corollary describes the class of non-reduced (i.e. not necessarily reduced) quadratic 2-step Lie algebras (see Theorem 3.2 in [Ovando, 2007b]; the corank here is d_2).

Corollary 3.2.6. *For every $d \geq 3$, there exist quadratic 2-step Lie algebras of type d over any field of characteristic zero. Up to isometric isomorphisms the algebras in this class are of the form $(\mathfrak{n}(M_1,\dots,M_{d_1}) \oplus \mathfrak{a}, \varphi \perp \phi)$ where $d = d_1 + d_2$, $4 \neq d_1 \geq 3$, $\{M_1,\dots,M_{d_1}\}$ is a non-degenerate d_1-quadratic family of matrices and $(\mathfrak{a},\phi)$, is a quadratic and abelian Lie algebra of dimension $d_2 \geq 0$.*

Proof. The result follows from Theorem 2.2.9, Theorem 3.2.2 and Proposition 3.2.4. □

Theorem 3.2.5 and Corollary 3.2.6 extend to arbitrary fields of characteristic zero Theorem 3.6 and Theorem 3.2 respectively in [Ovando, 2007b]. Our approach for these results is more algorithmic and let us develop computational methods in the following section.

3.3 Computational algorithms for quadratic families

Up to this point, we have introduce a new approach for constructing reduced 2-step quadratic nilpotent Lie algebras of type $d \geq 3$. This method, which relies in the theoretical results developed in previous Section 3, Theorem 3.2.2 and Corollary 3.2.6, can be computationally addressed. The theorem itself provides the algorithms and the corollary assures that they will be successful. According to Definition 3.2.3, we will design several algorithms in order to build d-quadratic families of matrices (Algorithms 1, 2 and 3) and a final algorithm to obtain the multiplication table of the quadratic Lie algebra (Algorithm 4). Along the section we will use the following notation:

- $M[a, b]$ denotes the entry in row a and column b for a given matrix M;
- $M[a :: b, c :: d]$ denotes the submatrix obtained from A by using the whole set of rows from a up to b and the set of columns from c up to d, both included.

Moreover we consider the first row/column is the one numbered as 1.

The matrices of a d-quadratic family $\{M_1, \dots, M_d\}$ are skew-symmetric. From Theorem 3.2.2 and its proof, each matrix M_i determines an inner derivation $\operatorname{ad} v_i$ (adjoint derivation) of the quadratic Lie algebra $\mathfrak{n}(M_1, \dots, M_d)$. The set $\{v_1, \dots, v_d\}$ is a minimal generator set of

$$\mathfrak{n}(M_1, \dots, M_d) = \operatorname{span}\langle v_1, \dots, v_d, z_1, \dots, z_d\rangle = \operatorname{span}\langle v_i, [v_i, v_j]\rangle,$$

and the adjoint $\operatorname{ad} v_i$ is represented by the matrix $\hat{M}(\operatorname{ad} v_i)$ as it is described by the matrix in (3.8). The computational procedure to obtain quadratic reduced Lie algebras splits into three steps:

- First, we need to build the skew-symmetric family of matrices $M_1, \ldots, M_d$ satisfying conditions (1) and (2) of Definition 3.2.3 and the matrix $\mathcal{F}(M_1, \ldots, M_d) = [M_{1<j} M_{2<j} \ldots M_{d-1<j}]$ associated to this family.

- Secondly, we need to verify that the rank of $\mathcal{F}(M_1, \ldots, M_d)$ is maximal, so the rank must be d, which is precisely the non-degenerate condition.

Any rank d matrix $\mathcal{F}(M_1, \ldots, M_d)$ gives the non-degenerate d-quadratic family $\{M_1, \ldots, M_d\}$. Then:

- Finally we build the multiplication table of the quadratic algebra attached to $\{M_1, \ldots, M_d\}$.

The algebras will be introduced using the basis $\{v_1, \ldots, v_d, z_1, \ldots z_d\}$. In all cases, and according to Theorem 3.2.2 and Remark 3.2.3, the invariant bilinear form is the canonical metabolic form φ defined by the formulas:

$$\varphi(v_i, v_j) = \varphi(z_i, z_j) = 0 \quad \text{and} \quad \varphi(v_i, z_j) = \delta_{ij}.$$

3.3.1 General algorithms for quadratic 2-step Lie algebras

Algorithm 1 $\mathrm{Skewsymmetric}(d, s)$ algorithm

Input: A natural d indicating the dimension of the square matrix to generate.
Input: An integer s indicating the initial subindex of the variables to use.
Output: A skewsymmetric $d \times d$ matrix whose variables are a_s, $a_{s+1}, \ldots$

Let M be a empty $d \times d$ matrix
Let $num = s$ ▷ num keeps the current variable value
for $i = 1$ **to** $i = d$ **do**
  Let $v = (a_{num}, \ldots, a_{num+d+i-1})$ a vector
  $M[i, i+1 :: d] = v$
  $M[i+1 :: d, i] = -v$.
  $num = num + d - i$
end for
return M

Example 3.3.1. Algorithm 1, for values $d = 4$ and $s = 1$, returns the matrix,

$$\mathrm{Skewsymmetric}(4,1) = \begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix}.$$

This is the general shape of a 4×4 skew-symmetric matrix.

To build the adjoint matrices we will need to know how many subscripts we have used before in order not to use them again. In a generic $d \times d$ matrix

$$\sum_{i=1}^{d} i = \frac{d(d+1)}{2}$$

different variables are used. So, in the m^{th} adjoint matrix, due to the relationships among them, we will have used

$$\sum_{i=d}^{m} \frac{i(i+1)}{2} = \frac{(1+m-nd)(2m+m^2+d+md+d^2)}{6}.$$

variables. We call these quantities $\mathrm{varIn}(d)$ and $\mathrm{varUntil}(d,m)$ respectively. Now, with this notation

$$\mathrm{varUntil}(d,m) = \sum_{i=d}^{m} \mathrm{varIn}(i).$$

Algorithm 2 $\mathrm{Adjoint}(d,i)$ algorithm

Input: A $d \in \mathbb{N}$ indicating the dimension of the square adjoint matrix to generate.

Input: An integer i, such that $1 \leq i \leq d$ indicating which adjoint is it, $\mathrm{ad}\, v_i$, assuming $i = 1$ is the adjoint associated to the first element in the basis, and $i = d$ the one related to the last one.

Output: The adjoint matrix associated to the element v_i of the chosen basis.

Let M_i be a empty $d \times d$ matrix
for $j = 1$ **to** $j = i$ **do** ▷ The part deduced from previous adjoints
    $M_i[1 :: d, j] = -\mathrm{Skewsymmetric}(d,j)[1 :: d, i]$
    $M_i[j, j :: d] = -M_i[j :: d, j]$
end for
$M_i[i+1 :: d, i+1 :: d] = \mathrm{Skewsymmetric}(d-i, 1 + \mathrm{varUntil}(d-i, d-2))$
return M_i

Note that this algorithm is defined calling several times similar functions. Although it is not efficient, if it is needed an improved version can be easily develop. In order to achieve it without too many changes, it is highly recommended to store the already calculated adjoint matrices in order not to repeat operations.

Example 3.3.2. Algorithm 2, for value $d = 5$, returns the skew matrices $M_i = \mathrm{Adjoint}(5, i)$ for $i = 1, \dots, 5$,

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ 0 & -a_1 & 0 & a_4 & a_5 \\ 0 & -a_2 & -a_4 & 0 & a_6 \\ 0 & -a_3 & -a_5 & -a_6 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 0 & -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & a_7 & a_8 \\ a_2 & 0 & -a_7 & 0 & a_9 \\ a_3 & 0 & -a_8 & -a_9 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & a_1 & 0 & -a_4 & -a_5 \\ -a_1 & 0 & 0 & -a_7 & -a_8 \\ 0 & 0 & 0 & 0 & 0 \\ a_4 & a_7 & 0 & 0 & a_{10} \\ a_5 & a_8 & 0 & -a_{10} & 0 \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & a_2 & a_4 & 0 & -a_6 \\ -a_2 & 0 & a_7 & 0 & -a_9 \\ -a_4 & -a_7 & 0 & 0 & -a_{10} \\ 0 & 0 & 0 & 0 & 0 \\ a_6 & a_9 & a_{10} & 0 & 0 \end{pmatrix} \quad M_5 = \begin{pmatrix} 0 & a_3 & a_5 & a_6 & 0 \\ -a_3 & 0 & a_8 & a_9 & 0 \\ -a_5 & -a_8 & 0 & a_{10} & 0 \\ -a_6 & -a_9 & -a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Once we are able to obtain the adjoint matrices we can construct our matrix $\mathrm{RankedMatrix}(d) := \mathcal{F}(M_1, \dots, M_d)$. This matrix is the union of certain columns of the adjoint matrices: from each adjoint, considering we are at the i^{th} adjoint, we take columns $i + 1, i + 2, \dots, d$.

Algorithm 3 $\mathrm{RankedMatrix}(d)$ algorithm

Input: A natural d indicating the type of the algebra

Output: The matrix we use to check if the rank is maximal

Let C be a empty $d \times \frac{d^2-d}{2}$ matrix
for $i = 1$ **to** $i = d-1$ **do** ▷ Obtaining the possible linearly idependant columns
    $C[1 :: d, \frac{2d(i-1)-i^2+i+2}{2} :: \frac{i(2d-i-1)}{i}] = \mathrm{Adjoint}(d,i)[1 :: d, 1+i :: d]$
end for
return C

Example 3.3.3. For $d = 5$ the matrix $\mathrm{RankedMatrix}(5) = \mathcal{F}(M_1, \dots, M_d)$, for $M_i = \mathrm{Adjoint}(5,i)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & -a_7 & -a_8 & -a_9 \\ -a_1 & 0 & a_4 & a_5 & 0 & a_7 & a_8 & 0 & 0 & -a_{10} \\ -a_2 & -a_4 & 0 & a_6 & -a_7 & 0 & a_9 & 0 & a_{10} & 0 \\ -a_3 & -a_5 & -a_6 & 0 & -a_8 & -a_9 & 0 & -a_{10} & 0 & 0 \end{pmatrix}$$

$\mathrm{RankedMatrix}(5)$ is obtained from columns $2, 3, 4$ and 5 of M_1, columns $3, 4$ and 5 of M_2, columns 4 and 5 of M_3 and the final column 5 of M_4.

So far we can create $\mathcal{F}(M_1, \dots, M_d)$ matrices for every dimension d. Now we have to study for which values of the parameters a_i the rank of these matrices is d. In order to generate examples, this can be done by using any symbolic computational program (for example Mathematica or SageMath among others).

One general way to compute the rank is choosing the minors of the matrix of size $d \times d$ and getting all the conditions the variables a_i have to satisfy to have at least one minor whose determinant is not null, making the rank maximum. Therefore, the problem is simply finding the $\binom{\frac{d(d-1)}{2}}{d}$ minors and calculating their determinants. Unfortunately, we are working with factorial complexity so increasing the dimension makes the complexity extremely huge. For example, in the case $d = 5$ we already have 252 minors. Other way is using Gauss method, although the possible nullity of variables might make it difficult too. However, as explained in the proof of Proposition 3.2.4, with only the first two adjoints, $d-1+d-2=2d-3$ columns, this can be always achieved. This is much more efficient as we turn $\binom{\frac{d(d-1)}{2}}{d}$ into $\binom{2d-3}{d}$ minors. For example, in $d=5$ this is just 21. Even more, as the first adjoint reaches rank $d-1$ or $d-2$ depending whether d is odd or even respectively. This gives us another improved approach using $\binom{d-1}{\lfloor d/2 \rfloor}$ minors for the first adjoint, and $\binom{d-2}{2}$ or $\binom{d-2}{1}$ for the second. In dimension five we reduce this way the number of minors to 4 from $\binom{10}{5}=252$.

Finally, according to Theorem 3.2.2 and Remark 3.2.3, the complete table of products attached to a d-quadratic family is obtained from the expression

$$([v_1,v_2],[v_1,v_3],\ldots,[v_1,v_d],[v_2,v_3],\ldots,[v_2,v_d],\ldots,[v_{d-1},v_d]) \\ = (z_1,\ldots,z_d)\cdot \mathrm{RankedMatrix}(d).$$

So, the table generation algorithm is easy to describe:

Algorithm 4 Product(i,j,d) algorithm

Input: A natural i indicating the vector v_i where $i<j$
Input: A natural i indicating the vector v_j
Input: A natural d indicating the type of the algebra
Output: The product $[v_i,v_j]$ expressed in basis $\{z_1,\ldots,z_d\}$

1: Let Z be a $1\times d$ matrix whose entries are z_i
2: **return** $Z\cdot \mathrm{RankedMatrix}(d)\left[\frac{d(i-1)-i(i+1)+j}{2}\right]$

Therefore, assuming $i<j$ we have that

$$[v_i,v_j]=\mathrm{Product}(i,j,d).$$

Example 3.3.4. The multiplication table for a quadratic and reduced Lie algebra obtained from a 5-quadratic family of matrices $\{M_1,\ldots,M_5\}$ in the standard basis $\{v_1,\ldots,v_5,z_1,\ldots,z_5\}$ is

$$\begin{aligned}
[v_1,v_2] &= -a_1z_3-a_2z_4-a_3z_5,\\
[v_1,v_3] &= a_1z_2-a_4z_4-a_5z_5,\\
[v_1,v_4] &= a_2z_2+a_4z_3-a_6z_5,\\
[v_1,v_5] &= a_3z_2+a_5z_3+a_6z_4,\\
[v_2,v_3] &= -a_1z_1-a_7z_4-a_8z_5,\\
[v_2,v_4] &= -a_2z_1+a_7z_3-a_9z_5,
\end{aligned}$$

$$
\begin{aligned}
[v_2, v_5] &= -a_3 z_1 + a_8 z_3 + a_9 z_4,\\
[v_3, v_4] &= -a_4 z_1 - a_7 z_2 - a_{10} z_5,\\
[v_3, v_5] &= -a_5 z_1 - a_8 z_2 + a_{10} z_4,\\
[v_4, v_5] &= -a_6 z_1 - a_9 z_2 - a_{10} z_3.
\end{aligned}
$$

Example 3.3.5. In the case $d = 6$, by applying Algorithms 1, 2 and 3, we arrive at the matrix $\mathrm{RankedMatrix}(6) = \mathcal{F}(M_1, \ldots, M_6)$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & -a_8 & -a_9 & -a_{10}\\
0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & -a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16}\\
-a_1 & 0 & a_5 & a_6 & a_7 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & -a_{17} & -a_{18} & -a_{19}\\
-a_2 & -a_5 & 0 & a_8 & a_9 & -a_{11} & 0 & a_{14} & a_{15} & 0 & a_{17} & a_{18} & 0 & 0 & -a_{20}\\
-a_3 & -a_6 & -a_8 & 0 & a_{10} & -a_{12} & -a_{14} & 0 & a_{16} & -a_{17} & 0 & a_{19} & 0 & a_{20} & 0\\
-a_4 & -a_7 & -a_9 & -a_{10} & 0 & -a_{13} & -a_{15} & -a_{16} & 0 & -a_{18} & -a_{19} & 0 & -a_{20} & 0 & 0
\end{pmatrix}
$$

From previous matrix and following , Algorithm 4 gives us all the quadratic algebras (see Remark 3.2.3 for a complete description) described by the multiplication table (only those tables related to matrices $\mathrm{RankedMatrix}(6)$ such that $rank\mathrm{RankedMatrix}(6) = 6$ are valid):

$$
\begin{aligned}
[v_1, v_2] &= -a_1 z_3 - a_2 z_4 - a_3 z_5 - a_4 z_6,\\
[v_1, v_3] &= a_1 z_2 - a_5 z_4 - a_6 z_5 - a_7 z_6,\\
[v_1, v_4] &= a_2 z_2 + a_5 z_3 - a_8 z_5 - a_9 z_6,\\
[v_1, v_5] &= a_3 z_2 + a_6 z_3 + a_8 z_4 - a_{10} z_6,\\
[v_1, v_6] &= a_4 z_2 + a_7 z_3 + a_9 z_4 + a_{10} z_5,\\
[v_2, v_3] &= -a_1 z_1 - a_{11} z_4 - a_{12} z_5 - a_{13} z_6,\\
[v_2, v_4] &= -a_2 z_1 + a_{11} z_3 - a_{14} z_5 - a_{15} z_6,\\
[v_2, v_5] &= -a_3 z_1 + a_{12} z_3 + a_{14} z_4 - a_{16} z_6,\\
[v_2, v_6] &= -a_4 z_1 + a_{13} z_3 + a_{15} z_4 + a_{16} z_5,\\
[v_3, v_4] &= -a_5 z_1 - a_{11} z_2 - a_{17} z_5 - a_{18} z_6,\\
[v_3, v_5] &= -a_6 z_1 - a_1 z_2 + a_{17} z_4 - a_{19} z_6,\\
[v_3, v_6] &= -a_7 z_1 - a_{13} z_2 + a_{18} z_4 + a_{19} z_5,\\
[v_4, v_5] &= -a_8 z_1 - a_{14} z_2 - a_{17} z_3 - a_{20} z_6,\\
[v_4, v_6] &= -a_9 z_1 - a_{15} z_2 - a_{18} z_3 + a_{20} z_5,\\
[v_5, v_6] &= -a_{10} z_1 - a_{16} z_2 - a_{19} z_3 - a_{20} z_4.
\end{aligned}
$$

3.3.2 Quadratic and reduced 2-step algebras of type 5, 6, 7 and 8

By applying Algorithms 1, 2 and 3, we get all possible $\mathrm{RankedMatrix}(d)$ as we have shown in the examples of subsection 3.3.1. Table 3.3 displays the whole set of $\mathrm{RankedMatrix}(d)$ matrices for $d = 5, 6, 7, 8$. Then, from Theorem 3.2.2, we arrive at the following classification result.

Theorem 3.3.1. *The quadratic reduced 2-step Lie algebras of type $d = 5, 6, 7, 8$ over a field $\mathbb{F}$ of characteristic zero are given as $(\mathfrak{n}(a, d), \varphi)$ where $\mathfrak{n}(a, d) = \mathrm{span}\langle v_1, \dots, v_d, z_1, \dots, z_d\rangle$, φ is the metabolic canonical form of matrix $B(d) = \begin{pmatrix} 0_d & I_d \\ I_d & 0_d \end{pmatrix}$, and the skewsymmetric Lie product of $\mathfrak{n}(a, d)$ follows from the formula:*

$$([v_1, v_2], [v_1, v_3], \dots, [v_1, v_d], [v_2, v_3], \dots, [v_2, v_d], \dots, [v_{d-1}, v_d]) \\ = (z_1, \dots, z_d) \cdot \mathcal{F}(M_1, \dots, M_d).$$

with $\mathcal{F}(M_1, \dots, M_d)$ any matrix $\mathrm{RankedMatrix}(d)$ of rank d as described in Table 3.3 and determined by some entries $a = (a_i) \in \mathbb{F}^{r(d)}$, where $r(5) = 10, r(6) = 20, r(7) = 35$ and $r(8) = 56$.

Remark 3.3.2. In general, $r(d) = \mathrm{VarUntil}(1, d - 2)$.

The multiplication table for quadratic and reduced 2-step algebras for $d = 5, 6$ are explicitly given in Example 3.3.4 and Example 3.3.5. We miss out here the product tables in the cases $d = 7, 8$; both tables are obtained from the matrix expression $(z_1, \dots, z_7) \cdot \mathrm{RankedMatrix}(7)$ if $d = 7$ and $(z_1, \dots, z_8) \cdot \mathrm{RankedMatrix}(8)$ if $d = 8$. By using Theorem 3.2.5 for a given quadratic algebra $(\mathfrak{n}(a, d), \varphi)$, we can check the whole set of isometrically isomorphic quadratic algebras as we illustrate in the next example.

Example 3.3.6. The matrices C_1 and C_2 are $\mathrm{RankedMatrix}(5)$ of rank 5 given by $\alpha_1, \alpha_2 \in \mathbb{F}^{10}$ with

$$\alpha_1 = (-9, -60, -47, -15, 186, -53, -24, -174, -86, -206) \quad \text{and} \\ \alpha_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

So,

$$C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 9 & 60 & 47 & 15 & -186 & 53 \\ 0 & -9 & -60 & -47 & 0 & 0 & 0 & 24 & -174 & 86 \\ 9 & 0 & -15 & 186 & 0 & -24 & 174 & 0 & 0 & 206 \\ 60 & 15 & 0 & -53 & 24 & 0 & -86 & 0 & -206 & 0 \\ 47 & -186 & 53 & 0 & -174 & 86 & 0 & 206 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

We have the matrix relation $Q^t C_2 \hat{Q} = C_1$ where

$$Q = \begin{pmatrix} 2 & -2 & 4 & -1 & -2 \\ -4 & -2 & 3 & -3 & 5 \\ 3 & 3 & 0 & 3 & 4 \\ -2 & 1 & 4 & -4 & 0 \\ 3 & -2 & -1 & -1 & 5 \end{pmatrix}.$$

and $\hat{Q}$ is obtained from Q by applying the formula (3.12). Even more, for any regular matrix Q, so $\det Q \neq 0$),

$$Q = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \\ q_6 & q_7 & q_8 & q_9 & q_{10} \\ q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{16} & q_{17} & q_{18} & q_{19} & q_{20} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \end{pmatrix}.$$

we have that $Q^t C_2 \hat{Q} = \text{RankedMatrix}(5)$ with entries a_i for $1 \leq i \leq 10$, where

$$\begin{aligned} a_1 &= q_2q_8q_{11} - q_{18}q_{22}q_{11} - q_1q_8q_{12} + q_3(q_6q_{12} - q_7q_{11}) - q_2q_6q_{13} + q_1q_7q_{13} \\ &\quad -q_{13}q_{17}q_{21} + q_{12}q_{18}q_{21} + q_{13}q_{16}q_{22} + (q_{11}q_{17} - q_{12}q_{16})q_{23} \\ a_2 &= q_2q_9q_{11} - q_{19}q_{22}q_{11} - q_1q_9q_{12} + q_4(q_6q_{12} - q_7q_{11}) - q_2q_6q_{14} + q_1q_7q_{14} \\ &\quad -q_{14}q_{17}q_{21} + q_{12}q_{19}q_{21} + q_{14}q_{16}q_{22} + (q_{11}q_{17} - q_{12}q_{16})q_{24} \\ a_3 &= q_2q_{10}q_{11} - q_{20}q_{22}q_{11} - q_1q_{10}q_{12} + q_5(q_6q_{12} - q_7q_{11}) - q_2q_6q_{15} + q_1q_7q_{15} \\ &\quad -q_{15}q_{17}q_{21} + q_{12}q_{20}q_{21} + q_{15}q_{16}q_{22} + (q_{11}q_{17} - q_{12}q_{16})q_{25} \\ a_4 &= q_3q_9q_{11} - q_{19}q_{23}q_{11} - q_1q_9q_{13} + q_4(q_6q_{13} - q_8q_{11}) - q_3q_6q_{14} + q_1q_8q_{14} \\ &\quad -q_{14}q_{18}q_{21} + q_{13}q_{19}q_{21} + q_{14}q_{16}q_{23} + (q_{11}q_{18} - q_{13}q_{16})q_{24} \end{aligned}$$

$$
\begin{aligned}
a_5 &= q_3q_{10}q_{11} - q_{20}q_{23}q_{11} - q_1q_{10}q_{13} + q_5(q_6q_{13} - q_8q_{11}) - q_3q_6q_{15} + q_1q_8q_{15} \\
&\quad -q_{15}q_{18}q_{21} + q_{13}q_{20}q_{21} + q_{15}q_{16}q_{23} + (q_{11}q_{18} - q_{13}q_{16})q_{25} \\
a_6 &= q_4q_{10}q_{11} - q_{20}q_{24}q_{11} - q_1q_{10}q_{14} + q_5(q_6q_{14} - q_9q_{11}) - q_4q_6q_{15} + q_1q_9q_{15} \\
&\quad -q_{15}q_{19}q_{21} + q_{14}q_{20}q_{21} + q_{15}q_{16}q_{24} + (q_{11}q_{19} - q_{14}q_{16})q_{25} \\
a_7 &= q_3q_9q_{12} - q_{19}q_{23}q_{12} - q_2q_9q_{13} + q_4(q_7q_{13} - q_8q_{12}) - q_3q_7q_{14} + q_2q_8q_{14} \\
&\quad -q_{14}q_{18}q_{22} + q_{13}q_{19}q_{22} + q_{14}q_{17}q_{23} + (q_{12}q_{18} - q_{13}q_{17})q_{24} \\
a_8 &= q_3q_{10}q_{12} - q_{20}q_{23}q_{12} - q_2q_{10}q_{13} + q_5(q_7q_{13} - q_8q_{12}) - q_3q_7q_{15} + q_2q_8q_{15} \\
&\quad -q_{15}q_{18}q_{22} + q_{13}q_{20}q_{22} + q_{15}q_{17}q_{23} + (q_{12}q_{18} - q_{13}q_{17})q_{25} \\
a_9 &= q_4q_{10}q_{12} - q_{20}q_{24}q_{12} - q_2q_{10}q_{14} + q_5(q_7q_{14} - q_9q_{12}) - q_4q_7q_{15} + q_2q_9q_{15} \\
&\quad -q_{15}q_{19}q_{22} + q_{14}q_{20}q_{22} + q_{15}q_{17}q_{24} + (q_{12}q_{19} - q_{14}q_{17})q_{25} \\
a_{10} &= q_4q_{10}q_{13} - q_{20}q_{24}q_{13} - q_3q_{10}q_{14} + q_5(q_8q_{14} - q_9q_{13}) - q_4q_8q_{15} + q_3q_9q_{15} \\
&\quad -q_{15}q_{19}q_{23} + q_{14}q_{20}q_{23} + q_{15}q_{18}q_{24} + (q_{13}q_{19} - q_{14}q_{18})q_{25}
\end{aligned}
$$

By applying Theorem 3.2.5, quadratic Lie algebra $(\mathfrak{n}(\alpha_1, 5), \varphi)$ and algebra $(\mathfrak{n}(\alpha, 5), \varphi)$ for $\alpha = (a_1, \ldots, a_{10})$ are isometrically isomorphic to the quadratic algebra $(\mathfrak{n}(\alpha_2, 5), \varphi)$ with basis $\{v_1, \ldots, v_5, z_1, \ldots, z_5\}$ and non-zero products:

$$
\begin{aligned}
&[v_1, v_2] = -z_3, \qquad [v_2, v_3] = -z_1, \qquad [v_3, v_5] = z_4, \\
&[v_1, v_3] = z_2, \qquad [v_3, v_4] = -z_5, \qquad [v_4, v_5] = -z_3.
\end{aligned}
$$

3.3.3 Computational limitations

In previous Subsection 3.3.1 we have noted the limitations caused by the rank calculus. In the following Table 3.2 we can see the minimum number of variables needed to obtain maximum rank for small values of d.

Dimension	**Condition to make the rank maximum**
$d = 1$	Invalid
$d = 2$	Invalid
$d = 3$	There is just one variable, a_1, and it has to be non-null.
$d = 4$	Impossible.
$d = 5, 6$	We need at least two non-null variables.
$d = 7, 8, 9$	We need at least three non-null variables.

Table 3.2: Minimum number of variables needed to get maximum rank in $\mathrm{RankedMatrix}(d)$.

For example if $d = 6$, among its 20 variables, only the couples (a_i, a_{21-i}) (both variables being non-null) with the rest of them null, make the rank maximum. In the case $d = 5$ each variable appears in three possible couples that makes the rank maximum, while in $n = 7$ there is no couples of variables, we need at least three non-null parameters to get rank 7. For $d = 7, 8, 9$ the non-null triples (a_1, a_{10}, a_{15}), (a_1, a_{12}, a_{56}) and (a_1, a_{65}, a_{85}) provide quadratic Lie algebras of types 7, 8 and 9 respectively. Our experimental work with the software developed points out that the different isomorphisms classes of quadratic 2-step of type up to 8 could be achieved by using $\mathrm{RankedMatrix}(d)$ with at most three non-zero variables.

3.4 Summary

Along this chapter we have started seen how to build any mixed quadratic Lie algebra as a double extension of a solvable one, and this solvable as a double extension of a nilpotent algebra. Thanks to these results, we have been able to reduce the problem of structure and classification of quadratic Lie algebras to studying the nilpotent variety, as the semisimple case is already completely solved. This reduction was achieved thanks to Theorem 3.1.8 in combination with the ideals inclusion study at the beginning of Section 3.1. These results give us a good starting point for finding double extensions by means of isotropic ideals. This extends some previous known results of one-dimensional and simple double extensions (see Corollaries 3.1.11 and 3.1.12).

From here, we have seen different ways of approaching the nilpotent case. In this chapter we have presented the use of quotients of free nilpotent Lie algebras. Thanks to the UMP, and the categorical approach, from Section 3.1.3, all quadratic nilpotent Lie algebras can be obtained by means of bilinear invariant forms of free nilpotent Lie algebras.

The general problem of building a nilpotent quadratic Lie algebra with arbitrary d generators is not easy, neither the construction of isometric isomorphisms. But classification results in some special families (small type d or small index of nilpotency) could be of interest. This is the case of quadratic 2-step nilpotent Lie algebras.

d	**RankedMatrix(d)** (only matrices of rank d are valid)
5	$\begin{pmatrix} 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 \\ 0 & a_1 & a_2 & a_3 & 0 & 0 & 0 & -a_7 & -a_8 & -a_9 \\ -a_1 & 0 & a_4 & a_5 & 0 & a_7 & a_8 & 0 & 0 & -a_{10} \\ -a_2 & -a_4 & 0 & a_6 & -a_7 & 0 & a_9 & 0 & a_{10} & 0 \\ -a_3 & -a_5 & -a_6 & 0 & -a_8 & -a_9 & 0 & -a_{10} & 0 & 0 \end{pmatrix}$
6	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & -a_8 & -a_9 & -a_{10} \\ 0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & -a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15} & -a_{16} \\ -a_1 & 0 & a_5 & a_6 & a_7 & 0 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & -a_{17} & -a_{18} & -a_{19} \\ -a_2 & -a_5 & 0 & a_8 & a_9 & -a_{11} & 0 & a_{14} & a_{15} & 0 & a_{17} & a_{18} & 0 & 0 & -a_{20} \\ -a_3 & -a_6 & -a_8 & 0 & a_{10} & -a_{12} & -a_{14} & 0 & a_{16} & -a_{17} & 0 & a_{19} & 0 & a_{20} & 0 \\ -a_4 & -a_7 & -a_9 & -a_{10} & 0 & -a_{13} & -a_{15} & -a_{16} & 0 & -a_{18} & -a_{19} & 0 & -a_{20} & 0 & 0 \end{pmatrix}$
7	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & -a_8 & -a_9 & -a_{10} & -a_{11} & -a_{12} & -a_{13} & -a_{14} & -a_{15} \\ 0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 & 0 & 0 & -a_{16} & -a_{17} & -a_{18} & -a_{19} & -a_{20} & -a_{21} & -a_{22} & -a_{23} & -a_{24} & -a_{25} \\ -a_1 & 0 & a_6 & a_7 & a_8 & a_9 & 0 & a_{16} & a_{17} & a_{18} & a_{19} & 0 & 0 & 0 & 0 & -a_{26} & -a_{27} & -a_{28} & -a_{29} & -a_{30} & -a_{31} \\ -a_2 & -a_6 & 0 & a_{10} & a_{11} & a_{12} & -a_{16} & 0 & a_{20} & a_{21} & a_{22} & 0 & a_{26} & a_{27} & a_{28} & 0 & 0 & 0 & -a_{32} & -a_{33} & -a_{34} \\ -a_3 & -a_7 & -a_{10} & 0 & a_{13} & a_{14} & -a_{17} & -a_{20} & 0 & a_{23} & a_{24} & -a_{26} & 0 & a_{29} & a_{30} & 0 & a_{32} & a_{33} & 0 & 0 & -a_{35} \\ -a_4 & -a_8 & -a_{11} & -a_{13} & 0 & a_{15} & -a_{18} & -a_{21} & -a_{23} & 0 & a_{25} & -a_{27} & -a_{29} & 0 & a_{31} & -a_{32} & 0 & a_{34} & 0 & a_{35} & 0 \\ -a_5 & -a_9 & -a_{12} & -a_{14} & -a_{15} & 0 & -a_{19} & -a_{22} & -a_{24} & -a_{25} & 0 & -a_{28} & -a_{30} & -a_{31} & 0 & -a_{33} & -a_{34} & 0 & -a_{35} & 0 & 0 \end{pmatrix}$
8	$\left(\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 & -a_8 & -a_9 & -a_{10} & -a_{11} & -a_{12} & \cdots \\ 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{22} & -a_{23} & -a_{24} & -a_{25} & -a_{26} & -a_{27} & \cdots \\ -a_1 & 0 & a_7 & a_8 & a_9 & a_{10} & a_{11} & 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & 0 & 0 & 0 & 0 & 0 & -a_{37} & \cdots \\ -a_2 & -a_7 & 0 & a_{12} & a_{13} & a_{14} & a_{15} & -a_{22} & 0 & a_{27} & a_{28} & a_{29} & a_{30} & 0 & a_{37} & a_{38} & a_{39} & a_{40} & 0 & \cdots \\ -a_3 & -a_8 & -a_{12} & 0 & a_{16} & a_{17} & a_{18} & -a_{23} & -a_{27} & 0 & a_{31} & a_{32} & a_{33} & -a_{37} & 0 & a_{41} & a_{42} & a_{43} & 0 & \cdots \\ -a_4 & -a_9 & -a_{13} & -a_{16} & 0 & a_{19} & a_{20} & -a_{24} & -a_{28} & -a_{31} & 0 & a_{34} & a_{35} & -a_{38} & -a_{41} & 0 & a_{44} & a_{45} & -a_{47} & \cdots \\ -a_5 & -a_{10} & -a_{14} & -a_{17} & -a_{19} & 0 & a_{21} & -a_{25} & -a_{29} & -a_{32} & -a_{34} & 0 & a_{36} & -a_{39} & -a_{42} & -a_{44} & 0 & a_{46} & -a_{48} & \cdots \\ -a_6 & -a_{11} & -a_{15} & -a_{18} & -a_{20} & -a_{21} & 0 & -a_{26} & -a_{30} & -a_{33} & -a_{35} & -a_{36} & 0 & -a_{40} & -a_{43} & -a_{45} & -a_{46} & 0 & -a_{49} & \cdots \end{matrix}\right.$ $\left.\begin{matrix} \cdots & -a_{13} & -a_{14} & -a_{15} & -a_{16} & -a_{17} & -a_{18} & -a_{19} & -a_{20} & -a_{21} \\ \cdots & -a_{28} & -a_{29} & -a_{30} & -a_{31} & -a_{32} & -a_{33} & -a_{34} & -a_{35} & -a_{36} \\ \cdots & -a_{38} & -a_{39} & -a_{40} & -a_{41} & -a_{42} & -a_{43} & -a_{44} & -a_{45} & -a_{46} \\ \cdots & 0 & 0 & 0 & -a_{47} & -a_{48} & -a_{49} & -a_{50} & -a_{51} & -a_{52} \\ \cdots & a_{47} & a_{48} & a_{49} & 0 & 0 & 0 & -a_{53} & -a_{54} & -a_{55} \\ \cdots & 0 & a_{50} & a_{51} & 0 & a_{53} & a_{54} & 0 & 0 & -a_{56} \\ \cdots & -a_{50} & 0 & a_{52} & -a_{53} & 0 & a_{55} & 0 & a_{56} & 0 \\ \cdots & -a_{51} & -a_{52} & 0 & -a_{54} & -a_{55} & 0 & -a_{56} & 0 & 0 \end{matrix}\right)$

Table 3.3: Matrices $\mathcal{F}(M_1, \ldots, M_d)$ attached to a d-quadratic family of matrices $\{M_1, \ldots, M_d\}$.

The complete *finite list* of quadratic 2-step algebras up to dimension 17, as far as we know, was unknown. Although it will be solved in the next chapter. In [Kath, 2007] we can find the classification of quadratic nilpotent Lie algebras up to dimension 10. So, the reduced 2-step included in Kath's classification (a finite number up to isomorphisms) are of type $d = 3, 5$.

In order to get a list for $d = 5, 6, 7, 8$, we present a new approach based on d-quadratic family of matrices. This is a family of skewsymmetric matrices $\{M_1, \dots, M_d\}$ of order $d \times d$ such that the j^{th} column of M_i is the additive inverse of the i^{th} column of M_j. The entries of these families of matrices, let us to described 2-step quadratic Lie algebras by means of basis and structure constants and the attached non-degenerate bilinear form in terms of canonical metabolic invariant bilinear forms according to Theorem 3.2.2. And, using the previous categorical approach, we identify the classes of isometric isomorphisms in Theorem 3.2.5. This way, we extend the results about structure and existence given in [Ovando, 2007b].

Once the theoretical support of the new method is completely described, we give a computational implementation of the algorithm induced from the cotangent structure encoded in the quadratic family. This allows us to easily construct many examples of quadratic 2-step Lie algebras and leads us to Theorem 3.3.1 which includes the structure of quadratic and reduced 2-step nilpotent Lie algebras of dimension up to 17 are $d = 3, 5, 6, 7, 8$, and Example 3.2.3 which provides the unique 2-step nilpotent reduced algebra of type $d = 3$ up to isometries. But Theorem 3.3.1 does not contain a finite list because of the isometric isomorphisms among the different Lie algebras. The isomorphism problem, a not easy task (see for example [Benito et al., 2017]), can be treated by means of Theorem 3.2.5. This complex task will be tackled in the next chapter.

Equivalent constructions of 2-step quadratic Lie algebras

4

CHAPTER

As we have seen along this memoir, when we are working with 2-step quadratic Lie algebras, we have three main different approaches. In Section 2.2.2, we introduced the classical general methods: double extension and T^*-extension. On the other hand, in the previous chapter, we ended up explaining another technique based on quadratic families of matrices that encode, as structure constants, our algebras. Once here, we are ready to establish an equivalent characterization among these three different construction methods using multilinear tools to provide a constructive equivalence theorem (Theorem 4.2.1) that relates all three methods.

In 1962, S. T. Tsou (see [Tsou, 1962]) established an existence theorem for real quadratic Lie algebras of arbitrary type. The proof of this result, that was announced in [Tsou and Walker, 1957, Section 7]), involves structure constants, trivectors and solutions of non-linear systems of equations, so multilinear algebra. These ideas are in the base of the proposed scheme, as well as results on the structure and classification of quadratic nilpotent Lie algebras given in [Benito et al., 2017] or previous chapters in this thesis.

Moreover, this equivalence reduces, in a direct and natural way, the classification up to isometries of quadratic 2-step nilpotent Lie algebras to that of 3-alternating forms up to equivalence (Theorem 4.3.1) following the ideas of [Noui and Revoy, 1997]. Our equivalence also shows that invariant forms of the subclass of reduced quadratic 2-step are metabolic. Even more, the class of quadratic 2-step Lie algebras agree with the class of T^*-extensions of abelian Lie algebras given by non-degenerate (equivalently linearly surjective) 2-cocycles (Corollary 4.1.14). This assertion collects and expands Proposition 11 in [Duong, 2013].

In addition, we will provide simple rules for switching among the four different structures. The ideas and results of this chapter has been recently published in [Benito and Roldán-López, 2023a].

4.1 Methods

Now we are going to recap the different methods for obtaining quadratic Lie algebras, but restricted to the 2-step case.

4.1.1 Double extension

In Section 2.2.2.1, the double extension procedure was introduced. Just as a remainder, given a quadratic Lie algebra (A, f) we define its double extension by (B, ϕ) as the quadratic algebra (A_B, f_B) where $A_B = B \oplus A \oplus B^*$, the Lie product is given by

$$\begin{aligned}[b_1 + a_1 + \beta_1, b_2 + a_2 + \beta_2] = [b_1, b_2]_B + \phi(b_1)(a_2) - \phi(b_2)(a_1) + [a_1, a_2]_A \\ + w(a_1, a_2) + \mathrm{ad}^*(b_1)(\beta_2) - \mathrm{ad}^*(b_2)(\beta_1).\end{aligned}$$

And the bilinear form is defined as

$$f_B(b_1 + a_1 + \beta_1, b_2 + a_2 + \beta_2) = \beta_1(b_2) + \beta_2(b_1) + f(a_1, a_2)$$

for $b_i \in B$, $a_i \in A$ and $\beta_i \in B^*$.

But, when focusing in 2-step nilpotent Lie algebras the method has some special properties and can be tuned a bit more. First, in general, when double extending a nilpotent Lie algebra, in case the result is nilpotent, we always keep or increase its nilpotency index as next lemma proves.

Lemma 4.1.1. *Let (A_B, f_B) be the double extension of (A, f) by (B, ϕ). If A_B is t-step nilpotent then A is n-step with $n \leq t$.*

Proof. The proof is straightforward. As every $a \in A$ can be seen as even an element of A_B, and $[a,a']_{A_B} = [a,a']_A + w(a,a')$. If $[a,a']_{A_B} = 0$, then $[a,a']_A = 0$, and $w(a,a') = 0$ because the first part lays on A, while the second belongs to B^*. ☐

This result cannot be improved as an abelian quadratic Lie algebra can generate a n-step nilpotent one, for every $n \in \mathbb{N}$. To see it, we have the following example which also serves as a contradiction of a statement found in [Ovando, 2007b, p. 913], which says 2-step nilpotent double extensions come always from 2-step nilpotent algebras.

Example 4.1.1. Let us take the abelian quadratic Lie algebra (A, f) of dimension $2n$ with basis $\{e_{-n}, e_{-n+1}, \dots, e_{-1}, e_1, \dots, e_n\}$, where

$$f(e_i, e_j) = \begin{cases} 1 & \text{if } |i-j| = n+1, \\ 0 & \text{if } |i-j| \neq n+1. \end{cases}$$

Let us consider the f-skew-symmetric derivation $d\colon A \to A$ where $d(e_{-i}) = e_{-i+1}$, $d(e_i) = -e_{i-1}$ for $i = 2, \dots, n$ and $d(e_1) = d(e_{-1}) = 0$. Now, we can build the double extension (A_B, f_B) of (A, f) by (ϕ, B) where $B = \mathbb{F}b$ and $\phi(b) = d$. This new algebra satisfies

$$(A_B)^t = \operatorname{span}\langle e_{-n+t-1}, \dots, e_{-1}, e_1, \dots, e_{n-t+1}\rangle.$$

Thus, $(A_B)^n$ is the bidimensional ideal linearly generated by $\{e_{-1}, e_1\}$ and $(A_B)^{n+1} = 0$. Therefore, A_B is n-step nilpotent.

And even more, a nilpotent Lie algebra can be double-extended to produced a non-nilpotent algebra as we saw in Example 2.2.7 (oscillator), or as in Examples 5.2.2 (mixed or solvable) found later.

Now, according to Corollary 3.1.12, in our nilpotent (solvable) non-abelian quadratic algebra (L, f) we can find a non-zero element $z \in L^2 \cap Z(L)$ such that our algebra is a double extension of algebra

$$\left(\frac{(\mathbb{F}z)^\perp}{\mathbb{F}z}, \hat{f}\right),$$

which is also nilpotent quadratic of dimension $\dim L - 2$. Iterating this one-dimensional process, we get that the class of solvable (nilpotent) quadratic

Lie algebras is a direct sum of abelian and one-dimensional double extensions of solvable (nilpotent) ones. This result also appears in [Medina and Revoy, 1985, Théorème III]. So, although Theorem 2.2.20 gives us the general double extension method, for our goal, a subset of these extensions, the one-dimensional ones, is enough. This leads us to the following definition.

Definition 4.1.1. We call (A_b, f_b) the *one-dimensional double extension* of (A, f) by (b, d) to the double extension (A_B, f_B) of (A, f) by (B, ϕ) where $B = \mathbb{F}b$ has dimension 1 and $\phi(b) = d$.

Now, when considering the vector space $A_b = \mathbb{F}b \oplus A \oplus \mathbb{F}\beta$, where $\beta(b) = 1$ (dual 1-form of b), the Lie bracket from equation (2.18) turns into

$$[b_i b + a_i + \beta_i \beta, b_j b + a_j + \beta_j \beta] := b_i d(a_j) - b_j d(a_i) + [a_i, a_j] + f(d(a_i), a_j)\beta,$$

for every scalar $b_i, b_j, \beta_i, b_j \in \mathbb{F}$ and $a_i \in A$. While the symmetric bilinear form from equation (2.19) can be written as

$$f_b(b_i b + a_i + \beta_i \beta, b_j b + a_j + \beta_j \beta) := b_i \beta_j + b_j \beta_i + f(a_i, a_j).$$

This simplification also allows us to compute the derived algebra

$$A_b^2 = \operatorname{Im} d + \operatorname{span}\langle [a_1, a_2]_A + f(d(a_1), a_2)\beta : a_1, a_2 \in A \rangle. \tag{4.1}$$

As a consequence of Theorem 2.2.9, we are only interested in studying reduced algebras. That is why we should be able to describe the centre and, therefore, the reducibility of A_b depending on if d is either an inner or an outer f-skew-symmetric derivation of A.

Lemma 4.1.2. *Let (A_b, f_b) be the one-dimensional double extension of (A, f) by (b, d). Then*

$$Z(A_b) = (Z(A) \cap \ker d) \oplus B^*$$

if and only if $d \notin \operatorname{Inner} A$. Otherwise, $d = \operatorname{ad} x$ for some $x \in A$ and

$$Z(A_b) = (Z(A) \cap \ker d) \oplus B^* \oplus \mathbb{F}(b - x).$$

Proof. If we calculate the centre we obtain

$$\begin{aligned}
Z(A_b) &= \{b_1 b + a_1 + \beta_1\beta : b_1 d(a_2) - b_2 d(a_1) + [a_1, a_2] = 0, \\
&\qquad\qquad\qquad\qquad f(d(a_1), a_2) = 0 \,\forall\, a_2 \in A, b_2 \in \mathbb{F}\} \\
&= \{b_1 b + a_1 + \beta_1\beta : b_1 d(a_2) + [a_1, a_2] = 0, d(a_1) = 0 \,\forall\, a_2 \in A\} \\
&= \{a_1 + \beta_1\beta : [a_1, a_2] = 0, d(a_1) = 0 \,\forall\, a_2 \in A\} \\
&\qquad\qquad + \operatorname{span}\left\langle b + \frac{1}{b_1}\, a_1 : d = -\operatorname{ad}\left(\frac{1}{b_1}\, a_1\right), b_1 \neq 0 \right\rangle \\
&= \begin{cases} (Z(A) \cap \ker d) \oplus B^* & \text{if } d \notin \operatorname{Inner} A, \\ (Z(A) \cap \ker d) \oplus B^* \oplus \mathbb{F}(b - x) & \text{if } d = \operatorname{ad} x. \end{cases}
\end{aligned}$$

Note that $d = \operatorname{ad} x = \operatorname{ad} y$ if and only if $x - y \in Z(A) \cap \ker d$. □

Since $A_b^2 \subseteq (\operatorname{Im} d + A^2) \oplus B^*$ then $b - x \notin A_b^2$ when $d = \operatorname{ad} x$. Therefore $Z(A) \not\subseteq A_b^2$ and our algebra is non-reduced.

Corollary 4.1.3. *Let (A_b, f_b) be the one-dimensional double extension of (A, f) by (b, d). Then (A_b, f_b) is non-reduced when $d \in \operatorname{Inner} A$.*

This direct consequence of our lemma is mentioned in [Figueroa-O'Farrill and Stanciu, 1996, Proposition 5.1].

Corollary 4.1.4. *Let (A_b, f_b) be the one-dimensional double extension of (A, f) by (b, d). Then (A_b, f_b) is reduced if and only if $d \notin \operatorname{Inner} A$, $A_b^2 = (\operatorname{Im} d + A^2) \oplus B^*$ and $Z(A) \cap \ker d \subseteq \operatorname{Im} d + A^2$.*

Proof. Since A_b is reduced if and only if $Z(A_b) \subseteq A_b^2$, the result is straight forward applying Lemma 4.1.2 and equation (4.1). □

Now, as our aim in this chapter is to study the 2-step case, we will see which restrictions A and d do need to satisfy in the following proposition.

Proposition 4.1.5. *Let (A_b, f_b) be the one-dimensional double extension of some quadratic Lie algebra (A, f) by (b, d). Then A_b is 2-step if and only if*

$$0 \neq \operatorname{Im} d + A^2 \subseteq Z(A) \cap \ker d.$$

Proof. First, we have that

$$A_b^3 = d^2(A) + d(A^2) + \operatorname{span}\langle [d(a_1), a_2] + f(d^2(a_1), a_2)\beta : a_i \in A\rangle \\ + \operatorname{span}\langle f(d([a_1, a_2]), a_3)\beta : a_i \in A\}.$$

As this must be zero, we need

$$\begin{cases} d(A) \subseteq Z(A), \\ d(d(A)) = d^2(A) = 0, \\ d([A, A]) = d(A^2) = 0. \end{cases} \tag{4.2}$$

The conditions in equation (4.2), as d is a derivation, can be expressed in one line as

$$\operatorname{Im} d + A^2 \subseteq Z(A) \cap \ker d. \tag{4.3}$$

At this point, $A_b^3 = 0$, and we need to check if $A_b^2 \neq 0$ in case A is abelian. This, using equation (4.1), translates into $d \neq 0$ finishing the proof. □

Remark 4.1.6. Note that every homomorphism $d\colon A \to A$ that satisfies condition (4.3) is indeed a derivation as $\operatorname{Im} d \subseteq Z(A)$ and $A^2 \subseteq \ker d$.

Corollary 4.1.7. *Let (A_b, f_b) be the one-dimensional double extension of an abelian quadratic Lie algebra (A, f) by (b, d). Then A_b is 2-step if and only if $d \neq 0$ and $d^2 = 0$.*

As we have previously noted in the deconstruction of Chapter 3, solvable Lie algebras can be obtained by iterating one-dimensional double extensions. This multistep procedure can be implemented in a nested way, which is the idea in our following construction.

Chained one-dimensional double extensions construction

Now, we are going to consider a chain of one-dimensional double extensions $\{(A_k, f_k)\}_{k=0}^n$. We start by introducing the following notation for every algebra of our chain:

$$A_{k+1} = B_{k+1} \oplus A_k \oplus B_{k+1}^*,$$

where $B_k = \mathbb{F}b_k$ and $B_k^* = \mathbb{F}b_k^*$ are 1-dimensional and $\dim A_k = 2k$. Now, we can also define $A_{k+1} = A_{k+1,1} \oplus A_{k+1,2}$ where

$$\begin{aligned} A_{k+1,1} &= B_{k+1} \oplus A_{k,1}, \\ A_{k+1,2} &= A_{k,2} \oplus B_{k+1}^*. \end{aligned}$$

Applying this definition recursively we obtain

$$A_{k+1,1} = \bigoplus_{i=1}^{k+1} B_i, \qquad A_{k+1,2} = \bigoplus_{i=1}^{k+1} B_i^*. \tag{4.4}$$

All this algebras A_k are associated to an invariant bilinear form f_k. Moreover, over them, we define derivations $d_k \colon A_k \to A_k$ such that $d_k \in \mathrm{Der}_{f_k}(A_k)$ to do the double extensions. Hence, (A_{k+1}, f_{k+1}) is the one-dimensional double extension of (A_k, f_k) by (b_{k+1}, d_k), starting with $A_0 = \{0\}$ and $f_0 = 0$. Note this is a really convenient notation. First, it gives us a basis for A_k:

$$\{b_k, b_{k-1}, \ldots, b_1, b_1^*, b_2^*, \ldots, b_k^*\},$$

where the order of this basis is given by the chain itself. Even more, if we divide the set separating b_i from b_j^* elements, we get the bases for $A_{k,1}$ and $A_{k,2}$ respectively. All together, we can see this build as a telescopic construction in Figure 4.1.

$$\xleftarrow{\quad A_{i,1} \quad} \Big| \xrightarrow{\quad A_{i,2} \quad}$$

$$\underbrace{B_k \oplus \cdots \oplus \overbrace{B_2 \oplus \underbrace{B_1 \oplus B_1^*}_{A_1} \oplus B_2^*}^{A_2} \oplus \cdots \oplus B_k^*}_{A_k}$$

Figure 4.1: Telescopic view of the chained one-dimensional double extension.

Let us now define for $k = 0, \ldots, n-1$

$$\begin{aligned} w_{k+1} \colon A_k \times A_k &\to B_{k+1}^* \\ (a, b) &\mapsto f_k(d_k(a), b)\, b_{k+1}^*. \end{aligned} \tag{4.5}$$

So, in basis

$$\{b_{k+1}, b_k, \ldots, b_1, b_1^*, \ldots, b_k^*, b_{k+1}^*\},$$

we can give the Lie bracket $[\cdot,\cdot]_{k+1}$ of algebra A_{k+1} which is

$$\begin{aligned} [b_{k+1}^*,\cdot]_{k+1} &= 0, & [b_i,b_j]_{k+1} &= [b_i,b_j]_k + w_{k+1}(b_i,b_j),\\ [b_{k+1},b_i]_{k+1} &= d_k(b_i), & [b_i^*,b_j^*]_{k+1} &= [b_i^*,b_j^*]_k + w_{k+1}(b_i^*,b_j^*),\\ [b_{k+1},b_i^*]_{k+1} &= d_k(b_i^*), & [b_i,b_j^*]_{k+1} &= [b_i,b_j^*]_k + w_{k+1}(b_i,b_j^*), \end{aligned} \tag{4.6}$$

for $1 \le i,j \le k$. While the bilinear form satisfies

$$\begin{cases} f_{k+1}(b_{k+1},b_{k+1}^*) = 1,\\ f_{k+1}(b_{k+1},B_{k+1}\oplus A_k) = f_{k+1}(b_{k+1}^*,A_k\oplus B_{k+1}^*) = 0,\\ f_{k+1}\big|_{A_k\times A_k} = f_k. \end{cases}$$

Remark 4.1.8. Note $A_{k+1,2} = (A_{k+1,2})^{\perp}$. So, $A_{k+1,2}$ is a lagrangian for f_{k+1}, and, therefore, f_{k+1} is a metabolic form.

Remark 4.1.9. From Lemma 4.1.1, we have A_{k+1} can be t-nilpotent only if A_k is n-nilpotent with $n \le t$. Hence, combining this result in the case $t = 2$ with Proposition 4.1.5, we conclude that A_{k+1} is 2-step if and only if

- A_k is abelian and $0 \neq \operatorname{Im} d_k \subseteq \ker d_k$, or
- A_k is 2-step, and $\operatorname{Im} d_k \subseteq Z(A_k)\cap \ker d_k$ and $A_k^2 \subseteq \ker d_k$.

This remark is useful when searching for 2-step quadratic Lie algebras, and it leads us to the following definition:

Definition 4.1.2. Let $\{(A_k,f_k)\}_{k=0}^n$ be a chain of algebras obtained from successive one-dimensional double extensions from the previous one in the chain by $\{b_{k+1},d_k\}_{k=0}^{n-1}$ starting from $A_0 = \{0\}$ and $f_0 = 0$. We say the chain satisfies:

- the *non-null property* (*NNP*) if there exists k such that $d_k \neq 0$,
- and the *2-step property* (*2SP*) if $\operatorname{Im} d_k \subseteq A_{k,2} \subseteq \ker d_k$ for every $k \ge 1$.

In any $\{(A_k,f_k)\}_{k=0}^n$ chain of one-dimensional double extensions, we have $d_0 = d_1 = 0$ and, therefore, A_1 and A_2 are abelian quadratic algebras of dimension 2 and 4 respectively. But for greater dimensions we can obtain 2-step algebras. If our chain satisfies NNP and 2SP properties from Definition 4.1.2, we can easily check it by applying 2SP inductively

$$A_k^2 \subseteq A_{k,2} \subseteq Z(A_k).$$

Therefore, every step or link of this chain satisfies equation (4.3). Hence, its final quadratic Lie algebra (A_n, f_n) for $n \geq 3$ is 2-step by using the NNP as observed in Remark 4.1.9. In addition, we have chosen these properties as we want to end up with a reduced 2-step Lie algebra A_n whose square or centre is $A_{n,2}$ and, as images and kernels determine its product, this is the most natural way to obtain it. The important result we are going to prove later in Theorem 4.2.1 is that we can obtain all reduced quadratic 2-step Lie algebras as the final quadratic Lie algebra (A_n, f_n) of some chain of one-dimensional double extensions that satisfies NNP and 2SP.

Proposition 4.1.10. *Let $\{(A_k, f_k)\}_{k=0}^{n}$ be a chain of one-dimensional double extensions by $\{(b_{k+1}, d_k)\}_{k=0}^{n-1}$ satisfying NNP and 2SP. And let define*

$$D_{ijk} := \operatorname{sgn}(\sigma) f_{\sigma(k)-1}(d_{\sigma(k)-1}(b_{\sigma(i)}), b_{\sigma(j)})$$

for some permutation σ such that $1 \leq \sigma(i) < \sigma(j) < \sigma(k)$ or $D_{ijk} = 0$ if some subindexes repeat. Then $n \geq 3$ and

(a) *(A_n, f_n) is a $2n$-dimensional 2-step quadratic Lie algebra such that $A_{n,2} \subseteq Z(A_n)$, $[b_i, b_j]_n = \sum_{k=1}^{n} D_{ijk}\, b_k^*$, and the invariant bilinear form f_n is given by $f_n(b_i, b_j^*) = \delta_{ij}$ and $f_n(b_i, b_j) = f_n(b_i^*, b_j^*) = 0$.*

(b) *(A_n, f_n) is reduced if and only if*

$$A_{n,2} = \operatorname{span}\left\langle \sum_{k=1}^{n} \hat{w}_k(b_i, b_j) : 1 \leq j < i \leq n \right\rangle$$

where $\hat{w}_k$ is the alternating extension of w_k defined in equation (4.5).

Proof. First of all, we can observe D_{ijk} definition resembles the idea d_{k-1} is f_{k-1}-skew-symmetric because $D_{ijk} = -D_{ijk}$ and $D_{iik} = 0$ as $f_{k-1}(d_k(b_i), b_j)$ for $i, j \leq k$ in characteristic different from 2.

Next, from previous arguments after Definition 4.1.2, we have that (A_n, f_n) is a quadratic 2-step Lie algebra. Now, applying multiplication table in equation (4.6) recursively and using $A_{k,2} \subseteq \ker d_k$ by 2SP, we obtain

$$\begin{cases} [b_i, b_j]_{k+1} = w_{k+1}(b_i, b_j) + w_k(b_i, b_j) + \ldots + w_{i+1}(b_i, b_j) + d_{i-1}(b_j), \\ [b_i^*, \cdot\,]_{k+1} = 0, \end{cases} \tag{4.7}$$

for $1 \leq j < i \leq k+1$. We also have for $1 \leq i, j \leq k+1$

$$\begin{cases} f_{k+1}(b_i, b_j^*) = \delta_{ij}, \\ f_{k+1}(b_i, b_j) = f_{k+1}(b_i^*, b_j^*) = 0. \end{cases}$$

Moreover, 2SP also implies that $d_{i-1}(A_{i-1}) \subseteq A_{i-1,2} = \operatorname{span}\langle b_1^*, \ldots, b_{i-1}^* \rangle$ and for $j < i$,

$$d_{i-1}(b_j) = \sum_{k=1}^{i-1} f_{i-1}(d_{i-1}(b_j), b_k) b_k^* = \sum_{k=1}^{i-1} D_{jki}\, b_k^* = \sum_{k=1}^{i-1} D_{ijk}\, b_k^*,$$

by using D_{ijk} definition in the last equality. So product (4.7) in (A_n, f_n) turns into

$$\begin{cases} [b_i, b_j]_n = \sum_{k=1}^{n} D_{ijk}\, b_k^*, \\ [b_i^*, \cdot\,]_n = 0. \end{cases}$$

And even more,

$$f_n([b_i, b_j]_n, b_k) = D_{ijk}.$$

Now A_n is a 2-step nilpotent Lie algebra, thus being reduced is equivalent to $Z(A_n) = A_n^2$. Now if we define $\hat{w}_k(b_i, b_j) = w_k(b_i, b_j)$ when $i, j < k$ and $\hat{w}_k(b_i, b_j) = \operatorname{sgn}(\sigma) w_{\sigma(k)}(b_{\sigma(i)}, b_{\sigma(j)})$ where σ is some permutation of $\{i, j, k\}$ such that $\sigma(k) = \max\{i, j, k\}$,

$$\operatorname{ad}_{A_n} b_i(b_j) = \sum_{k=0}^{n-1} \hat{w}_{k+1}(b_i, b_j).$$

Hence $A_n^2 = \operatorname{span}\langle \sum_{k=1}^{n} \hat{w}_k(b_i, b_j) : 1 \leq j < i \leq n \rangle$ and applying equation (2.14) for $k = n$ we finish the proof. □

All these relations and notation will serve us later to prove the equivalence between the different approaches for constructing these algebras.

4.1.2 T*-extension

As we have seen in Subsection 2.2.2.2, the T^*-extension is a one-step method that takes a non-associative algebra, which may be non-quadratic, and produces a quadratic algebra whose dimension is double the original. Applied to Lie algebras, let B be a Lie algebra, the quadratic Lie algebra (T^*B_w, q_B) is defined as the vector space $B \oplus B^*$ with product

$$[b+\beta,\ b'+\beta'] := [b,b']_B + w(b,b') + \mathrm{ad}^*(b)(\beta') - \mathrm{ad}^*(b')(\beta), \tag{4.8}$$

where w is a cyclic 2-cocycle, ad^* is the coadjoint representation, and with the bilinear form

$$q_B(b+\beta, b'+\beta') := \beta(b') + \beta'(b)$$

for $b \in B$ and $\beta \in B^*$. Once we have remembered the general construction, we can start by seeing what do B and w need to satisfy in order to obtain a 2-step quadratic Lie algebra as we have already done in the double extension. First, analogously to Lemma 4.1.1, we have the following result about the nilpotency order of the extension. It comes from [Bordemann, 1997, Theorem 3.1] but adapting indices to our situation.

Proposition 4.1.11. *If B is a k-step nilpotent Lie algebra, then for every cyclic 2-cocycle $w\colon B\times B \to B^*$ the T^*-extension T^*_wB is n-step nilpotent where $k \leq n \leq 2k$.*

Remark 4.1.12. This result cannot be improved. Indeed, in the following sections, we build 2-step quadratic Lie algebras from abelian ones (see Corollary 4.1.15).

Now, let us find which is the centre and the square of these algebras. In general,

$$Z(T^*_wB) = \{b+\beta : b \in Z(B) \text{ and } w(b,b') + \beta \circ \mathrm{ad}\, b' = 0 \ \ \forall\, b' \in B\},$$

and

$$(T^*_wB)^2 = \mathrm{span}\langle [b,b']_B + w(b,b') : b, b' \in B\rangle + \mathrm{span}\langle \beta \circ \mathrm{ad}\, b : b \in B, \beta \in B^*\rangle, \tag{4.9}$$

Lemma 4.1.13. *For any V subspace of B, let define $V^\circ := \{\beta \in B^* : \beta(V) = 0\}$ and $V^\perp$ its orthogonal subspace in T^*_wB with respect to quadratic form q_B. Then, $V^\perp = B \oplus V^\circ$ and $(V^\circ)^\perp = B^* \oplus V$ and:*

(a) $Z(B)^\circ = \mathrm{span}\langle \beta \circ \mathrm{ad}\, b : b \in B, \beta \in B^*\rangle \subseteq (T^*_wB)^2$,

(b) $(B^2)^\circ = \{\beta \in B^* : \beta \circ \mathrm{ad}\, b = 0 \ \ \forall\, b \in B\}$,

(c) $Z(T^*_wB) \cap B = Z(B) \cap \mathrm{Rad}\, w$ *and* $Z(T^*_wB) \cap B^* = (B^2)^\circ$,

(d) $\operatorname{span}\langle w(b,b') : b,b' \in B\rangle \subseteq (\operatorname{Rad} w)^\circ$.

Proof. Let $X = \operatorname{span}\langle \beta \circ \operatorname{ad} b : b \in B, \beta \in B^*\rangle$ and $x \in B$ such that $0 = q_B(x, \beta \circ \operatorname{ad} b) = \beta([b,x]_B), \forall \beta \in B^*, \forall b \in B$. Previous equality is equivalent to $[b,x]_B = 0 \; \forall b \in B$, i.e., $x \in Z(B)$. Hence $Z(B) = X^\perp \cap B$ and item (a) follows from $Z(B)^\perp = X \oplus B$, $X \subseteq Z(B)^\circ$ and equation (4.9). Now, from equation (2.14), $(Z(T^*_w B) \cap B^*)^\perp = (T^*_w B)^2 + B^* = B^2 \oplus B^* = ((B^2)^\circ)^\perp$ which implies item (b) and second assertion in item (c). Note, the other equality in item (c) is straightforward. Finally, if $\beta \in (\operatorname{Rad} w)^\perp \cap B^*$, then $\beta(a) = 0$ $\forall a \in B$ such that $w(a,b) = 0, \forall b \in B$. Since w is cyclic, for a fixed $a \in B$, $w(a,b)(b') = w(b,b')(a)$ and we get item (d) when $a \in \operatorname{Rad} w$. $\square$

From Lemma 4.1.13 we get immediately that $Z(T^*_0 B) = Z(B) \oplus (B^2)^\circ$ and $(T^*_0 B)^2 = B^2 \oplus Z(B)^\circ$. So, $T^*_0 B$ is reduced 2-step if and only if B is reduced 2-step. Even more, this lemma also shows that we can build quadratic 2-step Lie algebras from abelian ones in an easy way:

Corollary 4.1.14. *Let B be an abelian Lie algebra. Then $(T^*_w B)^2 = \operatorname{span}\langle w(b,b') : b,b' \in B\rangle$ and $Z(T^*_w B) = \operatorname{Rad} w \oplus B^*$. So, $T^*_w B$ is 2-step if and only if w is not null. Moreover, it is equivalent:*

(a) *$(T^*_w B, q_B)$ is reduced,*

(b) *w is non-degenerate,*

(c) *$B^* = \operatorname{span}\langle w(b,b') : b,b' \in B\rangle$.*

Proof. The first part follows easily from B being abelian, item (d) of Lemma 4.1.13, and the general description of $(T^*_0 B)^2$. Now, $(T^*_w B, q_B)$ is reduced if and only if $(T^*_w B)^2 = B^* = Z(T^*_w B)$. Hence, items (a) and (c) are equivalent. Finally, $Z(T^*_w B) = B^*$ if w is non-degenerate, and then $(T^* B, q_B)^2 = Z(T^*_w B)^\perp = B^*$ by using equation (2.14). $\square$

Corollary 4.1.15. *Let (A,f) a quadratic Lie algebra over an arbitrary field of characteristic zero. Then, A is 2-step reduced nilpotent if and only if A is isometrically isomorphic to a $T^*_w B$ extension of an abelian Lie algebra B where w is non-degenerate.*

Proof. If A is 2-step and reduced, $Z(A) = A^2 = Z(A)^\perp$ is a lagrangian ideal and, from Theorem 2.2.23, algebra A is isometrically isomorphic to $T^*_w B$ and $B \cong A/A^2$, so B is abelian. The converse follows from Corollary 4.1.14. $\square$

Remark 4.1.16. Corollary 4.1.15 expands and provides an alternatively proof to [Duong, 2013, Proposition 11], which is restricted to $\mathbb{C}$. It also shows the condition of w being non-degenerate can be changed by that of the dual space B^* being the linear span of the image of w.

It is precisely these abelian extensions the ones we are going to use in order to obtain our algebras. In this case, the product defined in equation (4.8) can be simplified to

$$[b + \beta,\ b' + \beta'] := w(b, b').$$

Remark 4.1.17. In [García-Delgado et al., 2020] the authors make a detailed study of invariant metrics on central extensions of quadratic Lie algebras.

4.1.3 Quadratic families and Lie algebras

According to Section 3.1.3, the classification of quadratic nilpotent Lie algebras can be reduced, in some categorical way, to the study of symmetric invariant bilinear forms on free nilpotent Lie algebras. Moreover, by the UMP, any t-step nilpotent Lie algebra of type d is a homomorphic image of $\mathfrak{n}_{d,t}/I$ with I an ideal such that $\mathfrak{n}_{d,t}^t \subsetneq I \subsetneq \mathfrak{n}_{d,t}^2$. This special relation allows us to build quadratic nilpotent Lie algebras by means of free nilpotent as seen in Lemma 3.1.17.

For 2-step quadratic Lie algebras, this classification process can be reformulated by using the notion of n-quadratic family as seen in Section 3.2. This is a special set of skew-symmetric matrices, see Definition 3.2.3, which encodes the structural constants of a quadratic 2-step Lie algebra of type n and dimension $2n$.

Let (A, φ) be a quadratic 2-step Lie algebra. Since $A^2 \subseteq Z(A)$, A is reduced if and only if $Z(A) = A^2$, and, from equation (2.14), $\dim A = 2n$ and $n = \operatorname{codim} A^2 = \dim A^2$. Otherwise, Theorem 2.2.9 tells us A decomposes as an orthogonal sum of ideals $\mathfrak{n} \oplus \mathfrak{a}$ where $\mathfrak{n}$ is 2-step reduced (so even dimensional) and $\mathfrak{a}$ is abelian. We assume in the sequel (A, φ) is a quadratic 2-step Lie algebra of dimension $2n$.

If even 2-step, as stated in Theorem 3.2.2, we can find a basis of the form $\{v_1, \dots, v_n, z_1, \dots, z_n\}$, where the Lie bracket is

$$[v_i, v_j] := \sum_{k=1}^{n} m_{ijk}\, z_k,$$
$$[z_i, \cdot\,] := 0.$$

While the bilinear form satisfies

$$\varphi(v_i, v_j) = 0, \qquad \varphi(z_i, z_j) = 0, \qquad \varphi(v_i, z_j) = \delta_{ij}.$$

This means its structure constants are determined by a non-degenerate family of n-quadratic matrices $\{M_i : 1 \le i \le n\}$. Here, m_{ijk} is the entry (k, j) of M_i, which is the same as saying m_{ijk} is the entry $(n+k, j)$ of the matrix of the inner derivation $\operatorname{ad} v_i$. And, by properties of the inner derivations or of the n-quadratic family, we have

$$m_{ijk} = m_{jki} = m_{kij} = -m_{ikj} = -m_{jik} = -m_{kji}.$$

Even more, $\varphi([v_i, v_j], v_k) = m_{ijk}$, and the non-degeneration of the family is equivalent to

$$\sum_{i=1}^{k} \operatorname{Im} \operatorname{ad} v_i = A^2 = \operatorname{span}\langle z_i : i = 1, \dots, n\rangle = Z(A).$$

And, we can recover the Lie product of A from the matrix equation:

$$\begin{multline}([v_1, v_2], \dots, [v_1, v_n], [v_2, v_3], \dots, [v_2, v_n], \dots, [v_{n-1}, v_n]) = \\ (z_1, \dots, z_n) \cdot \mathcal{F}(M_1, \dots, M_n). \quad (4.10)\end{multline}$$

In this construction, we also have Theorem 3.2.5 to solve the problem of isomorphisms of quadratic 2-step reduced algebras in terms of matrix relations between the non-degenerate n-quadratic families attached to them.

4.2 Equivalence theorem

Despite the methods double-extension and T^*-extension, introduced in the Section 2.2.2, and n-quadratic families from Section 3.2.2 are apparently totally different at first glance, all three of them end up constructing the same type of algebras and, therefore, it makes sense they are equivalent in some way. The relationship among them appears in the following theorem.

Theorem 4.2.1. *Let B be a vector space with basis $\mathcal{B} = \{b_1, \dots, b_n\}$, $n \geq 3$ and, let $w\colon B \times B \to B^*$ be a bilinear form where $w(b_i, b_j)(b_k) = c_{ijk}$. In the vector space $\mathfrak{L} = B \oplus B^*$ we define the following product and bilinear form ϕ for $b, b' \in B$ and $\beta, \beta' \in B^*$:*

$$[b + \beta, b' + \beta'] = w(b, b'), \qquad \phi(b + \beta, b' + \beta') = \beta(b') + \beta'(b).$$

Then, it is equivalent:

(a) *$(\mathfrak{L}, \phi)$ is a 2-step quadratic Lie algebra.*

(b) *w is a non-zero cyclic 2-cocycle and $(\mathfrak{L}, \phi) = (T^*_w B, q_B)$.*

(c) *For $\{b_1^*, \dots, b_n^*\}$ dual basis of $\mathcal{B}$, the chain of one-dimensional double extensions $\{(A_k, f_k)\}_{k=0}^n$, starting with $A_0 = \{0\}$ and $f_0 = 0$, given by the sucession $\{(b_{k+1}, d_k)\}_{k=0}^{n-1}$, where $A_k = \operatorname{span}\langle b_i, b_i^* : i = 1, \dots, k\rangle$, $d_{i-1}(b_j^*) = 0$, and $d_{i-1}(b_j) = \sum_{k=1}^{i-1} c_{ijk} b_k^*$ for $j < i \leq n$, satisfies properties NNP and 2SP, and $(\mathfrak{L}, \phi) = (A_n, f_n)$.*

(d) *The family of matrices $\{M_1, \dots, M_n\}_{i=1}^n$, where the entrance (k, j) of M_i is c_{ijk}, is a non-null n-quadratic family and defines algebra $(\mathfrak{L}, \phi) = (\mathbb{F}^{2n}, f)$ with product given by formula* (4.10).

Proof. The construction given in this theorem is exactly the T^*-extension one, so $(\mathfrak{L}, \phi)$ is $(T^*_w B, q_B)$, the T^*-extension of the abelian Lie algebra B by w. This proves the equivalence between (a) and (b) using Proposition 2.2.22.

Assume now item (b) holds and let decompose $A_k = A_{k,1} \oplus A_{k,2}$ as in equation (4.4). Note that $A_n = \mathfrak{L}$ (as vector spaces) and $f_k := \phi\big|_{A_k \times A_k}$ are as in Remark 4.1.8. In particular, $f_n = \phi$. So, the chain will be of one-dimensional double extensions if and only if $d_{i-1} \in \operatorname{Der}_{f_{i-1}} A_{i-1}$. Since B^* is ϕ-isotropic and $d_{i-1}(b_j^*) = 0$, this assertion is equivalent to $\lambda_{ijs} = 0$ where:

$$\lambda_{ijs} = f_{i-1}(d_{i-1}(b_j), b_s) + f_{i-1}(b_j, d_{i-1}(b_s)) = \varphi(d_{i-1}(b_j), b_s) + \varphi(b_j, d_{i-1}(b_s)).$$

Now from $d_{i-1}(A_{i-1}) \subseteq A_{i-1,2}$, $\lambda_{ijs} = 0$ if $s \geq i$. Otherwise $\lambda_{ijs} = c_{ijs} + c_{isj}$ and it is also null because of w is cyclic and skew. Moreover, since $A_{k,1} =$

$\mathrm{span}\langle b_1, \ldots, b_k \rangle$ and $A_{k,2} = \mathrm{span}\langle b_1^*, \ldots, b_k^* \rangle$, the chain satisfies property 2SP. Finally, from $w \neq 0$ we have that $c_{i_0 jk} \neq 0$ for some i_0 index, thus $d_{i_0 - 1} \neq 0$ and the chain satisfies NNP. Next, observe that from Proposition 4.1.10, the chain described in (c) ends up in a quadratic Lie algebra (A_n, f_n) such that

$$[b_i, b_j]_n(b_k) = D_{ijk} \quad \text{and} \quad [\beta, \cdot]_n = 0 \quad \forall\, \beta \in B^*,$$

where scalars D_{ijk} are defined in that lemma. But $[b_i, b_j]_A(b_k) = c_{ijk}$ and for $j, k < i$, with $j \neq k$ we have

$$c_{ijk} = f_{i-1}(d_{i-1}(b_j), b_k) \underset{j<k<i}{=} D_{jki} = \mathrm{sgn}((i\,j\,k))D_{ijk} = D_{ijk},$$
$$c_{ijk} = f_{i-1}(d_{i-1}(b_j), b_k) \underset{k<j<i}{=} D_{jki} = \mathrm{sgn}((j\,k\,i))D_{ijk} = D_{ijk}.$$

Now, from w being cyclic and skew we get $c_{ijk} = \mathrm{sgn}(\sigma) c_{\sigma(i)\sigma(j)\sigma(k)}$ for every permutation σ and $c_{iik} = 0$. So we have $[b_i, b_j]_A(b_k) = c_{ijk} = D_{ijk} = [b_i, b_j]_n(b_k)$. Hence, $[\cdot, \cdot]_n = [\cdot, \cdot]_A$ and $(\mathfrak{L}, \phi) = (A_n, f_n)$ as quadratic Lie algebras.

To prove that item (c) implies item (d) just apply Proposition 4.1.10 taking into account that $\mathrm{sgn}(\sigma) D_{\sigma(i)\sigma(j)\sigma(k)} = D_{ijk} = c_{ijk}$ and $[b_i, b_j]_n = \sum_{k=1}^n c_{ijk} b_k^*$. So, the entry c_{ijk} of every matrix M_i described in item (d) is the entry in the position $(n + k, j)$ of the matrix of the inner derivation $\mathrm{ad}\, b_i$. This proves the matrix family is n-quadratic. Finally, the definition of n-quadratic family yields to w being a non-zero cyclic 2-cocycle. □

Note that, with this theorem, we can also check the reduced conditions required in each method are equivalent among them.

Once at this point, we are going to see that we can move easily between the three methods directly from their respective constructions previously given in this paper.

- In a chain $\{(A_k, f_k)\}_{k=1}^n$ of one-dimensional double extensions, we consider a basis $\{b_n, \ldots, b_1, b_1^*, \ldots, b_n^*\}$ as before. Then

$$[b_i, b_j] = \sum_{k=1}^n D_{ijk} e_k^*, \qquad [b_i^*, \cdot] = 0,$$
$$f_n(b_i, b_j) = f_n(b_i^*, b_j^*) = 0, \qquad f_n(b_i, b_j^*) = \delta_{ij}.$$

- In $(T^*_w B, q_B)$ with basis $\{e_1, \dots, e_n, e^*_1, \dots, e^*_n\}$

$$\begin{aligned} [e_i, e_j] = w(e_i, e_j) = \sum_{k=1}^{n} w_{ijk} e^*_k, \qquad & [e^*_i, \cdot\,] = 0, \\ q_B(e_i, e_j) = q_B(e^*_i, e^*_j) = 0, \qquad & q_B(e_i, e^*_j) = \delta_{ij}. \end{aligned} \tag{4.11}$$

- An n-quadratic family $\{M_1, \dots, M_n\}$ where m_{ijk} is the entry (j,k) of M_i defines over the basis $\{v_1, \dots, v_n, z_1, \dots, z_n\}$ of A the quadratic Lie algebra (A, φ) where

$$\begin{aligned} [v_i, v_j] = \sum_{k=1}^{n} m_{ijk} z_k, \qquad & [z_i, \cdot\,] = 0, \\ \varphi(v_i, v_j) = \varphi(z_i, z_j) = 0, \qquad & \varphi(v_i, z_j) = \delta_{ij}. \end{aligned}$$

So the equivalence comes from just renaming

$$\begin{array}{ccccc} b_i & \longleftrightarrow & e_i & \longleftrightarrow & v_i, \\ b^*_i & \longleftrightarrow & e^*_i & \longleftrightarrow & z_i, \\ f_n & \longleftrightarrow & q_B & \longleftrightarrow & f, \\ D_{ijk} & \longleftrightarrow & w_{ijk} & \longleftrightarrow & m_{ijk}. \end{array}$$

Therefore, if we have a n-quadratic family of matrices with coefficients m_{ijk} we can define the equivalent $(T^*_w B, q_B)$ extension taking

$$w(b_i, b_j)(b_k) = w_{ijk} = m_{ijk}.$$

And we can also obtain a chain of one-dimensional double extensions taking

$$\begin{aligned} d_{i-1} \colon A_{i-1} &\to A_{i-1} \\ b_j &\mapsto \sum_{k=1}^{i-1} w_{ijk}\, b^*_k = \sum_{k=1}^{i-1} w(e_i, e_j)(e_k)\, b^*_k, \\ b^*_j &\mapsto 0. \end{aligned}$$

And vice versa, if we have built a chain, we can get the equivalent $T^*_w B$ if we take

$$w(e_i, e_j) = \sum_{k=1}^{n} D_{ijk}\, e^*_k = \sum_{k=1}^{n} f_n([b_i, b_j], b_k)\, e^*_k.$$

And, this also defines our n-quadratic family of matrices taking $m_{ijk} = D_{ijk}$.

Example 4.2.1. When we try to generate a generic 2-step quadratic Lie algebra of dimension n we end up with $n(n-2)(n-4)/48$ parameters, a number that grows pretty fast. Even in the 10-dimensional algebra we have 10 parameters in its general form, despite all of them are isometrically isomorphic. These parameters can be observed in any of the three equivalent constructions. For example, when constructing a 2SP and NNP chain of one-dimensional double extensions, we obtain the following derivations: $d_1 = 0_{2\times 2}$,

$$d_2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 2} \\ \hline -D_{123} & 0 & \\ 0 & D_{123} & 0_{2\times 2} \end{array}\right), d_3 = \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{0_{3\times 3}} & 0_{3\times 3} \\ \hline -D_{134} & -D_{124} & 0 & \\ -D_{234} & 0 & D_{124} & 0_{3\times 3} \\ 0 & D_{234} & D_{134} & \end{array}\right),$$

$$d_4 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{4\times 4}} & 0_{4\times 4} \\ \hline -D_{145} & -D_{135} & -D_{125} & 0 & \\ -D_{245} & -D_{235} & 0 & D_{125} & \\ -D_{345} & 0 & D_{235} & D_{135} & 0_{4\times 4} \\ 0 & D_{345} & D_{245} & D_{145} & \end{array}\right).$$

These same parameters appear in T^*-extensions or in a more condensed way in 5-quadratic families, where $\mathfrak{F}(M_1, \ldots, M_5)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & m_{123} & m_{124} & m_{125} & m_{134} & m_{135} & m_{145} \\ 0 & -m_{123} & -m_{124} & -m_{125} & 0 & 0 & 0 & m_{234} & m_{235} & m_{245} \\ m_{123} & 0 & -m_{134} & -m_{135} & 0 & -m_{234} & -m_{235} & 0 & 0 & m_{345} \\ m_{124} & m_{134} & 0 & -m_{145} & m_{234} & 0 & -m_{245} & 0 & -m_{345} & 0 \\ m_{125} & m_{135} & m_{145} & 0 & m_{235} & m_{245} & 0 & m_{345} & 0 & 0 \end{pmatrix}$$

All this complexity in terms of classification can be reduced using the next section. For instance, we will see all algebras in this example are isometrically isomorphic to the one where $D_{123} = D_{145} = 1$ or $m_{123} = m_{145} = 1$ and the rest of the entries are zero, named as $\mathfrak{L}_{5,1}$ in the following section.

4.3 Trivectors and 2-step classification

In this section, we follow the main ideas given in [Noui and Revoy, 1997, Section 3]. For basic notions on multilinear algebra see [Fulton and Harris, 1991, Appendix B].

The classification of quadratic 2-step Lie algebras of dimension $2n$ can be reduced to that of trilinear alternating forms or trivectors over a vector space V of dimension n. In this section we will explain *why and how* under the scope of previous construction methods of quadratic algebras. We point out that, whereas the problem of classifying bilinear alternating forms is elemental, the classification of trivectors seems tractable only for small values of n.

Let $\{e_1, \ldots, e_n\}$ be a basis of V. The exterior power $\Lambda^m V$ or $\mathrm{Alt}^m V$ is a vector space associated to a universal alternating multilinear form

$$\begin{aligned} \wedge\colon V \times \cdots \times V &\to \Lambda^m V \\ (v_1, \ldots, v_m) &\mapsto v_1 \wedge \ldots \wedge v_m. \end{aligned}$$

The dimension of $\Lambda^m V$ is $\binom{n}{m}$, and $\{e_{i_1} \wedge \cdots \wedge e_{i_m} : 1 \leq i_1 < \ldots < i_m \leq n\}$ is its standard basis. Every element of $\Lambda^m V$ is called a m-vector. So a *trivector* is simply an element of $\Lambda^3 V$. Thus, every trivector can be expressed as a linear combination of their corresponding basis $\{e_i \wedge e_j \wedge e_k : 1 \leq i < j < k \leq 3\}$.

If V^* is the dual space of V, $\varphi_i \in V^*, v_i \in V$, the map $\iota\colon \Lambda^m V^* \to (\Lambda^m V)^*$ given explicitly as

$$(\varphi_1 \wedge \ldots \wedge \varphi_m) \mapsto \left(v_1 \wedge \ldots \wedge v_m \mapsto \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} \varphi_{\sigma(i)}(v_i) = \det\left(\varphi_j(v_i)\right) \right)$$

is an isomorphism. The elements of $(\Lambda^m V)^*$ are named m-alternating forms or m-forms. We also note that ι^{-1} sends the linear form $(e_{i_1} \wedge \ldots \wedge e_{i_m})^*$ back into $e_{i_1}^* \wedge \ldots \wedge e_{i_m}^*$. Since $(\Lambda^m V)^*$ is isomorphic to $\Lambda^m V$, there is no difference between m-vectors and m-forms.

Now that we know what a trivector is, we can see its relationship with reduced quadratic 2-step Lie algebras. In order to see it, we can make use of T^*-extensions of abelian Lie algebras as mentioned in Corollary 4.1.15. Here, every algebra $T_w^* B$ obtained of the same dimension differs only in the mapping $w\colon B \times B \to B^*$, as B is an abelian algebra and the bilinear form is defined in the same way. At this point, we can define $\phi_w\colon \Lambda^3 B \to \mathbb{F}$ taking $\phi_w(b_1, b_2, b_3) = w(b_1, b_2)(b_3)$. Note $\phi_w \in (\Lambda^3 B)^* \cong \Lambda^3 B^*$ is a trivector thanks to the bilinear map w is cyclic, so satisfies equation (2.20), and skew-symmetric (see Remark 2.2.21). In fact, the set $\{w_{ijk} = w(e_i, e_j)(e_k) : i < j < k\}$ seen in equation (4.11) is simply the coordinates of the trivector ϕ_w in the standard dual basis, $(e_i \wedge e_j \wedge e_k)^* \sim e_i^* \wedge e_j^* \wedge e_k^*$. Therefore, every quadratic

2-step Lie algebra can be defined from a trivector and also a quadratic 2-step Lie algebra gives us a trivector. In this way, we arrive at the bijection

$$\Delta\colon \{w \in Z^2(B, B^*) : w \text{ is cyclic}\} \to (\Lambda^3 B)^*, \quad w \mapsto \phi_w \tag{4.12}$$

given by the expression $w(e_i, e_j)(e_k) = \phi_w(e_i, e_j, e_k)$.

Even more, $\ker \phi_w = \{x \in B : \phi_w(x, \cdot, \cdot) = 0\} = \operatorname{Rad} w$. Thus Δ sends a non-degenerate w into a trivector ϕ_w such that $\ker \phi_w = 0$, and conversely. The nullity of $\ker \phi_w$ is equivalent to say that ϕ_w is a trivector of (maximal) rank equal to $\dim B$. Following [Cohen and Helminck, 1988], the *rank* of a trivector $\phi \in \Lambda^3 V$ is $\operatorname{rank} \phi = \dim V - \dim \ker \phi$. The rank of ϕ agrees with the dimension of the smallest subspace W of V such that $\phi \in \Lambda^3 W$ (see [Noui and Revoy, 1994, Section 1]).

But the important point is that not only a bijection between quadratic 2-step Lie algebras and trivectors exits. It is the fact the bijection maps isometrically isomorphic 2-step T^*-extensions into equivalent trivectors with respect to the natural equivalence relation given by the action of the general linear group (see Definition 4.3.1).

Definition 4.3.1. We say two trivectors ϕ_1, $\phi_2 \in \Lambda^3 V$ are *equivalent* if there exist $\sigma \in \mathrm{GL}(V)$ such that $\phi_1(x, y, z) = \phi_2(\sigma(x), \sigma(y), \sigma(z))$ for every $x, y, z \in V$. Hence $\sigma \cdot \phi_1 = \phi_2$, letting σ act on the trivectors by means of $(\sigma \cdot \phi)(x, y, z) = \phi(\sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(z))$.

Theorem 4.3.1. *Let B be a Lie algebra and B^* its coadjoint module, $w, w_1, w_2 \in Z^2(B, B^*)$ and cyclic. The map Δ defined in equation* (4.12) *is an involutive bijection satisfying the following properties:*

(a) *w is non-degenerate if and only if $\operatorname{rank}(\phi_w) = \dim B$.*

(b) *If B is abelian and w_1, w_2 are non-degenerate, $T^*_{w_1} B$ and $T^*_{w_2} B$ are isometrically isomorphic if and only if ϕ_{w_1} and ϕ_{w_2} are equivalent trivectors.*

Proof. For arbitrary B, Δ is well defined according to Remark 2.2.21. Thus item (a) follows from previous comments. Before proving item (b), we recall that Lie bracket of $T^*_w B = B \overset{w}{\oplus} B^*$ is given by $[a + \alpha, b + \beta]_w = w(a, b)$ if B is abelian and, from Corollary 4.1.14, $Z(T^*_w B) = B^*$ if w is non-degenerate. Hence, assuming B abelian and w_1, w_2 non-degenerate, for a given isometrically isomorphism φ from $T^*_{w_1} B$ onto $T^*_{w_2} B$, we have $\varphi(Z(T_{w_1})) = Z(T_{w_2}) =$

B^*, thus $T^*_{w_2}B = B^* \overset{w_2}{\oplus} \varphi(B) = B^* \overset{w_2}{\oplus} B$. This implies that $\sigma = \pi_B \circ \varphi \in \mathrm{GL}(B)$, where π_B is the projection map from $T^*_{w_2}B$ onto B. Then,

$$\begin{aligned}\phi_{w_1}(x, y, z) = w_1(x, y)(z) = f([x, y]_{w_1}, z) = f([\varphi(x), \varphi(y)]_{w_2}, \varphi(z)) = \\ f([\sigma(x), \sigma(y)]_{w_2}, \sigma(z)) = w_2(\sigma(x), \sigma(y)(\sigma(z)) = \phi_2(\sigma(x), \sigma(y), \sigma(z)).\end{aligned}$$

Thus ϕ_{w_1} and ϕ_{w_1} are equivalent. On the contrary, if $\sigma \in \mathrm{GL}(B)$ such that $\phi_{w_1}(x, y, z) = \phi_{w_2}(\sigma(x), \sigma(y), \sigma(z))$, $w_1(b_1, b_2) \circ \sigma^{-1} = w_2(\sigma(b_1), \sigma(b_2))$ follows easily, and the map

$$\begin{aligned}\varphi\colon T_{w_1}B = B \oplus B^* \to T_{w_2}B = B \oplus B^* \\ b + \beta \mapsto \sigma(b) + \beta \circ \sigma^{-1}\end{aligned}$$

is an isometric isomorphism. □

Remark 4.3.2. Item (b) in Theorem 4.3.1 extends to arbitrary fields of characteristic zero several results found in [Duong, 2013, Theorem 2]. Moreover, Theorem 4.3.1 also extends and reorganized other results in that same article.

Corollary 4.3.3. *The map* Δ *defined in equation* (4.12) *provides a natural bijection between isomorphism classes of reduced quadratic 2-step nilpotent Lie algebras of dimension* $2n$ *and the equivalence classes of trivectors of rank* n*.*

This result has been established in [Noui and Revoy, 1997, 3.5 Théorème] and it is quite useful as classification tables for trivectors are available. In order to simplify notation, from now on, trivector $e_i^* \wedge e_j^* \wedge e_k^*$ will be denoted as ijk. So $123 + 456 \longleftrightarrow e_1^* \wedge e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \wedge e_6^*$.

The fact each quadric reduced 2-step Lie algebra can be associated to a trivector and vice versa means their classification, thanks to Theorem 4.3.1, is equivalent to the trivectors one. This allows us to obtain a list of these algebras, as trivectors have been already classified for low dimensions. A nice classification up to dimension 9, over the complex field $\mathbb{C}$, appears in [Vinberg and Èlashvili, 1988] by using a $\mathbb{Z}_3$-grading of the simple Lie algebra $\mathfrak{e}_8$. Cohen and Helminck (see [Cohen and Helminck, 1988]) classify trivectors up to dimension 7 over fields of cohomological dimension at most 1, which includes algebraically closed fields and finite fields. Recently, Borovoi, De Graaf and Vân Lê published the classification of real trivectors of dimension 9 in [Borovoi et al., 2022].

Over the complex field, we can know how many reduced quadratic 2-step Lie algebras are there up to isometrically isomorphisms using less than 9 generators. This data is showed in Table 4.1, where the dimension $2n \leq 18$ of the Lie algebra is related to the rank of the corresponding trivector.

Dimension	6	8	10	12	14	16	≥ 18
Number	1	0	1	2	5	13	∞

Table 4.1: Non-isometric reduced quadratic 2-step Lie algebras in $\mathbb{C}$ (source [Vinberg and Èlashvili, 1988]).

Moreover, we are also able to give a representative of each of these algebras and find its multiplication table. Along the following list we consider a quadratic algebra (A, f) of dimension $2n$ with basis $\{e_1, \ldots, e_n, e_1^*, \ldots, e_n^*\}$, where $A^2 = \operatorname{span}\langle e_1^*, \ldots, e_n^*\rangle$, $f(e_i, e_j) = f(e_i^*, e_j^*) = 0$, and $f(e_i, e_j^*) = \delta_{ij}$. Each algebra receives a name of the form $\mathfrak{L}_{n,k}$, where n is the type, also half the dimension, and k is the position it occupies in the list among all algebras of the same type/dimension. In addition, to simplify the list we only show non-zero products of the form $[e_i, e_j]$ where $i < j$. According to the map Δ described in equation (4.12), the rule to display the different multiplication tables is given by the coordinates of the trivectors $\phi_w = \sum w_{ijk} e_i^* \wedge e_j^* \wedge e_k^*$, so

$$[e_i, e_j] = w_{ijk} e_k^* \iff \phi_w(e_i, e_j, e_k) = w_{ijk} = w(e_i, e_j)(e_k). \tag{4.13}$$

In this way, any 2-step quadratic reduced Lie algebras of dimension less than 17 over the complex field up to isometric isomorphisms is given in the following list:

- One 6-dimensional algebra:
  - Algebra $\mathfrak{L}_{3,1}$ associated to trivector 123:

$$[e_1, e_2] = e_3^*, \qquad [e_1, e_3] = -e_2^*, \qquad [e_2, e_3] = e_1^*.$$

- One 10-dimensional algebra:
  - Algebra $\mathfrak{L}_{5,1}$ associated to trivector $123 + 145$:

$$\begin{aligned} &[e_1, e_2] = e_3^*, && [e_1, e_3] = -e_2^*, && [e_1, e_4] = e_5^*, \\ &[e_1, e_5] = -e_4^*, && [e_2, e_3] = e_1^*, && [e_4, e_5] = e_1^*. \end{aligned}$$

- Two 12-dimensional algebras:
  - Algebra $\mathfrak{L}_{6,1}$ associated to trivector $123 + 456$:

$$\begin{aligned} &[e_1, e_2] = e_3^*, && [e_1, e_3] = -e_2^*, && [e_2, e_3] = e_1^*, \\ &[e_4, e_5] = e_6^*, && [e_4, e_6] = -e_5^*, && [e_5, e_6] = e_4^*. \end{aligned}$$

  - Algebra $\mathfrak{L}_{6,2}$ associated to trivector $124 + 135 + 236$:

$$\begin{aligned} &[e_1, e_2] = e_4^*, && [e_1, e_3] = e_5^*, && [e_1, e_4] = -e_2^*, \\ &[e_1, e_5] = -e_3^*, && [e_2, e_3] = e_6^*, && [e_2, e_4] = e_1^*, \\ &[e_2, e_6] = -e_3^*, && [e_3, e_5] = e_1^*, && [e_3, e_6] = e_2^*. \end{aligned}$$

- Five 14-dimensional algebras:
  - Algebra $\mathfrak{L}_{7,1}$ associated to trivector $123 + 145 + 167$:

$$\begin{aligned} &[e_1, e_2] = e_3^*, && [e_1, e_3] = -e_2^*, && [e_1, e_4] = e_5^*, \\ &[e_1, e_5] = -e_4^*, && [e_1, e_6] = e_7^*, && [e_1, e_7] = -e_6^*, \\ &[e_2, e_3] = e_1^*, && [e_4, e_5] = e_1^*, && [e_6, e_7] = e_1^*. \end{aligned}$$

  - Algebra $\mathfrak{L}_{7,2}$ associated to trivector $127 + 134 + 256$:

$$\begin{aligned} &[e_1, e_2] = e_7^*, && [e_1, e_3] = e_4^*, && [e_1, e_4] = -e_3^*, \\ &[e_1, e_7] = -e_2^*, && [e_2, e_5] = e_6^*, && [e_2, e_6] = -e_5^*, \\ &[e_2, e_7] = e_1^*, && [e_3, e_4] = e_1^*, && [e_5, e_6] = e_2^*. \end{aligned}$$

  - Algebra $\mathfrak{L}_{7,3}$ associated to trivector $125 + 136 + 147 + 234$:

$$\begin{aligned} &[e_1, e_2] = e_5^*, && [e_1, e_3] = e_6^*, && [e_1, e_4] = e_7^*, \\ &[e_1, e_5] = -e_2^*, && [e_1, e_6] = -e_3^*, && [e_1, e_7] = -e_4^*, \\ &[e_2, e_3] = e_4^*, && [e_2, e_4] = -e_3^*, && [e_2, e_5] = e_1^*, \\ &[e_3, e_4] = e_2^*, && [e_3, e_6] = e_1^*, && [e_4, e_7] = e_1^*. \end{aligned}$$

  - Algebra $\mathfrak{L}_{7,4}$ associated to trivector $125 + 137 + 247 + 346$:

$$\begin{aligned} &[e_1, e_2] = e_5^*, && [e_1, e_3] = e_7^*, && [e_1, e_5] = -e_2^*, \\ &[e_1, e_7] = -e_3^*, && [e_2, e_4] = e_7^*, && [e_2, e_5] = e_1^*, \\ &[e_2, e_7] = -e_4^*, && [e_3, e_4] = e_6^*, && [e_3, e_6] = -e_4^*, \\ &[e_3, e_7] = e_1^*, && [e_4, e_6] = e_3^*, && [e_4, e_7] = e_2^*. \end{aligned}$$

- Algebra $\mathfrak{L}_{7,5}$ associated to trivector $123 + 147 + 257 + 367 + 456$:

$$\begin{array}{lll} [e_1, e_2] = e_3^*, & [e_1, e_3] = -e_2^*, & [e_1, e_4] = e_7^*, \\ [e_1, e_7] = -e_4^*, & [e_2, e_3] = e_1^*, & [e_2, e_5] = e_7^*, \\ [e_2, e_7] = -e_5^*, & [e_3, e_6] = e_7^*, & [e_3, e_7] = -e_6^*, \\ [e_4, e_5] = e_6^*, & [e_4, e_6] = -e_5^*, & [e_4, e_7] = e_1^*, \\ [e_5, e_6] = e_4^*, & [e_5, e_7] = e_2^*, & [e_6, e_7] = e_3^*. \end{array}$$

- Thirteen 16-dimensional algebras:
  - Algebra $\mathfrak{L}_{8,1}$ associated to trivector $156 + 178 + 234$:

$$\begin{array}{lll} [e_1, e_5] = e_6^*, & [e_1, e_6] = -e_5^*, & [e_1, e_7] = e_8^*, \\ [e_1, e_8] = -e_7^*, & [e_2, e_3] = e_4^*, & [e_2, e_4] = -e_3^*, \\ [e_3, e_4] = e_2^*, & [e_5, e_6] = e_1^*, & [e_7, e_8] = e_1^*. \end{array}$$

  - Algebra $\mathfrak{L}_{8,2}$ associated to trivector $127 + 138 + 145 + 236$:

$$\begin{array}{lll} [e_1, e_2] = e_7^*, & [e_1, e_3] = e_8^*, & [e_1, e_4] = e_5^*, \\ [e_1, e_5] = -e_4^*, & [e_1, e_7] = -e_2^*, & [e_1, e_8] = -e_3^*, \\ [e_2, e_3] = e_6^*, & [e_2, e_6] = -e_3^*, & [e_2, e_7] = e_1^*, \\ [e_3, e_6] = e_2^*, & [e_3, e_8] = e_1^*, & [e_4, e_5] = e_1^*. \end{array}$$

  - Algebra $\mathfrak{L}_{8,3}$ associated to trivector $125 + 137 + 248 + 346$:

$$\begin{array}{lll} [e_1, e_2] = e_5^*, & [e_1, e_3] = e_7^*, & [e_1, e_5] = -e_2^*, \\ [e_1, e_7] = -e_3^*, & [e_2, e_4] = e_8^*, & [e_2, e_5] = e_1^*, \\ [e_2, e_8] = -e_4^*, & [e_3, e_4] = e_6^*, & [e_3, e_6] = -e_4^*, \\ [e_3, e_7] = e_1^*, & [e_4, e_6] = e_3^*, & [e_4, e_8] = e_2^*. \end{array}$$

  - Algebra $\mathfrak{L}_{8,4}$ associated to trivector $137 + 168 + 236 + 245$:

$$\begin{array}{lll} [e_1, e_3] = e_7^*, & [e_1, e_6] = e_8^*, & [e_1, e_7] = -e_3^*, \\ [e_1, e_8] = -e_6^*, & [e_2, e_3] = e_6^*, & [e_2, e_4] = e_5^*, \\ [e_2, e_5] = -e_4^*, & [e_2, e_6] = -e_3^*, & [e_3, e_6] = e_2^*, \\ [e_3, e_7] = e_1^*, & [e_4, e_5] = e_2^*, & [e_6, e_8] = e_1^*. \end{array}$$

– Algebra $\mathfrak{L}_{8,5}$ associated to trivector $134 + 178 + 256 + 278$:

$$
\begin{aligned}
&[e_1, e_3] = e_4^*, && [e_1, e_4] = -e_3^*, && [e_1, e_7] = e_8^*,\\
&[e_1, e_8] = -e_7^*, && [e_2, e_5] = e_6^*, && [e_2, e_6] = -e_5^*,\\
&[e_2, e_7] = e_8^*, && [e_2, e_8] = -e_7^*, && [e_3, e_4] = e_1^*,\\
&[e_5, e_6] = e_2^*, && [e_7, e_8] = e_1^* + e_2^*.
\end{aligned}
$$

– Algebra $\mathfrak{L}_{8,6}$ associated to trivector $128 + 135 + 147 + 237 + 246$:

$$
\begin{aligned}
&[e_1, e_2] = e_8^*, && [e_1, e_3] = e_5^*, && [e_1, e_4] = e_7^*,\\
&[e_1, e_5] = -e_3^*, && [e_1, e_7] = -e_4^*, && [e_1, e_8] = -e_2^*,\\
&[e_2, e_3] = e_7^*, && [e_2, e_4] = e_6^*, && [e_2, e_6] = -e_4^*,\\
&[e_2, e_7] = -e_3^*, && [e_2, e_8] = e_1^*, && [e_3, e_5] = e_1^*,\\
&[e_3, e_7] = e_2^*, && [e_4, e_6] = e_2^*, && [e_4, e_7] = e_1^*.
\end{aligned}
$$

– Algebra $\mathfrak{L}_{8,7}$ associated to trivector $127 + 138 + 156 + 246 + 345$:

$$
\begin{aligned}
&[e_1, e_2] = e_7^*, && [e_1, e_3] = e_8^*, && [e_1, e_5] = e_6^*,\\
&[e_1, e_6] = -e_5^*, && [e_1, e_7] = -e_2^*, && [e_1, e_8] = -e_3^*,\\
&[e_2, e_4] = e_6^*, && [e_2, e_6] = -e_4^*, && [e_2, e_7] = e_1^*,\\
&[e_3, e_4] = e_5^*, && [e_3, e_5] = -e_4^*, && [e_3, e_8] = e_1^*,\\
&[e_4, e_5] = e_3^*, && [e_4, e_6] = e_2^*, && [e_5, e_6] = e_1^*.
\end{aligned}
$$

– Algebra $\mathfrak{L}_{8,8}$ associated to trivector $136 + 158 + 247 + 258 + 345$:

$$
\begin{aligned}
&[e_1, e_3] = e_6^*, && [e_1, e_5] = e_8^*, && [e_1, e_6] = -e_3^*,\\
&[e_1, e_8] = -e_5^*, && [e_2, e_4] = e_7^*, && [e_2, e_5] = e_8^*,\\
&[e_2, e_7] = -e_4^*, && [e_2, e_8] = -e_5^*, && [e_3, e_4] = e_5^*,\\
&[e_3, e_5] = -e_4^*, && [e_3, e_6] = e_1^*, && [e_4, e_5] = e_3^*,\\
&[e_4, e_7] = e_2^*, && [e_5, e_8] = e_1^* + e_2^*.
\end{aligned}
$$

– Algebra $\mathfrak{L}_{8,9}$ associated to trivector $145 + 167 + 238 + 246 + 357$:

$$
\begin{aligned}
&[e_1, e_4] = e_5^*, && [e_1, e_5] = -e_4^*, && [e_1, e_6] = e_7^*,\\
&[e_1, e_7] = -e_6^*, && [e_2, e_3] = e_8^*, && [e_2, e_4] = e_6^*,\\
&[e_2, e_6] = -e_4^*, && [e_2, e_8] = -e_3^*, && [e_3, e_5] = e_7^*,\\
&[e_3, e_7] = -e_5^*, && [e_3, e_8] = e_2^*, && [e_4, e_5] = e_1^*,\\
&[e_4, e_6] = e_2^*, && [e_5, e_7] = e_3^*, && [e_6, e_7] = e_1^*.
\end{aligned}
$$

– Algebra $\mathfrak{L}_{8,10}$ associated to trivector $128 + 167 + 236 + 247 + 345$:

$$\begin{aligned}
&[e_1, e_2] = e_8^*, && [e_1, e_6] = e_7^*, && [e_1, e_7] = -e_6^*,\\
&[e_1, e_8] = -e_2^*, && [e_2, e_3] = e_6^*, && [e_2, e_4] = e_7^*,\\
&[e_2, e_6] = -e_3^*, && [e_2, e_7] = -e_4^*, && [e_2, e_8] = e_1^*,\\
&[e_3, e_4] = e_5^*, && [e_3, e_5] = -e_4^*, && [e_3, e_6] = e_2^*,\\
&[e_4, e_5] = e_3^*, && [e_4, e_7] = e_2^*, && [e_6, e_7] = e_1^*.
\end{aligned}$$

– Algebra $\mathfrak{L}_{8,11}$ associated to trivector $128+136+157+247+256+345$:

$$\begin{aligned}
&[e_1, e_2] = e_8^*, && [e_1, e_3] = e_6^*, && [e_1, e_5] = e_7^*,\\
&[e_1, e_6] = -e_3^*, && [e_1, e_7] = -e_5^*, && [e_1, e_8] = -e_2^*,\\
&[e_2, e_4] = e_7^*, && [e_2, e_5] = e_6^*, && [e_2, e_6] = -e_5^*,\\
&[e_2, e_7] = -e_4^*, && [e_2, e_8] = e_1^*, && [e_3, e_4] = e_5^*,\\
&[e_3, e_5] = -e_4^*, && [e_3, e_6] = e_1^*, && [e_4, e_5] = e_3^*,\\
&[e_4, e_7] = e_2^*, && [e_5, e_6] = e_2^*, && [e_5, e_7] = e_1^*.
\end{aligned}$$

– Algebra $\mathfrak{L}_{8,12}$ associated to trivector $126+158+238+257+347+456$:

$$\begin{aligned}
&[e_1, e_2] = e_6^*, && [e_1, e_5] = e_8^*, && [e_1, e_6] = -e_2^*,\\
&[e_1, e_8] = -e_5^*, && [e_2, e_3] = e_8^*, && [e_2, e_5] = e_7^*,\\
&[e_2, e_6] = e_1^*, && [e_2, e_7] = -e_5^*, && [e_2, e_8] = -e_3^*,\\
&[e_3, e_4] = e_7^*, && [e_3, e_7] = -e_4^*, && [e_3, e_8] = e_2^*,\\
&[e_4, e_5] = e_6^*, && [e_4, e_6] = -e_5^*, && [e_4, e_7] = e_3^*,\\
&[e_5, e_6] = e_4^*, && [e_5, e_7] = e_2^*, && [e_5, e_8] = e_1^*.
\end{aligned}$$

– Algebra $\mathfrak{L}_{8,13}$ associated to trivector $123+178+257+368+456+478$:

$$\begin{aligned}
&[e_1, e_2] = e_3^*, && [e_1, e_3] = -e_2^*, && [e_1, e_7] = e_8^*,\\
&[e_1, e_8] = -e_7^*, && [e_2, e_3] = e_1^*, && [e_2, e_5] = e_7^*,\\
&[e_2, e_7] = -e_5^*, && [e_3, e_6] = e_8^*, && [e_3, e_8] = -e_6^*,\\
&[e_4, e_5] = e_6^*, && [e_4, e_6] = -e_5^*, && [e_4, e_7] = e_8^*,\\
&[e_4, e_8] = -e_7^*, && [e_5, e_6] = e_4^*, && [e_5, e_7] = e_2^*,\\
&[e_6, e_8] = e_3^*, && [e_7, e_8] = e_1^* + e_4^*.
\end{aligned}$$

However, despite there are an infinite number of non-isometrically isomorphic 2-step quadratic Lie algebras of dimension greater or equal than 18, for the 18-dimensional algebras we still have a classification based on seven families of trivectors, where each family depends on some parameters. This classification can be found in [Vinberg and Èlashvili, 1988]. Here, the authors explain how every trivector can be decompose as a sum of a semisimple trivector and a nilpotent one. Details of this concepts can be found in §1 of the paper. That semisimple part is a linear combination of four special trivectors, and it is that specific combination in addition to the nilpotent part what defines each family. Despite this classification involves parameters λ_i, this does not affect our conversion to 2-step quadratic Lie algebras, and the procedure described in (4.13) is still functional. We can see it in the following example.

Example 4.3.1. Let us take a trivector u in the sixth family, which decomposes as $u = p + e$ with $p = \lambda(123 + 456 + 789) \neq 0$, where λ is determined up to multiplication by a sixth root of the unity, and e is in table [Vinberg and Èlashvili, 1988, Table 5]. For example, we consider $e = 147 + 158$. So our trivector is

$$u = \lambda(123 + 456 + 789) + 147 + 158.$$

In this case the associated 18-dimensional Lie algebra $\mathfrak{L}$ is defined by products

$$\begin{array}{llll}
[e_1, e_2] = \lambda e_3^*, & [e_1, e_3] = -\lambda e_2^*, & [e_1, e_4] = e_7^*, & [e_1, e_5] = e_8^*, \\
[e_1, e_7] = -e_4^*, & [e_1, e_8] = -e_5^*, & [e_2, e_3] = \lambda e_1^*, & [e_4, e_5] = \lambda e_6^*, \\
[e_4, e_6] = -\lambda e_5^*, & [e_4, e_7] = e_1^*, & [e_5, e_6] = \lambda e_4^*, & [e_5, e_8] = e_1^*, \\
[e_7, e_8] = \lambda e_9^*, & [e_7, e_9] = -\lambda e_8^*, & [e_8, e_9] = \lambda e_7^*, &
\end{array}$$

when considering the basis $\{e_1, \ldots, e_9, e_1^*, \ldots, e_9^*\}$.

Remark 4.3.4. The idea of a nilpotent or semisimple trivector does not affect the nilpotency of the algebra obtained from it. By construction, the quadratic algebra we obtain is always 2-step.

Remark 4.3.5. In case we are interested in greater dimensions, things start to get much more difficult as there are not a finite number of trivectors-algebras. We can get bigger examples by applying Theorem 3.2.2 or Proposition 4.1.10. These results provide computational constructions based on non-degenerate n-quadratic matrices or chained one-dimensional double extensions.

4.4 Summary

In this chapter we have seen how the three methods: double-extension, T^*-extension and n-quadratic families apply over two-step nilpotent Lie algebras. While the last method was originally developed specially for them, the other two can be simplified when considered over this specific subfamily. For the double extension process, we can take successive one-dimensional double extensions with some restrictions on the derivations chain (NNP and 2SP from Definition 4.1.2) to build these algebras. Starting with the trivial quadratic algebra $(A_0 = 0, \varphi_0 = 0)$, Subsection 4.1.1 includes, as its main tool, a general multi-step metabolic extension process. This method appears described in Proposition 4.1.10 and visually represented in Figure 4.1. For the T^*-extension, we can limit ourselves to extension of abelian Lie algebras or, equivalently, simply vector spaces. This restriction is explained in Corollary 4.1.15, which extends to characteristic zero a result from [Duong, 2013].

Once the three methods have been adapted to our goal, the equivalence Theorem 4.2.1 appears. This result gives us 2-step quadratic Lie algebras by structure constants encoded in cyclic 2-cocycles and provides a simple rule for switching from one construction method to another.

In the final section, the bijective map described in equation (4.12) yields to the explicit relationship between cyclic 2-cocycles and trivectors. This bijection, introduced in [Noui and Revoy, 1997], leads to the bijection up to isomorphisms of reduced quadratic $2n$-dimensional 2-step Lie algebras and n-rank trivectors up to equivalence given in Corollary 4.3.3. This result is an immediate consequence of our Theorem 4.3.1. As an easy application, we list the 22 non-isometrically isomorphic reduced quadratic 2-step Lie algebras up to dimension 17 over the complex field. This extends the classification of these algebras which was, as far as we know and we mentioned in the previous chapter, unknown.

Tools and patterns

5

CHAPTER

Up to this point, in this dissertation, we have continuously reduced our field of study in order to obtain low dimensional classifications of quadratic 2-step Lie algebras. Now, it is time to grow in dimension producing extensions of those low dimensional Lie algebras and obtain solvable and mixed algebras. The first section of this chapter is devoted to the study of derivations, automorphisms and bilinear forms in nilpotent Lie algebras obtained via quotients of free nilpotent ones. This part is based on [Benito and Roldán-López, 2020] and ends up mentioning the classification given in [Benito et al., 2017].

In Section 5.2, we are going to see the structure of local Lie algebras (only one proper maximal ideal). Among them, we find oscillator algebras which produce an infinite family of solvable quadratic Lie algebras. All these algebras can be double extended using its derivations to produce mixed Lie algebras. This section follows the article [Benito and Roldán-López, 2023b] and the paper [Benito and Roldán-López, 2022c].

In the final Section 5.3, as a continuation of the ideal decomposition at the beginning of Chapter 3, we will end up seeing in detail and completely which ideal structure must a quadratic Lie algebra follow, not limiting in just some special ideals as before. These ideas, when focused on algebras with a finite number of ideals, have been developed in detail and it is published in [Benito and Roldán-López, 2022a]. Only half of this paper supports this chapter. All these restrictions greatly limit which lattices of ideals are valid candidates. Therefore, we will end this section studying the particular case in which all ideals form a chain. These algebras, as observed in the [Benito and Roldán-López, 2022b] which is indeed the preprint which supports the last part of this section, can be algorithmically obtained and among all them, just a few are quadratic.

5.1 Derivations, automorphisms and bilinear forms

As seen just before in Definition 2.1.27, $\mathfrak{n}_{d,t}$ represents the free nilpotent Lie algebra of type d and nilindex t. Starting out with the derivation algebra and the automorphism group of $\mathfrak{n}_{d,t}$, we get a natural description of derivations and automorphisms of any generic nilpotent Lie algebra of the same type and nilindex.

In the middle of the 20th century, the study of derivations and automorphisms of algebras was a central topic of research. It is well known that many linear algebraic Lie groups and their Lie algebras arise from the automorphism groups and the derivation algebras of certain non-associative algebras (see [McCrimmon, 2004]). In fact, for a given finite-dimensional real non-associative algebra A, the automorphism group $\operatorname{Aut} A$ is a closed Lie subgroup of the lineal group $\mathrm{GL}(A)$ and the derivation algebra $\operatorname{Der} A$ is the Lie algebra associated to $\operatorname{Aut} A$ (see [Sagle and Walde, 1973, Proposition 7.1 and 7.3, Chapter 7]).

Paying attention to Lie algebras, a lot of research papers on this topic are devoted to the study of the interplay between the structures of their derivation algebras, their groups of automorphisms and Lie algebras themselves (see [Varea and Varea, 2006] and references therein). Among them, we point out two simple but elegant results on this direction. According to [Borel and Serre, 1953], any Lie algebra that has an automorphism of prime period without non-zero fixed points is nilpotent. The same result is valid in case the Lie algebra has a non-singular derivation (see [Jacobson, 1955, Theorem 2]). So, automorphisms and derivations and the nature of their elements are interesting tools in the study of structural properties of general Lie algebras. Even more, according to Theorem 2.2.20, derivations are needed to make double extensions.

The deconstruction process in Section 3.1 lead us to pay attention to derivations and automorphisms of nilpotent quadratic Lie algebras as important tools on constructions and isomorphisms. These objects are the main motif of the first two sections, which describe the group of automorphisms and the algebra of derivations of any finite-dimensional t-step nilpotent Lie algebra $\mathfrak{n}$ generated by a m.s.g. U of d elements. The description will be given through the derivation algebra and the automorphism group of the free t-step nilpotent Lie algebra $\mathfrak{n}_{d,t}$ generated by U. Denoting by $\mathfrak{u} = \mathrm{span}\langle U\rangle$, the elements of the derivation algebra, $\mathrm{Der}\,\mathfrak{n}_{d,t}$, arise by extending and combining, in a natural way, linear maps from $\mathfrak{u}$ into $\mathfrak{u}$ and from $\mathfrak{u}$ to $\mathfrak{n}^2_{d,t}$. The group of automorphisms, $\mathrm{Aut}\,\mathfrak{n}_{d,t}$, is described through automorphism induced by elements of the general linear group $\mathtt{GL}(\mathfrak{u})$ and automorphisms provide by linear maps from $\mathfrak{u}$ to $\mathfrak{n}_{d,t}$ which induce the identity mapping on $\dfrac{\mathfrak{n}_{d,t}}{\mathfrak{n}^2_{d,t}}$.

Some comments on bilinear forms are included at the end of this section. This is the tool used in the classification of nilpotent quadratic Lie algebras given in in [Benito et al., 2017] and mentioned in Section 3.1.3 and listed in Table 3.1.

5.1.1 Linear maps and extensions

Recovering Definition 2.1.27 and thanks to UMP seeing in Proposition 2.1.9, any nilpotent Lie algebra can be seen as a quotient of some free nilpotent Lie algebra $\mathfrak{n}_{d,t}$. In addition, derivations and automorphisms of $\mathfrak{n}_{d,t}$ are completely determined by their effect on $\mathfrak{u}$. Conversely, any linear map from $\mathfrak{u}$ into $\mathfrak{n}_{d,t}$ (bijection from a basis of $\mathfrak{u}$ to any m.s.g.) determines a unique derivation (automorphism) of $\mathfrak{n}_{d,t}$. This assertion is covered by the next result and its corollary. A detailed proof can be found in [Satô, 1971, Propositions 2 and 3].

Proposition 5.1.1. *Let φ be any linear map from vector space $\mathfrak{u} = \mathrm{span}\langle x_1, \ldots, x_d\rangle$ into $\mathfrak{n}_{d,t}$, where $\{x_1, \ldots, x_d\}$ is a m.s.g. of $\mathfrak{n}_{d,t}$. Then:*

(a) *φ extends to a derivation of $\mathfrak{n}_{d,t}$ by declaring*

$$d_\varphi([x_{\alpha_1}, \ldots, x_{\alpha_r}]) = \sum_{1\le i\le r} [x_{\alpha_1}, \ldots, \varphi(x_{\alpha_i}), \ldots, x_{\alpha_r}].$$

(b) *φ extends to an algebra homomorphism of $\mathfrak{n}_{d,t}$ by declaring*

$$\Phi_\varphi([x_{\alpha_1}, \ldots, x_{\alpha_r}]) = [\varphi(x_{\alpha_1}), \ldots, \varphi(x_{\alpha_r})].$$

Moreover if $\operatorname{proj}_{\mathfrak{u}}$ *stands for the projection map from $\mathfrak{n}_{d,t}$ into $\mathfrak{u}$, we have Φ_φ is an automorphism if and only if* $\{\operatorname{proj}_{\mathfrak{u}}(\varphi(x_1)), \ldots, \operatorname{proj}_{\mathfrak{u}}(\varphi(x_n))\}$ *is a linearly independent set.*

Corollary 5.1.2. *Let $\mathfrak{n}_{d,t}$ be the free t-nilpotent Lie algebra on d-generators $\mathfrak{m} = \{x_1, \ldots, x_d\}$ and $\mathfrak{u} = \operatorname{span}\langle\mathfrak{m}\rangle$. The derivation algebra and the automorphism group of $\mathfrak{n}_{d,t}$ are described as* $\operatorname{Der}\mathfrak{n}_{d,t} = \{d_\varphi : \varphi \in \operatorname{Hom}(\mathfrak{u}, \mathfrak{n}_{d,t})\}$ *and* $\operatorname{Aut}\mathfrak{n}_{d,t} = \{\Phi_\varphi : \varphi \in \operatorname{Hom}(\mathfrak{u}, \mathfrak{n}_{d,t})$ *and* $\{\operatorname{proj}_{\mathfrak{u}}(\varphi(x_1)), \ldots, \operatorname{proj}_{\mathfrak{u}}(\varphi(x_d))\}$ *m.s.g.*$\}$.

Remark 5.1.3. The Levi factor $\mathfrak{S}_{d,t}$ of $\operatorname{Der}\mathfrak{n}_{d,t}$ is given by the maps d_φ for $\varphi \in \mathfrak{sl}(\mathfrak{u})$. Clearly, $\mathfrak{S}_{d,t}$ is isomorphic to the special lineal Lie algebra $\mathfrak{sl}_d(\mathbb{F})$. Elements of the nilpotent radical $\mathfrak{N}_{d,t}$ are linear maps d_φ where $\varphi \in \operatorname{Hom}(\mathfrak{u}, \mathfrak{n}^2_{d,t})$. And the solvable radical is just $\mathfrak{R}_{d,t} = k \cdot \operatorname{Id}_{d,t} \oplus \mathfrak{N}_{d,t}$ where $\operatorname{Id}_{d,t}(a_s) = s \cdot a_k$ for any $a_s \in \mathfrak{u}^s$ (see [Benito and de-la-Concepción, 2014, Proposition 2.4]).

Remark 5.1.4. The group $\operatorname{Aut}\mathfrak{n}_{d,t}$ is the semidirect product of the general linear group $\operatorname{GL}(d,t)$, obtained from the automorphisms Φ_φ where $\varphi \in \operatorname{GL}(\mathfrak{u})$, and the nilpotent group $\operatorname{NL}(d,t)$, whose elements are Φ_σ and $\sigma = \operatorname{Id}_{\mathfrak{u}} + \delta$ where $\delta \in \operatorname{End}(\mathfrak{u}, n^2_{d,t})$ (see [Benito et al., 2017, Proposition 3.1]).

For any ideal I of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq I \subseteq \mathfrak{n}^2_{d,t}$, let denote by $\operatorname{Der}_I \mathfrak{n}_{d,t}$ and $\operatorname{Der}_{\mathfrak{n}_{d,t},I}\mathfrak{n}_{d,t}$ the subset of derivations which map I into itself, and $\mathfrak{n}_{d,t}$ into I respectively. Both sets are subalgebras of $\operatorname{Der}\mathfrak{n}_{d,t}$, even more, $\operatorname{Der}_{\mathfrak{n}_{d,t},I}\mathfrak{n}_{d,t}$ is an ideal inside $\operatorname{Der}_I \mathfrak{n}_{d,t}$, and the following result comes from [Satô, 1971, Proposition 5]:

Theorem 5.1.5. *Let I be an ideal of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq I \subseteq \mathfrak{n}^2_{d,t}$, the algebra of derivations of $\dfrac{\mathfrak{n}_{d,t}}{I}$ is isomorphic to $\dfrac{\operatorname{Der}_I \mathfrak{n}_{d,t}}{\operatorname{Der}_{\mathfrak{n}_{d,t},I}\mathfrak{n}_{d,t}}$, where $\operatorname{Der}_I \mathfrak{n}_{d,t}$ and $\operatorname{Der}_{\mathfrak{n}_{d,t},I}\mathfrak{n}_{d,t}$ maps I and $\mathfrak{n}_{d,t}$ into I respectively.*

In a similar vein to the previous theorem, it is possible to arrive at a structural description of automorphisms of homomorphic images of free nilpotent algebras. For any ideal I of $\mathfrak{n}_{d,t}$, $\mathfrak{n}^t_{d,t} \not\subseteq I \subseteq \mathfrak{n}^2_{d,t}$, let denote by $\operatorname{Aut}_I \mathfrak{n}_{d,t}$ the subset of automorphisms which map I into itself. It is easily checked that $\operatorname{Aut}_I \mathfrak{n}_{d,t}$ is a subgroup of $\operatorname{Aut}\mathfrak{n}_{d,t}$. Consider now the map

$$\theta\colon \operatorname{Aut}_I \mathfrak{n}_{d,t} \to \operatorname{Aut}\frac{\mathfrak{n}_{d,t}}{I}, \quad \theta(\Phi)(x+I) = \Phi(x) + I.$$

By using $\Phi(I) = I$ and Φ homomorphism, we can easily check that θ is well defined. Now, a straightforward computation shows that θ is a group homomorphism with kernel

$$\ker\theta = \{\Phi \in \operatorname{Aut}\mathfrak{n}_{d,t} : \operatorname{Im}(\Phi - \operatorname{Id}) \subseteq I\}.$$

Then, we have the following result:

Theorem 5.1.6. *For any ideal I of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}_{d,t}^t \not\subseteq I \subseteq \mathfrak{n}_{d,t}^2$, the set $\operatorname{Aut}_I^\circ \mathfrak{n}_{d,t} = \{\Phi \in \operatorname{Aut}\mathfrak{n}_{d,t} : \operatorname{Im}(\Phi - \operatorname{Id}) \subseteq I\}$ is a normal subgroup of the group of automorphisms of $\dfrac{\mathfrak{n}_{d,t}}{I}$. Moreover $\operatorname{Aut}\dfrac{\mathfrak{n}_{d,t}}{I}$ is isomorphic to $\dfrac{\operatorname{Aut}_I \mathfrak{n}_{d,t}}{\operatorname{Aut}_I^\circ \mathfrak{n}_{d,t}}$, where $\operatorname{Aut}_I \mathfrak{n}_{d,t}$ maps I into I.*

Proof. From previous comments, we only need to prove that the map θ is onto. Let $\rho_I \colon \mathfrak{n}_{d,t} \to \frac{\mathfrak{n}_{d,t}}{I}$ be the canonical projection and let $\{f_1 + I, \ldots, f_k + I\}$ be a basis of $\frac{\mathfrak{n}_{d,t}}{I}$ and $\{e_1 + I, \ldots, e_d + I\}$ a m.s.g. of $\frac{\mathfrak{n}_{d,t}}{I}$. Then $\{e_1, \ldots, e_d\}$ is also a m.s.g. of $\mathfrak{n}_{d,t}$. If we take a generic automorphism $\hat{A} \in \operatorname{Aut}\dfrac{\mathfrak{n}_{d,t}}{I}$,

$$\hat{A}(e_i + I) = \sum_{j=1}^{k} \alpha_{ij} f_j + I, \text{ and declare } A(e_i) = \sum_{j=1}^{k} \alpha_{ij} f_j,$$

A extends to a linear homomorphism, $A \colon \mathfrak{e} \to \mathfrak{n}_{d,t}$, where $\mathfrak{e} = \operatorname{span}\langle e_1, \ldots, e_d\rangle$. Let Φ_A be the homomorphism given by Proposition 5.1.1. We check that $\theta(\Phi_A) = \hat{A}$ noting that, for a generic element $[[\ldots[a_1, a_2], \ldots, a_l]$ where $a_i \in \mathfrak{e}$, up to linear combinations, we have

$$\begin{aligned}
\rho_I \circ \Phi_A[[\ldots[a_1, a_2], \ldots, a_l] &= [[\ldots[\rho_I \circ A(a_1), \rho_I \circ A(a_2)], \ldots, \rho_I \circ A(a_l)] \\
&= [[\ldots[\hat{A} \circ \rho(a_1), \hat{A} \circ \rho_I(a_2)], \ldots, \hat{A} \circ \rho_I(a_l)] \\
&= \hat{A} \circ \rho_I[[\ldots[a_1, a_2], \ldots, a_l].
\end{aligned}$$

The second equality follows because for every $a_i = \sum_{j=1}^{d} \beta_{ji} e_j$,

$$\begin{aligned}
\rho_I \circ A(a_i) &= \rho_I \circ A\left(\sum_{j=1}^{d} \beta_{ji} e_j\right) = \sum_{j=1}^{d} \beta_{ji} \rho_I \circ A(e_j) \\
&= \sum_{j=1}^{d} \beta_{ji} \rho_I \left(\sum_{l=1}^{k} \alpha_{jl} f_l\right) = \sum_{j=1}^{d} \beta_{ji} \sum_{l=1}^{k} (\alpha_{jl} f_l + I) = \sum_{j=1}^{d} \beta_{ji} \hat{A}(e_i + I) \\
&= \sum_{j=1}^{d} \beta_{ji} \hat{A} \circ \rho_I(e_i) = \hat{A} \circ \rho_I(a_i).
\end{aligned}$$

Now $\ker \rho_I = I$ and $\rho_I \circ \Phi_A = \hat{A} \circ \rho_I$ implies $\Phi_A(I) = I$ and, Φ_A automorphism, follows by using the equivalence given in Proposition 5.1.1 and the fact that $\hat{A}$ is an automorphism. □

5.1.2 Techniques and examples

Now we introduce several examples which illustrate some techniques (among other things):

1. The way to describe a generic d-generated t-nilpotent Lie algebra as an homomorphic image of $\mathfrak{n}_{d,t}$.
2. The way to compute automorphisms and derivations regarding Proposition 5.1.1 and Theorems 5.1.5 and 5.1.6.
3. The recognition of some structural patterns of nilpotent algebras depending on the nature of their derivations and automorphisms.

Some of these techniques, using Hall basis, have been computationally implemented in Section 6.2.2.

In the sequel, if a map φ is given in a matrix form $A = (a_{ij})$ attached to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$, then $\varphi(v_i) = \sum_{j=1}^{n} a_{ji} v_j$.

The UMP lets us describe any t-nilpotent Lie algebra $\mathfrak{n}$ of type d as a homomorphic image of $\mathfrak{n}_{d,t}$ in a easy way. From any m.s.g. $\{e_1, \dots, e_d\}$ of $\mathfrak{n}$, the correspondence $x_i \mapsto e_i$ for $i = 1, \dots, d$ extends uniquely to a surjective algebra homomorphism $\theta_{\mathfrak{n}} \colon \mathfrak{n}_{d,t} \to \mathfrak{n}$ and $\mathfrak{n} \cong \dfrac{\mathfrak{n}_{d,t}}{\ker \theta_{\mathfrak{n}}}$. We will compute ideals of this type in our following example.

Example 5.1.1. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the 8-dimensional and 5-dimensional Lie algebras described through the basis $\{e_1, \dots, e_8\}$ and $\{u_1, \dots, u_5\}$ by the following multiplication table ($[a, b] = -[b, a]$ and $[a, b] = 0$ is not in the table):

$$
\begin{array}{lll}
[e_1, e_2] = e_5, & [e_2, e_3] = e_8, & [e_3, e_5] = -e_7, \\
[e_1, e_3] = e_6, & [e_2, e_4] = e_6, & [e_4, e_6] = -e_8, \\
[e_1, e_4] = e_7, & [e_2, e_6] = -e_7, & [u_1, u_3] = u_5, \\
[e_1, e_5] = -e_8, & [e_3, e_4] = -e_5, & [u_2, u_4] = u_5,
\end{array}
$$

The lower central series of these algebras are:

$$\mathfrak{n}_1^2 = \operatorname{span}\langle e_5, e_6, e_7, e_8 \rangle, \quad \mathfrak{n}_1^3 = \operatorname{span}\langle e_7, e_8 \rangle, \quad \mathfrak{n}_1^4 = 0,$$

and

$$\mathfrak{n}_2^2 = \operatorname{span}\langle u_5 \rangle, \quad \mathfrak{n}_2^3 = 0.$$

Consider now the maps $\theta_{\mathfrak{n}_1} \colon x_i \to e_i$ for $i = 1, \ldots, 4$ from $\mathfrak{n}_{4,3}$ onto $\mathfrak{n}_1$ and $\theta_{\mathfrak{n}_2} \colon x_i \to u_i$ for $i = 1, \ldots, 4$ from $\mathfrak{n}_{4,2}$ onto $\mathfrak{n}_2$. Both correspondences extend to homomorphisms of algebras as in the proof in [Satô, 1971, Proposition 4] ($\theta[x_{\alpha_1} \ldots x_{\alpha_s}] = [\theta(x_{\alpha_1}) \ldots \theta(x_{\alpha_s})]$). It is not hard to see that:

$$\begin{aligned}
\ker \theta_{\mathfrak{n}_1} = \operatorname{span}\langle & [x_3, x_4] + [x_1, x_2], [x_2, x_4] - [x_1, x_3], [x_2, x_3] - [[x_1, x_3], x_1], \\
& [x_1, x_4] + [[x_1, x_2], x_2], [[x_3, x_4], x_4] - [[x_1, x_3], x_1], [[x_3, x_4], x_3], \\
& [[x_3, x_4], x_2] + [[x_1, x_2], x_2], [[x_3, x_4], x_1], [[x_2, x_4], x_4], [[x_1, x_4], x_2], \\
& [[x_2, x_4], x_3] + [[x_1, x_2], x_2], [[x_2, x_4], x_2], [[x_2, x_4], x_1] - [[x_1, x_3], x_1], \\
& [[x_2, x_3], x_3], [[x_2, x_3], x_2], [[x_2, x_3], x_1], [[x_1, x_4], x_4], [[x_1, x_4], x_3], \\
& [[x_1, x_4], x_1], [[x_1, x_3], x_3] + [[x_1, x_2], x_2], [[x_1, x_3], x_2], [[x_1, x_2], x_1]\rangle,
\end{aligned}$$

and

$$\ker \theta_{\mathfrak{n}_2} = \operatorname{span}\langle [x_3, x_4], [x_2, x_3], [x_1, x_4], [x_1, x_2], [x_1, x_3] - [x_2, x_4]\rangle.$$

We point out that $\ker \theta_{\mathfrak{n}_2}$ is an homogeneous ideal in the $\mathbb{N}$-graded structure of $\mathfrak{n}_{4,2}$ and $\ker \theta_{\mathfrak{n}_1}$ is not an homogeneous ideal of $\mathfrak{n}_{4,3}$. Therefore, $\mathfrak{n}_2$ inherits the grading of $\mathfrak{n}_{4,2}$ and is quasi-cyclic, but $\mathfrak{n}_1$ does not inherit that of $\mathfrak{n}_{4,3}$.

In 1955, N. Jacobson proved in [Jacobson, 1955, Theorem 3] that any Lie algebra of characteristic zero with a non-singular derivation is nilpotent. The author also noted that the validity of the converse was an open question. Two years later, J. Dixmier and W.G. Lister supplied in [Dixmier and Lister, 1957] a negative answer to the question by means of the algebra $\mathfrak{n}_1$ that we have revisited in Example 5.1.1. Every derivation of $\mathfrak{n}_1$ is nilpotent, so the elements of $\operatorname{Der} \mathfrak{n}_1$ are nilpotent maps, and therefore, $\operatorname{Der} \mathfrak{n}_1$ is a nilpotent Lie algebra. It can be also proved that $\operatorname{Aut} \mathfrak{n}_1$ is not a nilpotent group (see [Leger and Luks, 1972]). The existence of $\mathfrak{n}_1$ is the starting point of the study of the so called *characteristically nilpotent Lie algebras*, that is, Lie algebras in which any derivation is nilpotent. Over fields of characteristic zero, this class of algebras matches to the class of algebras in which every semisimple automorphism is of finite order (see [Leger and Tôgô, 1959, Theorem 3]) or the class of algebras in which the algebra of derivations is nilpotent (see [Leger and Tôgô, 1959, Theorem 1]).

Example 5.1.2. According to Proposition 5.1.1, all derivations and automorphisms of $\mathfrak{n}_{2,4}$ in Hall basis $\mathcal{H}_{2,4}$ can be easily obtained by iterating the Leibniz rule $\varphi([a,b]) = [\varphi(a),b])] + [a,\varphi(b)]$ and the law $\varphi([a,b]) = [\varphi(a),\varphi(b)]$. The matrices representing the elements of $\operatorname{Aut}\mathfrak{n}_{2,4} = \mathrm{GL}(2,4) \rtimes \mathrm{NL}(2,4)$ are product of matrices of the following shapes:

$$\left(\begin{array}{cc|c|cc|c} a_1 & a_2 & \multirow{2}{*}{$0_{2\times 1}$} & \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \multirow{2}{*}{$0_{2\times 3}$} \\ a_3 & a_4 & & & & \\ \hline \multicolumn{2}{c|}{0_{1\times 2}} & \epsilon & \multicolumn{2}{c|}{0_{1\times 2}} & 0_{1\times 3} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \multirow{2}{*}{$0_{2\times 1}$} & \epsilon a_1 & \epsilon a_2 & \multirow{2}{*}{$0_{2\times 3}$} \\ & & & \epsilon a_3 & \epsilon a_4 & \\ \hline \multicolumn{2}{c|}{0_{3\times 2}} & 0_{3\times 1} & \multicolumn{2}{c|}{0_{3\times 2}} & \epsilon\cdot A' \end{array}\right) \in \mathrm{GL}(2,4),$$

$$\left(\begin{array}{cc|c|cc|c} \multicolumn{2}{c|}{I_2} & 0_{2\times 1} & \multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 3} \\ \hline b_1 & b_2 & 1 & \multicolumn{2}{c|}{0_{1\times 2}} & 0_{1\times 3} \\ \hline c_1 & c_2 & b_2 & \multicolumn{2}{c|}{\multirow{2}{*}{I_2}} & \multirow{2}{*}{$0_{2\times 3}$} \\ c_3 & c_4 & -b_1 & & & \\ \hline d_1 & d_2 & c_2 & b_2 & 0 & \\ d_3 & d_4 & c_4 - c_1 & -b_1 & b_2 & I_3 \\ d_5 & d_6 & -c_3 & 0 & -b_1 & \end{array}\right) \in \mathrm{NL}(2,4).$$

Here vertical and horizontal bars are visual separators between homogeneous components of $\mathfrak{n}_{2,4}$, I_k denotes the $k\times k$ identity matrix, $0_{k\times n}$ the null matrix of order $k\times n$, $\epsilon = a_1a_4 - a_2a_3 \neq 0$ and

$$A' = \begin{pmatrix} a_1^2 & a_1a_2 & a_2^2 \\ 2a_1a_3 & a_1a_4 + a_2a_3 & 2a_2a_4 \\ a_3^2 & a_3a_4 & a_4^2 \end{pmatrix}.$$

From the decomposition $\operatorname{Der}\mathfrak{n}_{2,4} = \mathfrak{S}_{2,4} \oplus \mathbb{F}\cdot \mathrm{Id}_{2,4} \oplus \mathfrak{N}_{2,4}$, the matrices that represent derivations of $\mathfrak{n}_{2,4}$ are sum of matrices of three different types:

$$\left(\begin{array}{cc|c|cc|ccc} a_1 & a_2 & \multirow{2}{*}{$0_{2\times 1}$} & \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times 3}$}} \\ a_3 & -a_1 & & & & & & \\ \hline \multicolumn{2}{c|}{0_{1\times 2}} & 0 & \multicolumn{2}{c|}{0_{1\times 2}} & \multicolumn{3}{c}{0_{1\times 3}} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \multirow{2}{*}{$0_{2\times 1}$} & a_1 & a_2 & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times 3}$}} \\ & & & a_3 & -a_1 & & & \\ \hline & & & & & 2a_1 & a_2 & 0 \\ \multicolumn{2}{c|}{0_{3\times 2}} & 0_{3\times 1} & \multicolumn{2}{c|}{0_{3\times 2}} & 2a_3 & 0 & 2a_2 \\ & & & & & 0 & a_3 & -2a_1 \end{array}\right) \in \mathfrak{S}_{2,4},$$

$$
\lambda\,\mathrm{Id}_{2,4} = \left(\begin{array}{cc|c|cc|ccc}
\lambda & 0 & 0_{2\times 1} & \multicolumn{2}{c|}{0_{2\times 2}} & \multicolumn{3}{c}{0_{2\times 3}} \\
0 & \lambda & & & & & & \\ \hline
\multicolumn{2}{c|}{0_{1\times 2}} & 2\lambda & \multicolumn{2}{c|}{0_{1\times 2}} & \multicolumn{3}{c}{0_{1\times 3}} \\ \hline
\multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 1} & 3\lambda & 0 & \multicolumn{3}{c}{0_{2\times 3}} \\
 & & & 0 & 3\lambda & & & \\ \hline
 & & & & & 4\lambda & 0 & 0 \\
\multicolumn{2}{c|}{0_{3\times 2}} & 0_{3\times 1} & \multicolumn{2}{c|}{0_{3\times 2}} & 0 & 4\lambda & 0 \\
 & & & & & 0 & 0 & 4\lambda
\end{array}\right)
$$

and

$$
\left(\begin{array}{cc|c|cc|c}
\multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 1} & \multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 3} \\ \hline
b_1 & b_2 & 0 & \multicolumn{2}{c|}{0_{1\times 2}} & 0_{1\times 3} \\ \hline
c_1 & c_2 & b_2 & \multicolumn{2}{c|}{0_{2\times 2}} & 0_{2\times 3} \\
c_3 & c_4 & -b_1 & & & \\ \hline
d_1 & d_2 & c_2 & b_2 & 0 & \\
d_3 & d_4 & c_4 - c_1 & -b_1 & b_2 & 0_{3\times 3} \\
d_5 & d_6 & -c_3 & 0 & -b_1 &
\end{array}\right) \in \mathfrak{N}_{2,4}.
$$

For any $0 \neq \lambda \in \mathbb{F}$, the linear map $\varphi_\lambda(x_i) = \lambda x_i$ provides the (semisimple) automorphism $\Phi_{\varphi_\lambda}([x_{\alpha_1} \dots x_{\alpha_r}]) = \lambda^r [x_{\alpha_1} \dots x_{\alpha_r}]$ and the (semisimple) derivation $d_{\varphi_\lambda}([x_{\alpha_1} \dots x_{\alpha_r}]) = r\lambda [x_{\alpha_1} \dots x_{\alpha_r}]$.

Remark 5.1.7. As $\operatorname{Der} \mathfrak{n}_{2,3} \cong \operatorname{Der}\left(\mathfrak{n}_{2,4}/\mathfrak{n}_{2,4}^4\right)$ the upper left 5×5 matrices in the previous example gives us these derivations in Hall basis. In a similar way, $\operatorname{Der} \mathfrak{h}_3 \cong \operatorname{Der} \mathfrak{n}_{2,2} \cong \operatorname{Der}\left(\mathfrak{n}_{2,4}/\mathfrak{n}_{2,4}^3\right)$ appears in the same corner. These derivation algebras will appear in detail in the following section.

Consider now the 5-dimensional Lie algebra $\mathfrak{n}_3$ with basis $\{z_1, \dots, z_5\}$ and non-zero products

$$[z_1, z_2] = z_3, \quad [z_1, z_3] = z_4, \quad [z_1, z_4] = [z_2, z_3] = z_5.$$

The lower central series is

$$\mathfrak{n}_3 \supseteq \mathfrak{n}_3^2 = \operatorname{span}\langle z_3, z_4, z_5\rangle \supseteq \mathfrak{n}_3^3 = \operatorname{span}\langle z_4, z_5\rangle \supseteq \mathfrak{n}_3^4 = \operatorname{span}\langle z_5\rangle \supseteq \mathfrak{n}_3^5 = 0.$$

So, the correspondence $x_i \mapsto z_i$ for $i = 1, 2$ extends to a surjective algebra homomorphism $\theta_{\mathfrak{n}_3} \colon \mathfrak{n}_{2,4} \to \mathfrak{n}_3$ and $\mathfrak{n}_3 \cong \dfrac{\mathfrak{n}_{2,4}}{\ker \theta_{\mathfrak{n}_3}}$. In this case, the kernel is the 3-dimensional ideal

$$\ker \theta_{\mathfrak{n}_3} = \operatorname{span}\langle [[[x_1, x_2], x_2], x_2], [[[x_1, x_2], x_2], x_1], \\ [[x_1, x_2], x_2] + [[[x_1, x_2], x_1], x_1]\rangle.$$

Example 5.1.3. Let denote $I = \ker\theta_{\mathfrak{n}_3}$. According to Theorems 5.1.5 and 5.1.6, derivations (automorphisms) of $\mathfrak{n}_3$ are a quotient of the set of derivations (automorphisms) of $\mathfrak{n}_{2,4}$ that leave I invariant. These sets are:

$$\mathrm{Der}_I\,\mathfrak{n}_{2,4}:\left(\begin{array}{cc|c|cc|ccc}
a_1 & a_2 & \multirow{2}{*}{$0_{2\times1}$} & \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times2}$}} & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times3}$}}\\
0 & \frac{1}{2}a_1 & & & & & & \\ \hline
b_1 & b_2 & \frac{3}{2}a_1 & \multicolumn{2}{c|}{0_{1\times2}} & \multicolumn{3}{c}{0_{1\times3}}\\ \hline
c_1 & c_2 & b_2 & \frac{5}{2}a_1 & a_2 & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times3}$}}\\
c_3 & c_4 & -b_1 & 0 & 2a_1 & & & \\ \hline
d_1 & d_2 & c_2 & b_2 & 0 & \frac{7}{2}a_1 & a_2 & 0\\
d_3 & d_4 & c_4-c_1 & -b_1 & b_2 & 0 & 3a_1 & 2a_2\\
d_5 & d_6 & -c_3 & 0 & -b_1 & 0 & 0 & \frac{5}{2}a_1
\end{array}\right),$$

and, for $a_4 \neq 0$, $\mathrm{Aut}_I\,\mathfrak{n}_{2,4}$ has the form

$$\left(\begin{array}{cc|c|cc|ccc}
a_4^2 & a_2 & \multirow{2}{*}{$0_{2\times1}$} & \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times2}$}} & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times3}$}}\\
0 & a_4 & & & & & & \\ \hline
b_1 & b_2 & a_4^3 & \multicolumn{2}{c|}{0_{1\times2}} & \multicolumn{3}{c}{0_{1\times3}}\\ \hline
c_1 & c_2 & a_4^2b_2-a_2b_1 & a_4^5 & a_2a_4^3 & \multicolumn{3}{c}{\multirow{2}{*}{$0_{2\times3}$}}\\
c_3 & c_4 & -a_4b_1 & 0 & a_4^4 & & & \\ \hline
d_1 & d_2 & a_4^2c_2-a_2c_1 & a_4^4b_2-a_2a_4^2b_1 & a_2\left(a_4^2b_2-a_2b_1\right) & a_4^7 & a_2a_4^5 & a_2^2a_4^3\\
d_3 & d_4 & c_4a_4^2-a_4c_1-a_2c_3 & -a_4^3b_1 & a_4^3b_2-2a_2a_4b_1 & 0 & a_4^6 & 2a_2a_4^4\\
d_5 & d_6 & -a_4c_3 & 0 & -a_4^2b_1 & 0 & 0 & a_4^5
\end{array}\right).$$

Note that the isomorphism $\frac{\mathfrak{n}_{2,4}}{I} \to \mathfrak{n}_3$ is provided by the correspondence $z_i' \mapsto z_i$ by taking $z_1' = x_1+I$, $z_2' = x_2+I$, $z_3' = [x_1,x_2]+I$, $z_4' = [x_1,[x_1,x_2]]+I$, $z_5' = [x_2,[x_1,x_2]]+I$. So $\mathcal{B}' = \{z_1', z_2', z_3', z_4', z_5'\}$ is a basis. Now, by using the isomorphisms in Theorem 5.1.5 and Theorem 5.1.6 and a minor change of basis, we get a complete description of derivations and automorphisms of

$\mathfrak{n}_3 \cong \dfrac{\mathfrak{n}_{2,4}}{I}$. Relative to the basis $\{z_1, z_2, z_3, z_4, z_5\}$:

$$\operatorname{Der} \frac{\mathfrak{n}_{2,4}}{I} : \left(\begin{array}{cc|c|c|c} a_1 & 0 & & & \\ a_3 & 2a_1 & \smash{0_{2\times1}} & \smash{0_{2\times2}} & \smash{0_{2\times1}} \\ \hline b_1 & b_2 & 3a_1 & 0 & 0 \\ \hline c_1 & c_2 & b_2 & 4a_1 & 0 \\ \hline d_1 & d_2 & c_2 - b_1 & a_3 + b_2 & 5a_1 \end{array}\right),$$

and, for $a_4 \neq 0$,

$$\operatorname{Aut} \frac{\mathfrak{n}_{2,4}}{I} : \left(\begin{array}{cc|c|c|c} a_4 & 0 & & & \\ a_2 & a_4^2 & \smash{0_{2\times1}} & \smash{0_{2\times1}} & \smash{0_{2\times1}} \\ \hline b_2 & b_1 & a_4^3 & 0 & 0 \\ \hline -c_4 & -c_3 & a_4 b_1 & a_4^4 & 0 \\ \hline d_6 - c_2 & d_5 - c_1 & a_2 b_1 - a_4(a_4 b_2 + c_3) & a_4^2(a_2 a_4 + b_1) & a_4^5 \end{array}\right).$$

From previous descriptions, it is clear that the map $\varphi_\lambda \colon x_i \to \lambda x_i$, for $i = 1, 2$, extends to a derivation of $\mathfrak{n}_{2,4}$ if and only if $\lambda = 0$ and φ_λ extends to an automorphism of $\mathfrak{n}_{2,4}$ if and only if $\lambda = 1$. We also remark that, I is not an homogeneous ideal, so $\mathfrak{n}_3$ does not inherit the natural $\mathbb{N}$-grading of $\mathfrak{n}_{2,4}$ and it is not quasi-cyclic. However $\Phi_\lambda \colon z_i \to \lambda^i z_i$ is an automorphism for all $0 \neq \lambda \in \mathbb{F}$ with eigenvalues λ^i for $1 \leq i \leq 5$. In case $\mathbb{F} = \mathbb{R}$ and $\lambda > 1$, Φ_λ is an (expanding) automorphism that provides the $\mathbb{N}$-grading $\mathfrak{n}_3 = \oplus_{i=1}^5 S(\lambda^i)$ where $S(\lambda^i) = \{v \in \mathfrak{n}_3 : \Phi_\lambda(v) = \lambda^i v\}$. In fact, the algebra $\mathfrak{n}_3$, introduced in [Leger, 1963] and [Johnson, 1975], provides an example of a non-quasi-cyclic Lie algebra that admits expanding automorphisms.

As in Example 5.1.3, in the following one, we get the conditions that determine derivations and automorphisms of $\mathfrak{n}_2$, the Lie algebra described in Example 5.1.1, by using $\operatorname{Der} \mathfrak{n}_{4,2}$ and $\operatorname{Aut} \mathfrak{n}_{4,2}$.

Example 5.1.4. Let now $I = \ker \theta_{\mathfrak{n}_2}$. Derivations and automorphisms of $\mathfrak{n}_{4,2}$ in Hall basis $\mathcal{H}_{4,2}$ are (here $\Delta^{k,l}_{i,j} = a_i a_j - a_k a_l$). $\mathrm{Der}\,\mathfrak{n}_{4,2}$ is as follows

$$\left(\begin{array}{cccc|cccccc}
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_9 & a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
b_1 & b_2 & b_3 & b_4 & a_1 + a_6 & a_7 & -a_3 & a_8 & -a_4 & 0 \\
b_5 & b_6 & b_7 & b_8 & a_{10} & a_1 + a_{11} & a_2 & a_{12} & 0 & -a_4 \\
b_9 & b_{10} & b_{11} & b_{12} & -a_9 & a_5 & a_6 + a_{11} & 0 & a_{12} & -a_8 \\
b_{13} & b_{14} & b_{15} & b_{16} & a_{14} & a_{15} & 0 & a_1 + a_{16} & a_2 & a_3 \\
b_{17} & b_{18} & b_{19} & b_{20} & -a_{13} & 0 & a_{15} & a_5 & a_6 + a_{16} & a_7 \\
b_{21} & b_{22} & b_{23} & b_{24} & 0 & -a_{13} & -a_{14} & a_9 & a_{10} & a_{11} + a_{16}
\end{array}\right),$$

and, for non-singular matrices with entries a_i, the structure of $\mathrm{Aut}\,\mathfrak{n}_{4,2}$ is

$$\left(\begin{array}{cccc|cccccc}
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_9 & a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
b_1 & b_2 & b_3 & b_4 & \Delta^{2,5}_{1,6} & \Delta^{3,5}_{1,7} & \Delta^{3,6}_{2,7} & \Delta^{4,5}_{1,8} & \Delta^{4,6}_{2,8} & \Delta^{4,7}_{3,8} \\
b_5 & b_6 & b_7 & b_8 & \Delta^{2,9}_{1,10} & \Delta^{3,9}_{1,11} & \Delta^{3,10}_{2,11} & \Delta^{4,9}_{1,12} & \Delta^{4,10}_{2,12} & \Delta^{4,11}_{3,12} \\
b_9 & b_{10} & b_{11} & b_{12} & \Delta^{6,9}_{5,10} & \Delta^{7,9}_{5,11} & \Delta^{7,10}_{6,11} & \Delta^{8,9}_{5,12} & \Delta^{8,10}_{6,12} & \Delta^{8,11}_{7,12} \\
b_{13} & b_{14} & b_{15} & b_{16} & \Delta^{2,13}_{1,14} & \Delta^{3,13}_{1,15} & \Delta^{3,14}_{2,15} & \Delta^{4,13}_{1,16} & \Delta^{4,14}_{2,16} & \Delta^{4,15}_{3,16} \\
b_{17} & b_{18} & b_{19} & b_{20} & \Delta^{6,13}_{5,14} & \Delta^{7,13}_{5,15} & \Delta^{7,14}_{6,15} & \Delta^{8,13}_{5,16} & \Delta^{8,14}_{6,16} & \Delta^{8,15}_{7,16} \\
b_{21} & b_{22} & b_{23} & b_{24} & \Delta^{10,13}_{9,14} & \Delta^{11,13}_{9,15} & \Delta^{11,14}_{10,15} & \Delta^{12,13}_{9,16} & \Delta^{12,14}_{10,16} & \Delta^{12,15}_{11,16}
\end{array}\right).$$

Let $d_A \in \mathrm{Der}\,\mathfrak{n}_{4,2}$ be and $\Phi_A \in \mathrm{Aut}\,\mathfrak{n}_{4,2}$. An easy computation shows that

$$d_A \in \mathrm{Der}_I\,\mathfrak{n}_{4,2} \;\Leftrightarrow\; \begin{cases} a_{12} = -a_5, \quad a_{13} = a_{10},\ a_7 = a_4, \\ a_{15} = -a_2, \quad a_{16} = a_1 - a_6 + a_{11}, \end{cases}$$

and

$$\Phi_A \in \operatorname{Aut}_I \mathfrak{n}_{4,2} \;\Leftrightarrow\; \begin{cases} \Delta^{2,9}_{1,10} + \Delta^{6,13}_{5,14} = \Delta^{3,10}_{2,11} + \Delta^{7,14}_{6,15} = 0, \\ \Delta^{4,9}_{1,12} + \Delta^{8,13}_{5,16} = \Delta^{4,11}_{3,12} + \Delta^{8,15}_{7,16} = 0, \\ \Delta^{3,9}_{1,11} + \Delta^{4,10}_{2,12} + \Delta^{7,13}_{5,15} + \Delta^{8,14}_{6,16} = 0. \end{cases}$$

Therefore, the correspondence $u_i \mapsto \lambda u_i$ for $i = 1, \dots, 4$, and $u_5 \mapsto 2\lambda u_5$ extends by linearity to a derivation of $\mathfrak{n}_2$ for all λ. The correspondence $u_i \mapsto \lambda u_i$ for $i = 1, \dots, 4$, and $u_5 \mapsto \lambda^2 u_5$ extends to an automorphism if $\lambda \neq 0$.

5.1.3 Bilinear forms and quadratic quotients

Analogous to linear maps, we can try to figure out which are the bilinear invariant forms in some quotients of Lie algebras.

In general, in a quotient L/I, we can try to define $\hat{f} \in \operatorname{Bi}_{\mathrm{inv}}(L/I)$ from $f \in \operatorname{Bi}_{\mathrm{inv}}(L)$ taking $\hat{f}(x+I, y+I) = f(x,y)$. This is well defined if and only if $I \subseteq \operatorname{Rad} f$. But there is even more, this induces a vector space isomorphism

$$\begin{aligned} \Omega\colon\ \operatorname{Bi}_{\mathrm{inv}}(L, I) &\to \operatorname{Bi}_{\mathrm{inv}}(L/I) \\ f &\mapsto \hat{f} \end{aligned}$$

where $\operatorname{Bi}_{\mathrm{inv}}(L, I) = \{f \in \operatorname{Bi}_{\mathrm{inv}}(L) : I \subseteq \operatorname{Rad} f\}$ and $\hat{f}$ is defined as above. It is also worth mentioning this same Ω works when using symmetric forms in both sides.

At this point, as we are interested in nondegenerate bilinear forms, we can try to find when the resulting $\hat{f}$ is nondegenerate. As

$$\operatorname{Rad} \hat{f} = \frac{\operatorname{Rad} f}{I},$$

$\hat{f}$ is nondegenerate if $I = \operatorname{Rad} f$, so there is a bijection between the sets

$$\{f \in \operatorname{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L) : \operatorname{Rad} f = I\} \longleftrightarrow \{\hat{f} \in \operatorname{Bi}^{\mathrm{s}}_{\mathrm{inv}}(L/I) : \operatorname{Rad} \hat{f} = 0\}.$$

This procedure can be used to find quadratic Lie algebras inside bigger algebras with degenerate bilinear forms. When considering only nilpotent Lie algebras, by the UMP, this leads us to find all possible quadratic Lie algebras computing all bilinear symmetric forms in $\mathfrak{n}_{d,t}$ and finding their respective radicals. Those radical will be all the different ideals such that the quotient is a quadratic Lie algebra. Precisely, this procedure is the one followed in Section 3.1.3 (originally in [Benito et al., 2017]). There, the authors obtain a classification of nilpotent quadratic Lie algebras of low dimension.

5.2 Local and oscillator algebras

In Lemma 2.2.18 we have seen Lie algebras with quadratic dimension one are simple or 1-dimensional. The next logical step is studying the ones with quadratic dimension two, which has been done in [Bajo and Benayadi, 2007]. When indecomposable, they are local Lie algebras. This variety of Lie algebras, which includes oscillator algebras, is an important family of Lie algebras. They present an interesting quadratic structure and serve as a good example to find extensions to other more general, even mixed, Lie algebras. Along this section $(\mathfrak{g}, \varphi)$ is a quadratic Lie algebra.

5.2.1 Local Lie algebras

Definition 5.2.1. A *local Lie algebra* is a Lie algebra with only one proper maximal ideal.

Lemma 5.2.1. *For a Lie algebra $\mathfrak{g}$ of dimension greater than one and not simple, the following are equivalent:*

(a) $\mathfrak{g}$ *is local,*

(b) $\mathfrak{J}(\mathfrak{g}) = \mathfrak{n}$ *with* $\mathfrak{n} = N(\mathfrak{g})$*,*

(c) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ *with* $\mathfrak{s}$ *one dimensional or simple and*

$$\mathfrak{n}^2 = \{x \in \mathfrak{n} : [y, x] \in \mathfrak{n}^2 \ \forall\, y \in \mathfrak{s}\}.$$

Remark 5.2.2. Assertion $\mathfrak{n}^2 = \{x \in \mathfrak{n} : [y, x] \in \mathfrak{n}^2 \ \forall\, y \in \mathfrak{s}\}$ in Lemma 5.2.1, is equivalent to saying that $\operatorname{ad} x$ is faithful on $\mathfrak{n}/\mathfrak{n}^2$ for all $x \in \mathfrak{n}\backslash\mathfrak{n}^2$ if $\mathfrak{s}$ is one-dimensional and, in the simple case, the $\operatorname{ad}\mathfrak{s}$-module $\mathfrak{n}/\mathfrak{n}^2$ has not trivial submodules.

When working with quadratic local algebras one of the best results about their characterization is the following one found in [Bajo and Benayadi, 2007, Theorem 3.1]

Proposition 5.2.3. *Up to isometric isomorphisms, any local and quadratic Lie algebra $(\mathfrak{g}, \varphi)$ is of one of the following types:*

- *When $\mathcal{J}(\mathfrak{g}) = 0$:*
  - (a) $\mathfrak{g} = \mathrm{span}\langle x \rangle$ *and* $\varphi(x, x) = 1$.
  - (b) $\mathfrak{g}$ *is simple and in algebraically closed fields there is* $\lambda \neq 0$ *such that* $\lambda\varphi(x, y) = \mathrm{Tr}(\mathrm{ad}\ x \circ \mathrm{ad}\ y)$ *is the Killing form.*
- *When $\mathcal{J}(\mathfrak{g}) = \mathfrak{n} \cong \mathfrak{s}^*$ is abelian:*
  - (c) $\mathfrak{g} = T_0^*\mathfrak{s}$ *where* $\mathfrak{s}$ *is a simple Lie algebra and admits the bilinear form* $\varphi(x + \alpha, y + \beta) = \alpha(y) + \beta(x)$ *with* $x, y \in \mathfrak{s}$ *and* $\alpha, \beta \in \mathfrak{s}^*$.
- *When $\mathcal{J}(\mathfrak{g}) = \mathfrak{n}$ is not abelian:*
  - (d) $\mathfrak{g}$ *is a solvable Lie algebra double extension of a nonzero and nilpotent quadratic Lie algebra* $(\mathfrak{a}, \psi)$ *by a* ψ*-skew derivation* δ *which is invertible on the centre* $Z(\mathfrak{a})$.
  - (e) $\mathfrak{g} = \mathfrak{g}^2$ *is a mixed Lie algebra double extension of a nonzero and nilpotent quadratic Lie algebra* $(\mathfrak{a}, \psi)$ *by a simple subalgebra* $\mathfrak{s}$ *through a representation* $\psi \colon \mathfrak{s} \to \mathrm{Der}_\varphi(\mathfrak{a})$ *such that* $Z(\mathfrak{a})$ *has non-trivial* $\mathfrak{s}$*-modules.*

Proof. If $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ is reductive, the only possibilities are (a) or (b). Assume then $\mathfrak{g}$ non-reductive. From [Bajo and Benayadi, 2007, Theorem 3.1, item i)] we arrive at item (d) and from [Bajo and Benayadi, 2007, Theorem 3.1, item ii)] $\mathfrak{n}^\perp = Z(\mathfrak{n})$ we get either (c) if $\mathfrak{n} = \mathfrak{n}^\perp$ or (e) otherwise. Note that if $\mathfrak{g}$ solvable, $0 \neq Z(\mathfrak{g}) \subseteq \mathfrak{n}$. Then $\mathfrak{g}^2 = \mathfrak{n} = [x, \mathfrak{n}] \oplus \mathfrak{n}^2$ and $\mathfrak{n}^2 = 0$ give us $\mathrm{ad}\ x|_\mathfrak{n}$ bijective, which is not possible. □

From previous proposition we can obtain the following conclusions. For the first cases:

- Simple and one-dimensional algebras are the only local quadratic and reductive Lie algebras (types (a) and (b)).

- Trivial T^*-extensions of simple Lie algebras (type (c) in the list) are local and quadratic. Recall that the T^*-extension is a more general construction where the trivial extension is the easier case.

Quadratic local algebras $(\mathfrak{g}, \varphi)$ of types described in items (d) or (e) have some common structural patterns but remarkable differences. In both cases, since $\mathfrak{n}$ is the unique maximal ideal, $\mathfrak{n}^\perp$ is the unique minimal ideal, and $\mathfrak{n} \neq \mathfrak{n}^\perp$ because $\mathfrak{n}$ is not abelian. In addition, $0 \neq \mathfrak{n}^\perp \subseteq \mathfrak{n}^2$ and, using $\mathfrak{g}/\mathfrak{n}$ is simple or 1-dimensional. From Theorem 3.1.8, $\mathfrak{g}$ can be built as a double extension as of the quadratic nilpotent algebra

$$\left(\frac{\mathfrak{n}}{\mathfrak{n}^\perp}, \hat{\varphi}|_{\frac{\mathfrak{n}}{\mathfrak{n}^\perp}}\right).$$

But they also have differences :

- $\mathfrak{g}$ of type (d) is solvable ($\mathfrak{s} = 0$) and $\mathfrak{n} = \mathfrak{g}^2$ is of codimension 1, $\mathfrak{n}^\perp = Z(\mathfrak{g}) = \operatorname{span}\langle z\rangle \subseteq \mathfrak{n}^2 \subset \mathfrak{n}$ and the double extension is induced through the φ-skew $\operatorname{ad} x$ for any $x \in \mathfrak{g}$ such that $\varphi(x, z) \neq 0$ and $\operatorname{ad} x|_{Z(\frac{\mathfrak{n}}{\mathfrak{n}^\perp})}$ is invertible. This is described in Corollary 3.1.12.
- $\mathfrak{g}$ of type (e) satisfies $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ and it is a perfect mixed algebra. Also, $\mathfrak{n}^\perp = Z(\mathfrak{n}) \subseteq \mathfrak{n}^2 \subset \mathfrak{n} = [\mathfrak{g}, \mathfrak{n}]$, $Z(\mathfrak{n}) \cong \mathfrak{s}^*$ as ad $\mathfrak{s}$-modules and the second term of the lower central series $Z_2(\mathfrak{n})$ has no trivial $\operatorname{ad}\mathfrak{s}$-modules. Here the double extension is induced by the adjoint representation of the Levi subalgebra, $\operatorname{ad}_{\frac{\mathfrak{n}}{\mathfrak{n}^\perp}} : \mathfrak{s} \to \operatorname{Der}_\varphi(\frac{\mathfrak{n}}{\mathfrak{n}^\perp})$. This appears in Corollary 3.1.11.

In both cases, by imposing $\mathfrak{n}^2 = \mathfrak{n}^\perp$, $\mathfrak{g}$ is a double extension of the abelian algebra $\frac{\mathfrak{n}}{\mathfrak{n}^\perp}$. This applies to the constructions given the in Section 5.2.2 for $\mathfrak{g}$ solvable, and Example 5.2.1 for $\mathfrak{g}$ perfect. Moreover, as the nilradical $\mathfrak{n}$ is an ideal, then using item (b) in Proposition 2.2.13 $[\mathfrak{n}, \mathfrak{n}^\perp] = [\mathfrak{n}, \mathfrak{n}^\perp] = 0$ so $\mathfrak{n}^3 = 0$ and $\mathfrak{n}$ is 2-step. Otherwise, $\mathfrak{n}^2 \neq \mathfrak{n}^\perp$ and local quadratic Lie algebras follow from nilpotent and nonabelian quadratic algebras. In our Examples 5.2.2 and 5.2.3 we will present some local algebras extending low nilindex quadratic algebras.

Example 5.2.1. Let $\mathfrak{s}$ be a simple Lie algebra and $(\mathfrak{v}, \varphi)$ an $\mathfrak{s}$-module without trivial submodules. Denote by $\rho \colon \mathfrak{s} \to \mathfrak{gl}(\mathfrak{v})$ the representation of $\mathfrak{s}$ on $\mathfrak{v}$. Assume that φ is a symmetric, nondegenerate and $\mathfrak{s}$-invariant bilinear form, i.e. $\rho(\mathfrak{s}) \subseteq \operatorname{Der}_\varphi(\mathfrak{v}) \subseteq \mathfrak{so}(\mathfrak{v}, \varphi)$ defined in equation (2.8). The existence of φ only is possible if the 2-symmetric tensor power, $S^2\mathfrak{v}$, has a trivial submodule. Let $\mathfrak{v}(\mathfrak{s}) = \mathfrak{s} \ltimes_\rho (\mathfrak{v} \oplus_\omega \mathfrak{s}^*)$, the double extension of $(\mathfrak{v}, \varphi)$ by $(\mathfrak{s}, \rho)$. The bracket and the bilinear form are given by equations (2.18) and (2.19) where $B = \mathfrak{s}$, $A = \mathfrak{v}$ is an abelian algebra and $\omega(u, v)(s) = \varphi(\rho(s)(u), v)$, for all $u, v \in \mathfrak{v}$ and $s \in \mathfrak{s}$. Taking $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{F})$ and $\mathfrak{v} = V_n$ (see equation (5.12)), the unique possibility is $n = 2m$ and the algebra $\mathfrak{v}(\mathfrak{s})$ is a $(2m + 7)$-dimensional local algebra with 4 ideals in chain.

By using results and techniques of previous Section 5.1 we can compute $\mathrm{Der}_\varphi\, \mathfrak{n}_{2,3}$ and $\mathrm{Der}_\phi\, \mathfrak{n}_{3,2}$ (see Chapter 6). Both algebras have also been given in [del Barco and Ovando, 2012].

Example 5.2.2. Let $(\mathfrak{n}_{2,3}, \varphi)$ be the algebra described in Example 2.2.4 with (Hall) basis $\{a_i\}_{i=1}^5$. Then, $\mathrm{Der}_\varphi\, \mathfrak{n}_{2,3}$ is 6-dimensional and it is given as the direct sum of the (nipotent) ideal $\mathrm{Inner}\, \mathfrak{n}_{2,3}$, and its Levi subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{F})$. Thus, any derivation decomposes as the next sum, where the first summand $D = D(m_i) \in \mathfrak{s}$ and the second $d = d(v_i) \in \mathrm{Inner}\, \mathfrak{n}_{2,3}$,

$$\left(\begin{array}{cc|c|cc} m_1 & m_2 & 0 & 0 & 0 \\ m_3 & -m_1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & m_1 & m_2 \\ 0 & 0 & 0 & m_3 & -m_1 \end{array}\right) + \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline v_2 & v_1 & 0 & 0 & 0 \\ \hline v_3 & 0 & v_1 & 0 & 0 \\ 0 & v_3 & -v_2 & 0 & 0 \end{array}\right)$$

Note that $\mathfrak{n}_{2,3}$ has two copies of the 2-dimensional natural $\mathfrak{s}$-module V_1 (both linear spans of $\langle a_1, a_2\rangle$ and $\langle a_4, a_5\rangle$) and one of the trivial module $V_0 = \langle a_3\rangle$. By double extension formulae (2.18) and (2.19) we get the 11-dimensional local (applying Lemma 5.2.1 or Proposition 5.2.3) perfect quadratic Lie algebra $\mathfrak{n}_{2,3}(\mathfrak{s}) = \mathfrak{s} \oplus \mathfrak{n}_{2,3} \oplus \mathfrak{s}^*$. The lattice of ideals of $\mathfrak{n}_{2,3}(\mathfrak{s})$ is a 6-chain. Taking any derivation $\delta = D(m_1, m_2, m_3)$ such that $m_3m_2 + m_1^2 \neq 0$ we get a 7-dimensional solvable local quadratic via the double extension by $\mathbb{F}\delta$.

Example 5.2.3. Consider now the quadratic $(\mathfrak{n}_{3,2}, \phi)$ from Example 2.2.5 with (Hall) basis $\{a_i\}_{i=1}^6$. In this case, $\mathrm{Der}_\varphi\, \mathfrak{n}_{3,2}$ is 10-dimensional and it is the direct sum of $\mathrm{Inner}\, \mathfrak{n}_{3,2}$ and a simple Lie subalgebra $\mathfrak{s} \cong \mathfrak{sl}_3(\mathbb{F})$, thus we have two summands, the first $D = D(m_1, \ldots, m_5) \in \mathfrak{s}$ and the second $d = d(v_1, v_2, v_3) \in$

Inner $\mathfrak{n}_{3,2}$:

$$\left(\begin{array}{ccc|ccc} m_1 & m_2 & m_3 & 0 & 0 & 0 \\ m_4 & m_5 & m_6 & 0 & 0 & 0 \\ m_7 & m_8 & -m_1-m_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & m_1+m_5 & m_6 & -m_3 \\ 0 & 0 & 0 & m_8 & -m_5 & m_2 \\ 0 & 0 & 0 & -m_7 & m_4 & -m_1 \end{array}\right) + \left(\begin{array}{ccc|c} 0 & 0 & 0 & \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ \hline v_1 & v_2 & 0 & \\ v_3 & 0 & v_2 & \cdots \\ 0 & v_3 & -v_1 & \end{array}\right)$$

Here $\mathfrak{n}_{3,2}$ decomposes as two copies of the 3-dimensional natural $\mathfrak{s}$-module. By double extension, we get (just applying Lemma 5.2.1) the 22-dimensional local perfect quadratic Lie algebra $\mathfrak{n}_{3,2}(\mathfrak{s}) = \mathfrak{s} \oplus \mathfrak{n}_{3,2} \oplus \mathfrak{s}^*$ and its lattice of ideals is a 5-chain.

Remark 5.2.4. According to Corollary 4.1.3, double extensions via inner derivations provide decomposable quadratic algebras.

As a final comment in this section, in 2014, A. Elduque and S. Benayadi classified real and complex indecomposable mixed quadratic Lie algebras of dimension less or equal than 13 (see [Benayadi and Elduque, 2014, Theorems 3.16 and 4.11]). The Levi subalgebra of this type of algebras in the complex case is the 3-dimensional split $\mathfrak{sl}_2(\mathbb{C})$ and over the reals it is either $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{su}_2(\mathbb{R})$. Combining irreducible modules for these algebras, smart reasoning and elementary linear and multilinear algebra tools and previous knowledge on basic structure of quadratic Lie algebras, the authors achieve a very clean classification. In the complex case, apart from the double extension process, constructions of quadratic Lie algebras as tensor products (Example 2.2.6) also appeared. The authors arrived at those tensors by using Jordan algebras of dimension $2, 3, 4$ and following the ideas given in [Allison, 1976, Theorem 1, Section 5]. Lie algebras in the complex classification are mostly local. We also point out that their lattices of ideals are mainly n-chains with $n \leq 6$. This Lie structure will be treated in Section 5.3.2.

5.2.2 Generalised Oscillator algebras

According to Lemma 2.2.15, the smallest solvable non-abelian quadratic Lie algebra has dimension 4. This four-dimensional algebra in the real case is known as harmonic oscillator algebra and it is the Lie algebra of the harmonic oscillator group (see [Hilgert et al., 1989, Example V.4.15], [Douglas and Premat, 2007], [Ovando, 2006] and [Ovando, 2007a]). This algebra is local and it is the first step of a countable series of solvable Lie algebras which support invariant Lorentzian forms. Generalizing this situation, we arrive at oscillator Lie $\mathbb{F}$-algebras as double extensions of metric spaces (abelian Lie algebras). The aim of this section is to present some structural features, invariant metrics and derivations of this class of algebras and to explore their possibilities of being extended to mixed quadratic Lie algebras.

Real oscillator algebras are the Lie algebras attached to connected, simply connected and non-simple Lie groups that admit a Lorentz invariant metric that makes them indecomposable (see [Medina, 1985, Theorem 4.1]). This type of algebras was introduced in [Hilgert and Hofmann, 1985] as double extensions of Hilbert spaces, and renamed as standard solvable *Lorentzian Lie algebras* A_{2m+2} in [Hilgert et al., 1989, Definition II.3.16]. Also, they are the class of real solvable non-abelian Lie algebras that carry an invariant inner product of metric signature $(2m+1, 1)$. This class is integrated by real Lie algebras of dimension $2m+2$ for $m \geq 1$. The term oscillator comes from quantum mechanics because A_{2m+2} describe a system of harmonic oscillator m-dimensional euclidean space. These algebras also support other nonassociative structures such as Poisson and Leibniz algebras and symmetric Leibniz bialgebras following [Camacho et al., 2019] and [Albuquerque et al., 2021]. The results included in the last of these articles let us derive some geometric consequences at the level of the oscillator Lie groups.

Oscillator algebras are quadratic, local and solvable Lie algebras with nilradical a Lie algebra of Heisenberg type according to [Hilgert et al., 1989, Proposition II.3.11]. Along this section, we will see that the notion and their structural properties can be extended to arbitrary fields.

5.2.2.1 Metric spaces and Oscillator algebras

In Example 2.2.7 we obtain the oscillator algebra from the double extension of a two-dimensional vector space. This process can be replicated to produce a infinite series of oscillator algebras simply by increasing the dimension of this vector space. The algebras in our following definition are just the ones described in item (d) of Proposition 5.2.3 when considering an abelian quadratic algebra $(\mathfrak{a}, \psi)$.

Definition 5.2.2. Let (W_n, φ) be a n-dimensional $\mathbb{F}$-vector space endowed with a symmetric and non-degenerate form φ. Consider now any skew-linear automorphism δ of W_n; it is only possible for $n = 2m$. The double extension $\mathbb{F} \cdot \delta \oplus W_{2m} \oplus \mathbb{F} \cdot \delta^*$ where δ^* is the dual 1-form of δ is a quadratic Lie algebra that we will call *generalized oscillator $\mathbb{F}$-algebra* on the triple $(W_{2m}, \varphi, \delta)$. We will refer to it as $\mathfrak{d}(W_{2m}, \varphi, \delta) =$ or $\mathfrak{d}_{2m+2}(\mathbb{F})$ to shorten.

Remark 5.2.5. Algebras $\mathfrak{d}_{2m+2}(\mathbb{R})$ are the real oscillator algebras introduced firstly in [Hilgert and Hofmann, 1985, Proposition 2.2] as Lorentzian semialgebras of Class II (see also [Hilgert et al., 1989, Proposition II.3.11]). In [Neeb, 1993, Definition II.7] and [Hilgert and Neeb, 1996] they appear as remarkable examples of Lie algebras with cone potencial. The study of other nonassociative structures on oscillator algebras in [Albuquerque et al., 2021, Section 5] yields to some geometric information on connections and metrics of oscillator Lie groups.

From Definition 5.2.2 and Theorem 2.2.20, the $\mathbb{F}$-algebras $\mathfrak{d}(W_{2m}, \varphi, \delta)$ are one-dimensional double extensions of a quadratic abelian algebra by nonsingular self-derivations with bracket product

$$[t\delta + u + s\delta^*, t'\delta + v + s'\delta^*]_{\mathfrak{d}_{2m+2}} = t\delta(v) - t'\delta(u) + \varphi(\delta(u), v)\delta^*. \tag{5.1}$$

The quadratic structure (there may be others) in $\mathfrak{d}(W, \varphi, \delta)$ is given by expression (2.19).

The next result condenses and expands the structural algebraic properties of the oscillator $\mathbb{R}$-algebras (Propositions II.3.11 and II.3.12 of [Hilgert et al., 1989]) to any field of characteristic zero.

Proposition 5.2.6. *The generalized $\mathbb{F}$-oscillator algebra $\mathfrak{d}_{2m+2} = \mathfrak{d}(W_{2m}, \varphi, \delta)$ is a solvable quadratic algebra under the invariant bilinear form φ_δ described as*

$$\varphi_\delta(t\delta + u + s\delta^*, t'\delta + v + s'\delta^*) = ts' + t's + \varphi(u, v).$$

The nilradical $N(\mathfrak{d}_{2m+2}) = \mathfrak{d}^2_{2m+2} = W_{2m} \oplus \mathbb{F} \cdot \delta^*$ is its only maximal ideal. In particular, $\mathfrak{d}_{2m+2}$ is a local and indecomposable quadratic algebra, and its centre,

$$Z(\mathfrak{d}_{2m+2}) = \mathfrak{d}^{(2)}_{2m+2} = \mathbb{F} \cdot \delta^* = Z(N(\mathfrak{d}_{2m+2})),$$

is the orthogonal subspace of $N(\mathfrak{d}_{2m+2})$, and it is also the only minimal ideal. Moreover, $N(\mathfrak{d}_{2m+2})$ is a Lie algebra of Heisenberg type in which the product is completely *determine by the automorphism δ through the formulas $[u, v] = \varphi(\delta(u), v)\delta^*$ for all $u, v \in W_{2m}$. The algebra is a split extension of $N(\mathfrak{d}_{2m+2})$ by $\mathbb{F} \cdot \delta$, so $[\delta, u] = \delta(u)$ for $u \in W_{2m}$.*

Proof. Since δ is a linear automorphism, following equation (4.1) and using Lemma 4.1.2, from expression (5.1) we get $\mathfrak{d}^2_{2m+2} = \operatorname{Im}\delta + \operatorname{span}\langle \varphi(\delta(u), v)\delta^* : u, v \in W_{2m}\rangle = W_{2m} \oplus \mathbb{F} \cdot \delta^*$ and $Z(\mathfrak{d}_{2m+2}) = (Z(W_{2m}) \cap \operatorname{Ker}\delta) \oplus \mathbb{F} \cdot \delta^* = \mathbb{F} \cdot \delta^*$. Now $\mathfrak{d}^{(3)}_{2m+2} = [\mathfrak{d}^{(2)}_{2m+2}, \mathfrak{d}^{(2)}_{2m+2}] = 0$, so $\mathfrak{d}_{2m+2}$ is a solvable algebra, that is, $R(\mathfrak{d}_{2m+2}) = \mathfrak{d}_{2m+2}$. And it is not nilpotent because of $\mathfrak{d}^3_{2m+2} = \mathfrak{d}^2_{2m+2}$. Then its Jacobson radical, $\mathcal{J}(\mathfrak{d}_{2m+2}) = \mathfrak{d}^2_{2m+2}$ according to equation (2.9) and comments around. As $\mathcal{J}(\mathfrak{d}_{2m+2}) \subseteq N(\mathfrak{d}_{2m+2}) \neq \mathfrak{d}_{2m+2}$ and it is the intersection of the whole set of maximal ideals, it is the only maximal ideal and $\mathcal{J}(\mathfrak{d}_{2m+2}) = N(\mathfrak{d}_{2m+2})$. The statement on the Lie bracket of two elements of W_{2m} follows from equation (5.1). To finish the proof, we use Lemma 5.2.1 and Definition 2.2.9. □

From previous proposition, we get $Z(\mathfrak{d}_{2m+2}) = \mathbb{F} \cdot \delta^*$, and $N(\mathfrak{d}_{2m+2}) = W_{2m} \oplus \mathbb{F} \cdot \delta^*$ and

$$\mathfrak{d}_{2m+2}(W_{2m}, \varphi, \delta) = \mathbb{F} \cdot \delta \oplus W_{2m} \oplus \mathbb{F} \cdot \delta^* = \mathbb{F} \cdot \delta \oplus \underbrace{N(\mathfrak{d}_{2m+2})}_{\mathfrak{h}_{2m+1}}.$$

Setting $d = \operatorname{ad}_{\mathfrak{d}_{2m+2}} \delta$, we have $d|_{W_{2m}} = \delta$ and $d(\delta^*) = 0$. Then, the nilradical is d-invariant and any oscillator algebra can be viewed as the split extension of an algebra of Heisenberg type $\mathfrak{h}_{2m+1}$ by a map $d \in \operatorname{Der}\mathfrak{h}_{2m+1}$ such that $\ker d = Z(\mathfrak{d}_{2m+2}) = Z(\mathfrak{h}_{2m+1})$ and

$$\mathfrak{h}_{2m+1} = \operatorname{Im} d \oplus \ker d.$$

Proposition 5.2.7. *Any oscillator $\mathbb{F}$-algebra can be obtained as a split extension of a Lie algebra of Heisenberg type $\mathfrak{h}_{2m+1}$ and a map $d \in \operatorname{Der}\mathfrak{h}_{2m+1}$ such that $\mathfrak{h}_{2m+1} = \operatorname{Im} d \oplus \ker d$ where $\ker d = Z(\mathfrak{h}_{2m+1})$ and the invariant vector space $\operatorname{Im} d$ is endowed with a symmetric and nondegenerate bilinear form φ for which $d|_{\operatorname{Im} d}$ is φ-skew. Moreover, a self-linear map D of the oscillator $\mathbb{F}$-algebra $\mathfrak{d}(W_{2m}, \varphi, \delta)$ is a derivation if and only if:*

(a) *$N(\mathfrak{d}(W_{2m}, \varphi, \delta))$ is D-invariant and $D|_{N(\mathfrak{d}(W_{2m},\varphi,\delta))} \in \operatorname{Der}\mathfrak{h}_{2m+1}$,*

(b) *$D(\delta^*) = \alpha\delta^*$ for some $\alpha \in \mathbb{F}$,*

(c) $D(\delta(a)) = [D(\delta), a] + [\delta, D(a)]$ *for all* $a \in W_{2m}$.

Proof. The first part follows from previous discussion. The second part is based on the fact that the nilradical and the centre of any Lie algebra are characteristic ideals, which means that they are invariant through derivations. So items (a), (b) and (c) are necessary conditions if D is a derivation. For the converse, it is straightforward to check that if D is a self-linear map of the oscillator algebra satisfying items (a), (b) and (c), D also satisfies the identity $d([x, y]) = [d(x), y] + [x, d(y)]$. $\square$

5.2.2.2 Double extensions of Oscillator algebras

Following Theorem 2.2.20, the knowledge of the set $\mathrm{Der}_\varphi \mathfrak{d}(W_{2m}, \varphi, \delta)$ allows to expand oscillator algebras to other quadratic algebras. The latter set is closely related to $\mathrm{Der}\,\mathfrak{h}_{2m+1}$ which is well-known set easy to describe. Starting from Proposition 5.2.7, in this section we will give an explicit description of the whole sets of derivations and skew-derivations of real oscillator algebras. And we will also describe their invariant forms. The results let us obtain two countable series of mixed quadratic Lie algebras based on real oscillator algebras.

According to [Benito and de-la-Concepción, 2013] and following Definition 2.2.9 and notation therein, the 2-graded decomposition $\mathfrak{h}_{2m+1} = W \oplus Z(\mathfrak{h}_{2m+1})$, where W is an arbitrary $2m$-dimensional $\mathbb{F}$-complement, induces a natural grading on $\mathrm{End}\,(\mathfrak{h}_{2m+1})$ and lets us describe the derivations of $\mathfrak{h}_{2m+1}$ as (for a matrix description see [Rubin and Winternitz, 1993]):

$$\mathrm{Der}\,\mathfrak{h}_{2m+1} = \underbrace{\{\delta : \delta\,|_W \in \mathfrak{sp}(W, b_z), \delta(Z(\mathfrak{h}_{2m+1}) = 0\}}_{\mathfrak{s} \cong \mathfrak{sp}(2m, \mathbb{F})} \oplus \overbrace{\mathbb{F} \cdot \widehat{\mathrm{Id}} \oplus \{\delta : \delta(W) \subseteq Z(\mathfrak{h}_{2m+1}), \delta(Z(\mathfrak{h}_{2m+1})) = 0\}}^{R(\mathrm{Der}\,\mathfrak{h}_{2m+1})} \quad (5.2)$$

Here $\widehat{\mathrm{Id}}$ means $\widehat{\mathrm{Id}}|_W = \mathrm{Id}_W$ and $\widehat{\mathrm{Id}}|_{Z(\mathfrak{h}_{2m+1})} = 2\,\mathrm{Id}_{Z(\mathfrak{h}_{2m+1})}$. So, the Levi subalgebra $\mathfrak{s}$ of $\mathrm{Der}\,\mathfrak{h}_{2m+1}$ is simple. And its solvable radical is a $(2m+1)$-dimensional Lie algebra with abelian nilpotent radical $N(\mathrm{Der}\,\mathfrak{h}_{2m+1}) = \{\delta : \delta(W) \subseteq \mathbb{F} \cdot z, \delta(z) = 0\}$. In matrix form, the general shape of a derivation in an ordered standard basis as described in Definition 2.2.9 is

$$\left(\begin{array}{c|c|c} M+\alpha I_m & P & 0_{m\times 1} \\ \hline Q & -M^t+\alpha I_m & 0_{m\times 1} \\ \hline c_1^t & c_2^t & 2\alpha \end{array}\right), \tag{5.3}$$

where $\alpha \in \mathbb{F}$, c_i are column matrices, M, P and Q are $m \times m$ matrices $P^t = P$ and $Q^t = Q$.

Returning to the real field and applying the spectral theorem, for any real φ-skew and invertible map of an euclidean space, $\delta\colon (W_{2m}, \varphi) \to (W_{2m}, \varphi)$, there is an orthonormal basis $\{e_1, \dots, e_{2m}\}$ such that $\delta(e_{2i-1}) = \lambda_i e_{2i}$ and $\delta(e_{2i}) = -\lambda_i e_{2i-1}$ and $\lambda_1, \dots, \lambda_{2m}$ are positive real numbers (w.l.o.g. we can assume $\lambda_i \leq \lambda_{i+1}$). So any oscillator $\mathbb{R}$-algebra of dimension $2m+2$ is determined by an m-fold $\lambda = (\lambda_1, \dots, \lambda_m)$ of positive scalars such that $0 < \lambda_1 \leq \cdots \leq \lambda_m$ (to shorten $\mathfrak{d}^{\lambda}_{2m+2}(\mathbb{R})$ for a fixed m-fold λ),

$$\mathfrak{d}_{2m+2}(\lambda_1, \dots, \lambda_m) = \mathbb{R} \cdot \delta_\lambda \oplus W_{2m} \oplus \delta^*_\lambda.$$

Applying Proposition 5.2.6, the structure constants respect to the basis $\delta_\lambda, e_1, \dots, e_{2m}, \delta^*_\lambda$ are determined by the entries of λ. Using $\varphi(e_i, e_j) = \delta_{ij}$ and $[u, v] = \varphi(\delta_\lambda(u), v)\delta^*_\lambda$ for all $u, v \in W$, we have

$$\begin{cases} [e_{2i-1}, e_{2i}] = -[e_{2i}, e_{2i-1}] = \lambda_i \delta^*_\lambda, \\ [e_p, e_q] = 0 \text{ if } (p, q) \neq (2i-i, 2i), (2i, 2i-1), \\ [\delta_\lambda, e_{2i-1}] = -[e_{2i-1}, \delta_\lambda] = \delta_\lambda(e_{2i-1}) = \lambda_i e_{2i}, \\ [\delta_\lambda, e_{2i}] = -[e_{2i}, \delta_\lambda] = \delta_\lambda(e_{2i}) = -\lambda_i e_{2i-1}, \\ [\mathfrak{d}_{2m+2}, \delta^*_\lambda] = 0. \end{cases} \tag{5.4}$$

From the basis-bracket description of $\mathfrak{d}^{\lambda}_{2m+2}(\mathbb{R})$, the following lemma restates [Hilgert et al., 1989, Proposition II.3.14].

Lemma 5.2.8. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mathfrak{d}^{\lambda}_{2m+2}(\mathbb{R}) = \mathbb{R} \cdot \delta_\lambda \oplus \mathrm{span}\langle e_1, \dots, e_{2m}\rangle \oplus \mathbb{R} \cdot \delta^*_\lambda$ the oscillator algebra with Lie bracket given in equations* (5.4)*. For any $t \in \mathbb{F}$ and $s \neq 0$ the symmetric bilinear form $\varphi_{t,s}$ given as the orthogonal sum*

$$\mathrm{span}\langle e_1, \dots, e_{2m}\rangle \perp \mathrm{span}\langle \delta_\lambda, \delta^*_\lambda\rangle,$$

with $\varphi_{t,s}(\delta_\lambda, \delta_\lambda) = t$, δ_λ and δ^*_λ isotropic, $\{e_1, \dots, e_{2m}\}$ orthogonal family and $\varphi_{t,s}(\delta_\lambda, \delta^*_\lambda) = s = \varphi_{t,s}(e_i, e_i)$ is invariant and nondegenerate. The set $\{\varphi_{t,s} : t, s \in$

$\mathbb{F}, s \neq 0\}$ *is the whole set of symmetric invariant and nondegenerate bilinear forms of* $\mathfrak{d}_{2m+2}^{\lambda}$. *In particular,* $\mathrm{Bi}_{inv}^{\mathrm{s}}(\mathfrak{d}_{2m+2}^{\lambda}) = \mathrm{span}\langle \varphi_{0,1}, \varphi_{1,1} \rangle$ *and the quadratic dimension of real oscillator algebras is two.*

Proof. Note that $\det \varphi_{t,s} = -s^{2m+2}$, so $\varphi_{t,s}$ is nondegenerate if and only if $s \neq 0$. The invariance of $\varphi_{t,s}$ is equivalently to

$$\varphi_{t,s}([x,a],b) + \varphi_{t,s}(a,[x,b]) = 0 \quad \forall\, x,a,b \in \mathfrak{d}_{2m+2}.$$

The equality follows by checking it for $x \in \{\delta_\lambda, e_i\}$ (only $\varphi_{0,1}$ and $\varphi_{1,1}$ need to be checked). Now let b an arbitrary invariant symmetric and nondegenerate form. From Proposition 5.3.6, we get $(\mathfrak{d}_{2m+2}^2)^{\perp} = Z(\mathfrak{d}_{2m+2}) = \mathbb{F}\delta_\lambda^*$ and $b(\delta_\lambda, \delta_\lambda^*) = s_0 \neq 0$ because b is non-degenerate. We also set $b(\delta_\lambda, \delta_\lambda) = t_0$. Since $\delta_\lambda(e_{2i-1}) = \lambda_i e_{2i}$ and $\delta_\lambda(e_{2i}) = -\lambda_i e_{2i-1}$, we get

$$b(\delta_\lambda, e_{2i}) = \frac{1}{\lambda_i} b(\delta_\lambda, \delta_\lambda(e_{2i-1})) = b(\delta_\lambda, [\delta_\lambda, e_{2i-1}]) = 0,$$

by using that b is invariant. So $b(\delta_\lambda, e_{2i}) = 0$ and $b(\delta_\lambda, e_{2i-1}) = 0$ in the same vein. Finally, from (5.4), $b(\delta_\lambda, [e_{2i-1}, e_{2j}]) = \delta_{ij}\lambda_i s_0$ and by invariance

$$\begin{cases} b(\delta_\lambda, [e_{2i-1}, e_{2j}]) = b(\delta_\lambda(e_{2i-1}), e_{2j}) = \lambda_i b(e_{2i}, e_{2j}), \\ b(\delta_\lambda, [e_{2i-1}, e_{2j}]) = -b(\delta_\lambda(e_{2j}), e_{2i-1}) = \lambda_j b(e_{2j-1}, e_{2i-1}). \end{cases}$$

Therefore, $b(e_{2i}, e_{2j}) = b(e_{2i-1}, e_{2j-1}) = \delta_{ij} s_0$. A similar reasoning yields

$$b(e_{2i}, e_{2j-1}) = \frac{1}{\lambda_i} b(\delta_\lambda(e_{2i-1}), e_{2j-1}) = \lambda_i b(\delta_\lambda, [e_{2i-1}), e_{2j-1}]) = 0,$$

so $b = \varphi_{t_0,s_0}$. The final assertion follows from Proposition 2.2.17 and the linearly depending relation $\varphi_{t_0,s_0} = (s_0 - t_0)\varphi_{0,1} + t_0 \varphi_{1,1}$. □

There are three classes of metric real oscillator algebras depending on $\lambda = (\lambda_1, \dots, \lambda_m)$ (see [Medina and Revoy, 1985, Section 4]):

- $\mathcal{O}$-I: all the entries of λ are different. Then, the set of $\varphi_{0,1}$-skew derivations is an abelian Lie algebra.
- $\mathcal{O}$-II: all the entries of λ are equals to λ_1. Up to isomorphisms for any $m \geq 1$ we have the series $\mathfrak{d}_{2m+2}(1, \dots, 1)$. Since $\lambda = (\lambda_1, \dots, \lambda_1)$, rescaling the basis of $\mathfrak{d}_{2m+2}^{\lambda}(\mathbb{R})$ in the form $\frac{1}{\lambda_1}\delta, e_1, \dots, e_{2m}, \lambda_1 \delta_\lambda^*$ we arrive at $\mathfrak{d}_{2m+2}(1, \dots, 1)$. The set of $\varphi_{0,1}$-skew derivations is the special unitary Lie algebra $\mathfrak{su}_m(\mathbb{R})$, a simple Lie algebra of type A (i.e. the complex extension $\mathfrak{su}_m(\mathbb{R}) \otimes \mathbb{C}$ is $\mathfrak{sl}_m(\mathbb{C})$).

- $\mathcal{O}$-III: there are at least two different entries $\lambda_i < \lambda_{i+k}$ and one of them of multiplicity greater or equal than 2. The set of $\varphi_{0,1}$-skew derivations is a reductive non-abelian Lie algebra.

Remark 5.2.9. This result shows real oscillator algebras as counterexamples of a conjecture made in [Walker, 1963]. In this paper, the author asserts the only real Lie algebras of quadratic dimension two are the direct sum of two simple central algebras, a simple but non-central algebra, or the T^*-extension of a simple central algebra. Although the author reaches this conjecture from Lie groups, in general, reductive Lie algebras formed by a simple central and 1-dimensional also have quadratic dimension two.

To end this section, we will compute explicitly the whole sets of derivations and $\varphi_{0,1}$-skew derivations of $\mathfrak{d}_{2m+2}(1,\ldots,1)$ (for short $\mathfrak{d}_{2m+2}$). In the sequel, we fixed a natural $m \geq 1$ and, in order to get a more symmetric block description of any derivation, all the self-linear maps of $\mathfrak{d}_{2m+2}$ will be given in a matrix level with respect to the ordered basis

$$\{\delta_\lambda, x_1, \ldots, x_m, y_1, \ldots, y_m, \delta_\lambda^* = z\}$$

with $x_i = e_{2i-1}$ and $y_i = e_{2i}$. In this way, $\{x_i, y_i, \delta_\lambda^*\}$ forms a standard basis of $\mathfrak{h}_{2m+1}$ as in Definition 2.2.9. For the rest of products we observe equations (5.4) with $\lambda_i = 1$. Let D be any derivation of $\mathfrak{d}_{2m+2}$. From

$$D(\delta_\lambda) = \gamma\delta_\lambda + \sum_{i=1}^{m} b_i x_i + \sum_{i=1}^{m} c_i y_i + \beta z,$$

items (a), (b) and (c) in Proposition 5.2.7 and matrix specification in equation (5.3), we arrive at the general matrix description

$$D = \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline b & M + \alpha I_m & P & 0 \\ \hline c & -P & -M^t + \alpha I_m & 0 \\ \hline \beta & -b^t & -c^t & 2\alpha \end{array}\right), \tag{5.5}$$

with $\beta, \alpha \in \mathbb{R}$, b, c $1 \times m$ matrices and $M^t = -M$ and $P^t = P$. The set of inner derivations $\operatorname{Inner} \mathfrak{d}_{2m+2} = \operatorname{span}\langle \operatorname{ad} \delta_\lambda, \operatorname{ad} x_i, \operatorname{ad} y_i : 1 \leq i \leq m\rangle$ is just the set of matrices as in equation (5.5) with $M = 0$, $\beta = \alpha = 0$ and $P = \mu I_m$. Using $\operatorname{ad}[x, y] = [\operatorname{ad} x, \operatorname{ad} y]$, we get the derived subalgebra of this algebra, $(\operatorname{Inner} \mathfrak{d}_{2m+2})^2 = \operatorname{span}\langle \operatorname{ad} x_i, \operatorname{ad} y_i : 1 \leq i \leq m\rangle$, which is clearly abelian.

Among the derivations of $(\mathfrak{d}_{2m+2}, \varphi_{0,1})$, we look for the $\varphi_{0,1}$-skew ones:

$$D \in \operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_{2m+2} \Longleftrightarrow \varphi_{0,1}(D(x), y) + \varphi_{0,1}(x, D(y)) = 0. \tag{5.6}$$

From equation (5.6), the skew-derivations are as in expression (5.5) with $\beta = \alpha = 0$. Hence, any derivation D decomposes into the generic sum of basic blocks of derivations:

$$D = \overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \alpha I_m & 0 & 0 \\ \hline 0 & 0 & \alpha I_m & 0 \\ \hline 0 & 0 & 0 & 2\alpha \end{array}\right)}^{\alpha D_{0,1,2}} + \underbrace{\overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & M & 0 & 0 \\ \hline 0 & 0 & M & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)}^{\mathfrak{s}_0} + \overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & P_0 & 0 \\ \hline 0 & -P_0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)}^{\in \mathfrak{s}_1}}_{\mathfrak{s}^2 = [\mathfrak{s}, \mathfrak{s}]_{\mathfrak{s}}} +$$

$$\underbrace{\overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mu I_m & 0 \\ \hline 0 & -\mu I_m & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right)}^{\mu \operatorname{ad} \delta_\lambda \in \mathfrak{s}_1} + \overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline b_1 & 0 & 0 & 0 \\ \hline b_2 & 0 & 0 & 0 \\ \hline 0 & -b_1^t & -b_2^t & 0 \end{array}\right)}^{\mathfrak{t} = (\operatorname{Inner} \mathfrak{d}_{2m+2})^2}}_{\operatorname{Inner} \mathfrak{d}_{2m+2}} + \overbrace{\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \beta & 0 & 0 & 0 \end{array}\right)}^{\beta D_{1,0,0}},$$

where $M^t = -M$ and $P_0^t = P_0$ are $m \times m$ traceless matrices. Let denote $\mathfrak{s}$ the set of $2m \times 2m$ matrices of the following shape:

$$\left(\begin{array}{c|c} M & P \\ \hline -P & M \end{array}\right) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & M \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & P_0 \\ \hline -P_0 & 0 \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & \mu I_m \\ \hline -\mu I_m & 0 \end{array}\right), \tag{5.7}$$

here $P = P_0 + \mu I_m$ and $\mu = \frac{\operatorname{Tr} P}{m}$. It is easily checked that $\mathfrak{s}$ is a vector space which is closed under the bracket $[x, y]_{\mathfrak{s}} = xy - yx$. Then, $\mathfrak{s}$ is a linear Lie subalgebra of the special linear algebra $\mathfrak{sl}_{2m}(\mathbb{R})$ and the direct sum decomposition given in equation (5.7) provides a $\mathbb{Z}_2$-graded decomposition $\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$. So, the even part $\mathfrak{s}_0$ is a Lie algebra, in this case isomorphic to the simple orthogonal algebra of skew-symmetric matrices $\mathfrak{so}_m(\mathbb{R})$ if $m \geq 3$.

Theorem 5.2.10. *Let $\mathfrak{d}^{\lambda}_{2m+2}(\mathbb{R}) = \mathbb{R} \cdot \delta_\lambda \oplus (W_{2m}, \varphi_{0,1}) \oplus \mathbb{R} \cdot \delta^*_\lambda$, $m \geq 2$, be the real oscillator Lie algebra of m-fold $\lambda = (1, \ldots, 1)$. The sets of derivations and skew-symmetric derivations can be described as follows:*

1. $\operatorname{Der} \mathfrak{d}_{2m+2} = \mathbb{R} \cdot D_{1,0,0} \oplus \mathbb{R} \cdot D_{0,1,2} \oplus [\mathfrak{s}, \mathfrak{s}]_{\mathfrak{s}} \oplus \operatorname{Inner} \mathfrak{d}_{2m+2}$ *where $D_{0,1,2}$ is the derivation given by $D_{0,1,2}(\delta_\lambda) = 0$, $D_{0,1,2}(v) = v$ for all $v \in W$ and $D_{0,1,2}(z) = 2z$ and $D_{1,0,0}(\delta_\lambda) = z$ and $D_{1,0,0}(\mathfrak{d}^2_{2m+2}) = 0$.*

2. $\operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_{2m+2} = \mathfrak{s} \oplus [\operatorname{Inner} \mathfrak{d}_{2m+2}, \operatorname{Inner} \mathfrak{d}_{2m+2}]$, *and* $\mathfrak{s}^2$ *is isomorphic to the special unitary simple Lie algebra* $\mathfrak{su}_m(\mathbb{R})$. *For* $m \geq 3$, $\mathfrak{s}_0$ *is the orthogonal simple algebra of* $m \times m$ *skew-matrices.*

For $m = 1$, $\operatorname{Der} \mathfrak{d}_4 = \mathbb{R} \cdot D_{0,1,2} \oplus \mathbb{R} \cdot D_{1,0,0} \oplus \operatorname{Inner} \mathfrak{d}_4$ *and* $\operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_4 = \operatorname{Inner} \mathfrak{d}_4$.

Proof. The result follows from previous matrix decompositions and discussion. Since the special unitary real Lie algebra can be realized as the vector space of traceless skew-Hermitinan $m \times m$ matrices, so $\mathfrak{su}_m(\mathbb{R}) = \{M + iP : M^t = -M, P^t = P, \operatorname{tr} P = 0\}$ and $\mathfrak{so}_m(\mathbb{R}) = \{M : M^t = -M\}$ is a subalgebra. It is easily checked that the map $M+P_0 \mapsto M-iP_0$ (here $M+P_0$ represents the two first summands in the decomposition (5.7)) is a Lie isomorphism from $\mathfrak{s}^2$ to $\mathfrak{su}_m(\mathbb{R})$. The same map proves that $\mathfrak{s}_0 \cong \mathfrak{so}_m(\mathbb{R})$. □

Remark 5.2.11. Let $J = H(M_m(\mathbb{R}, t))$ be the real simple unitary Jordan algebra of $m \times m$ symmetric matrices for $m \geq 2$. Up to isomorphisms, $\mathfrak{s}_0$ and $\mathfrak{s} = \mathbb{R} \cdot \operatorname{ad} \delta_\lambda \oplus [\mathfrak{s}, \mathfrak{s}]_\mathfrak{s}$ are just the algebra of derivations of J and the Lie multiplication algebra of J according to [Jacobson, 1968, Chapter VI, Section 9, Theorems 9 and 11]. And the $\mathbb{Z}_2$-graded decomposition $\mathfrak{s}^2 = \mathfrak{s}_0 \oplus \mathfrak{s}_1 \cap \mathfrak{s}^2$ is related to the compact symmetric space $SU(m)/SO(m)$ (see [Helgason, 1979, Table V, page 518]). We also point out that Jordan algebras were introduced by Pascual Jordan in 1933 to formalize the notion of an algebra of observables in quantum mechanics. The algebra $\mathfrak{d}_4(\mathbb{R})$ is the algebra of the observables of the quantum mechanical model of the harmonic oscillator.

The existence of $\mathfrak{s}^2 \cong \mathfrak{su}_m(\mathbb{R})$ and $\mathfrak{s}_0 \cong \mathfrak{so}_m(\mathbb{R})$ as simple subalgebras of $\operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_{2m+2}$ for $m \geq 2$ and $m \geq 3$ lets us construct, in parallel with the quadratic solvable series $(\mathfrak{d}_{2m+2}, \varphi_{0,1})$, the series of mixed quadratic algebras,

$$(\mathfrak{d}_{2m+2})_{\mathfrak{su}_m(\mathbb{R})} := \mathfrak{su}_m(\mathbb{R}) \oplus \mathfrak{d}_{2m+2} \oplus \mathfrak{su}_m(\mathbb{R})^*, \text{ and}$$

$$(\mathfrak{d}_{2m+2})_{\mathfrak{so}_m(\mathbb{R})} := \mathfrak{so}_m(\mathbb{R}) \oplus \mathfrak{d}_{2m+2} \oplus \mathfrak{so}_m(\mathbb{R})^*,$$

by following Theorem 2.2.20. In this case, we extend from the natural inclusion of $\mathfrak{s}$ into the subalgebra $\operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_{2m+2}$, so $\mathfrak{b} = \mathfrak{s}^2, \mathfrak{s}_0$ and $\phi = \iota$.

As pointed out in Remark 5.2.4, double extensions of the oscillator algebra $\mathfrak{d}_4(\mathbb{R})$ by means of its skew-derivations only produce decomposable quadratic algebras (see Corollary 4.1.3) that are orthogonal sums $\mathfrak{d}_4 \oplus \mathfrak{a}$, with $\mathfrak{a}$ metric abelian because $\operatorname{Der}_{\varphi_{0,1}} \mathfrak{d}_4 = \operatorname{Inner} \mathfrak{d}_4$. But this little algebra, also known as *diamond* algebra, has a self-interest of its own (see [Douglas and Premat, 2007] and [Casati et al., 2010] and references therein).

5.3 Lattices of ideals

In Chapter 3 we started analysing the relationships among some important ideals in quadratic Lie algebras. Indeed, these ideals and their location are so important they define families like the local one we have just seen. In local algebras, the existence of a unique maximal ideal implies the existence of a unique minimal ideal. This structural pattern is a consequence of the orthogonality imposed by the quadratic form. In fact, we have the mapping

$$\begin{aligned} \Phi\colon \mathrm{Ideals}(L) &\to \mathrm{Ideals}(L) \\ I &\mapsto I^{\perp}. \end{aligned} \tag{5.8}$$

As the orthogonal of an ideal is another ideal, this mapping is well defined. Moreover, it is an involutive anti-automorphism which turns $I + J$ into $I^{\perp} \cap J^{\perp}$ and vice versa. This pattern was pointed in [Hofmann and Keith, 1986, Corollary 1.4]. As stated in that corollary, this influences strongly the ideal structure of the algebra.

The set of ideals of a Lie algebra L is a *partially ordered set* (*poset*) ordered by inclusion. In the set of ideals of L, every two elements have unique supremum taking the sum of ideals, and infimum when considering the intersection, which coincides with the definition of a *lattice*.

In this section we will start seeing the main structural properties of ideals in Lie algebras, particularly on finite lattices of quadratic Lie algebras. Among all the variety, we will focus on lattices whose ideals form a chain, which are a peculiar example of local Lie algebras. For those ones, we will give procedures about how to obtain them, to end up seeing a list of these chains and when they are quadratic. Along this first subsection we follow, partially, [Benito and Roldán-López, 2022a].

5.3.1 Structure and properties

As observed in the introduction, when considering sum and intersection of ideals, the set of ideals form a lattice.

Definition 5.3.1. A poset $(L, \leq)$ is a *lattice* if $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ exist for all $a, b \in L$.

As seen in [Grätzer, 2011, Lemma 2], this definition is equivalent to $\sup H$ and $\inf H$ exist for every finite nonempty subset H of L. We should not confuse this to being complete.

Definition 5.3.2. A lattice L is called *complete* if every subset (not only finite nonempty subsets) of ideals has a join and a meet.

Lattices of ideals of Lie algebras are complete lattices. Moreover, these lattices are also *bounded*: the zero ideal is the smallest element, and the total algebra is the largest element. In terms of lattices, they are also referred as 0 and 1 respectively. Many fundamental properties of Lie algebras can be interpreted as facts about lattices of ideals (see [Benito, 1995] and for finite lattices [Benito and Roldán-López, 2022a]).

In addition, we can represent lattice of ideals, totally or partially, using a *Hasse diagram*, which represents a finite poset in the form of a drawing of its transitive reduction. For a poset $(S, \leq)$, the Hasse diagram represents the elements of S by nodes (small black circles in our figures). And the nodes representing the elements x and y of S are connected by a straight line or segment that goes upward from x to y whenever y *covers* x, that is, whenever $x < y$ and there is no z such that $x < z < y$. These segments may cross each other, but must not touch any vertices other than their endpoints. We have drawn some examples of these diagrams in Figure 5.1.

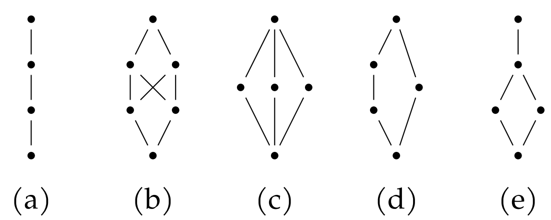

Figure 5.1: Examples of different Hasse diagrams. Not all of them are valid Hasse diagramas for Lie algebras ideals as we will see.

According to [Grätzer, 2011, Lemma 1, Section 1.4], from a Hasse diagram we can recapture the relation $\leq$ by noting that $x < y$ holds if and only if there exists a sequence of elements $c_0, c_1, \ldots, c_n$ such that $x = c_0$, $y = c_n$ and c_{i+1} covers c_i. Hence, the Hasse diagram of a finite poset determines the poset up to isomorphisms. A finite lattice is a poset attached to a Hasse diagram for which every pair of nodes has a unique supremum and infimum. The above remark implies that for each of the diagrams in Figure 5.1, the corresponding poset is a lattice with the exception of diagram (b).

On lattices of ideals of Lie algebras, we can observe that for any three ideals A, B, C of L such that $A \subseteq B$, we have the identity

$$B \cap (A + C) = A + (B \cap C). \tag{5.9}$$

In a general lattice, equation (5.9) can be rewritten exactly as the *Modular Law* in the following definition:

Definition 5.3.3. We say a lattice $\mathcal{L}$ is *modular* it satisfies the Modular Law, i.e., if for every $a, b, c \in \mathcal{L}$ such that $a < b$

$$\inf\{b, \sup\{a, c\}\} = \sup\{a, \inf\{b, c\}\}.$$

So, lattices of ideals of Lie algebras are also modular, apart from being complete and bounded. Unfortunately, despite the strong conditions a Lie algebra imposes on its lattice of ideals, the ideal lattice structure does not always determine the Lie algebra in a unique way. For instance, the one-dimensional Lie algebra and every simple Lie algebra have the same lattice of ideals consisting of a 2-element chain (see lattice h2.1 in Figure 5.6). This elementary example shows that there exist non-isomorphic Lie algebras with the same lattice of ideals. Even more, the lattice ideal structure is not preserved through scalar extensions as seen in Example 5.3.1. Nevertheless, the lattice structure of a Lie algebra does approach to the algebraic structure.

Example 5.3.1. The oscillator Lie algebra has 4-chain as a lattice when considered over $\mathbb{R}$, while it is related to the second lattice in Figure 5.2 when taken over $\mathbb{C}$.

In Figure 5.2, we can observe how the third lattice comes from a sort of duplication of the second one. This duplication can be achieved, for example, by a direct sum of a simple Lie algebra as a trivial extension. Indeed, this also

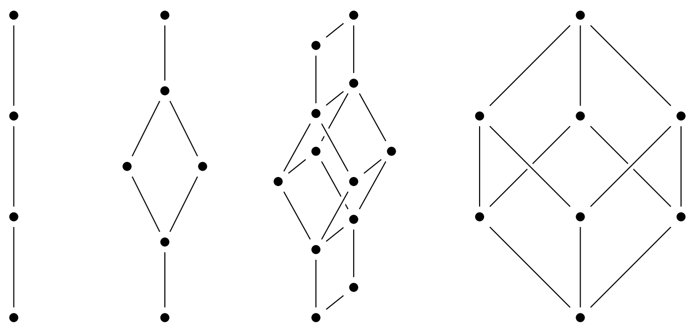

Figure 5.2: Examples of lattices of Lie algebras.

explains why $\mathfrak{sl}_2(\mathbb{F}) \oplus \mathfrak{sl}_2(\mathbb{F}) \oplus \mathfrak{sl}_2(\mathbb{F})$ has by its lattice the fourth one in the same figure.

When studying lattices of ideals, other definitions appear:

Definition 5.3.4. Let $\mathcal{L}$ be a bounded lattice attached to the poset $(S, \leq)$ with join and meet denoted by $\vee$ and $\wedge$ respectively. Then:

(a) $\mathcal{L}$ is a *complemented lattice* if each element has a complement, that is, for a given element a, there is an element b such that $a \vee b = 1$ and $a \wedge b = 0$.

(b) $\mathcal{L}$ is a *distributive lattice* if $\mathcal{L}$ satisfies either of the following equivalent distributive laws:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

(c) $\mathcal{L}$ is a *boolean lattice* if it is complemented and distributive.

The completely reducibility of reductive Lie algebras implies that their lattices of ideals are complemented. When their centre has dimension less or equal than one they are also distributive. Two typical examples of non-distributive lattices are the pentagon N_5 and the diamond M_3 whose Hasse diagrams are shown in Figure 5.3. The following theorem characterizes distributive and modular lattices by means of pentagons and diamonds.

Theorem 5.3.1. (See [Grätzer, 2011, Theorem 102]) *Every distributive lattice is modular. Moreover:*

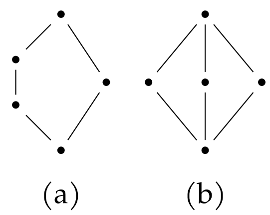

Figure 5.3: (a) is the pentagon N_5 non-modular lattice, and (b) is the diamond M_3 non-distributive lattice.

(a) *A lattice is modular if and only if it does not contain a pentagon.*

(b) *A lattice is distributive if and only if it does not contain a pentagon or a diamond.*

This relation between distributive lattices and the diamond lattice makes the following definition appears.

Definition 5.3.5. Let $\mathcal{L}$ be a lattice. We say $\mathcal{L}$ satisfies the *diamond property* (DP) if for two distinct *atoms*[1] a, b there exists another element c for which $\{0, a, b, c, a \vee b\}$ is the M_3 lattice in Figure 5.3.

As seen the previous theorem, if $\mathcal{L}$ satisfies DP then $\mathcal{L}$ is non-distributive.

In a totally ordered set $(S, \leq)$, not just a poset, any pair of elements are comparable, thus the infimum and supremum exist and therefore S is always a lattice. If this set contains n elements, we will say that S is an n-element *chain lattice* (n-chain for short). The Hasse diagram for any n-chain is given in Figure 5.4. Chains play an important role as they allow us to define the length of a general lattice.

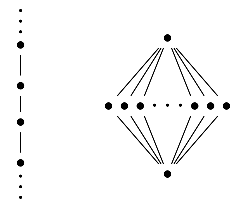

Figure 5.4: On the left the n-chain and on the right the M_n lattice. They have n and $n+2$ nodes respectively.

[1]Any element that covers 0 is named *atom*.

Definition 5.3.6. A general lattice is said to be of *length* k, where k is a natural number ($k = 0$ is possible), if there is a $(k+1)$-chain sublattice and all of its chain sublattices have a number of elements smaller than $k+1$.

Note that the length of an n-chain is $n-1$, just the number of "jumps". In Figure 5.4 we can also see the generalization M_n of M_3 lattice. M_n is the 2-length lattice with $n \geq 3$ atoms.

The following examples relate the set of ideals of some varieties of Lie algebras and the previous notions and examples of lattices. From now on, $\mathcal{L}_{\mathrm{Id}}(L)$ will denote the lattice of ideals of a Lie algebra L with product $[x, y]$.

Example 5.3.2 (n-chain lattice $C_n (n \geq 1)$). This lattice is distributive and non-complemented if $n \geq 3$. The trivial Lie algebra is the only algebra whose lattice of ideals is C_1. The variety of Lie algebras that have C_2 as lattice of ideals is the set of simple Lie algebras plus the 1-dimensional algebra. For any $n \geq 3$, Lie algebras whose lattice of ideals is an n-chain have been fully characterized in [Benito, 1992a]. A complete classification is given for solvable Lie algebras over algebraically closed fields.

Example 5.3.3 (n-subspace lattice $\operatorname{Sub} \mathbb{F}^n$ $(n \geq 0)$). The set of subspaces, denoted as $\operatorname{Sub} \mathbb{F}^n$, of the n-dimensional vector space $\mathbb{F}^n$ is a complemented and modular lattice of length n that satisfies the DP. This lattice is not finite and nondistributive if $n \geq 2$. The variety of Lie algebras whose lattice of ideals is isomorphic to $\operatorname{Sub} \mathbb{F}^n$ for some n, is the variety of finite dimensional abelian Lie algebras. This happens because any algebra L such that $\mathcal{L}_{\mathrm{Id}}(L)$ is isomorphic to $Sub\, \mathbb{F}^n$ decomposes into the sum of n minimal ideals $L = I_1 \oplus \cdots \oplus I_n$. The ideals in the decomposition are abelian because of the DP, and therefore $L^2 = 0$. This variety is trivially quadratic.

Example 5.3.4 (M_n lattice $(n \geq 3)$). This lattice has length 2 and it is complemented, modular and satisfies the DP, so it is nondistributive. There are no Lie algebras with a lattice of ideals isomorphic to M_n: If $\mathcal{L}_{\mathrm{Id}}(L)$ is isomorphic to M_n, DP implies $L^2 = 0$, so L is a 2-dimensional abelian Lie algebra and therefore $\mathcal{L}_{\mathrm{Id}}(L)$ is not a finite lattice, a contradiction. This assertion is a consequence of Theorem 5.3.4. When L is abelian of dimension two we obtain lattice $\operatorname{Sub} \mathbb{F}^2$ which behaves as M_n when n goes to infinity.

For a finite set A, the Hasse diagram of the lattice of

$$\operatorname{Pow} A = \{S : S \subseteq A\},$$

the set of subsets, also denoted as 2^A, has $2^{|A|}$ nodes. Here a line joins two nodes whenever the corresponding subsets differ in a single element. This way, we arrive at the hypercube lattice or n-cube lattice Q_n, the lattice of subsets of the set $\{1, 2, \ldots, n\}$, where Q_0 is the lattice of subsets of the empty set. We point out the following result about n-cube lattices and its relation with distributive and boolean lattices:

Theorem 5.3.2. [*Grätzer, 2011, Corollaries 109, 110*] *Let $\mathcal{L}$ be a finite lattice. Then:*

(a) *$\mathcal{L}$ is distributive if and only if $\mathcal{L}$ is isomorphic to a sublattice of the lattice of subsets of some finite set.*

(b) *$\mathcal{L}$ is boolean if and only if $\mathcal{L}$ is isomorphic to Q_n for some $n \geq 0$.*

Example 5.3.5 (n-cube lattice Q_n $(n \geq 0)$). In Figure 5.5 we have drawn the n-cube lattices of ideals of either a semisimple Lie algebra with $n = 1, 2, 3, 4, 5$ simple components or $L = S \oplus \mathbb{F}z$, where $\mathbb{F}z$ is the centre of L and S is a semisimple ideal with $n - 1 = 0, 1, 2, 3, 4$ simple components. The lattices Q_4 and Q_5 are called tesseract and penteract respectively. We also note that n-cube lattices are just the lattices of ideals of reductive Lie algebras, whose centre has dimension at most 1, and all these algebras are quadratic.

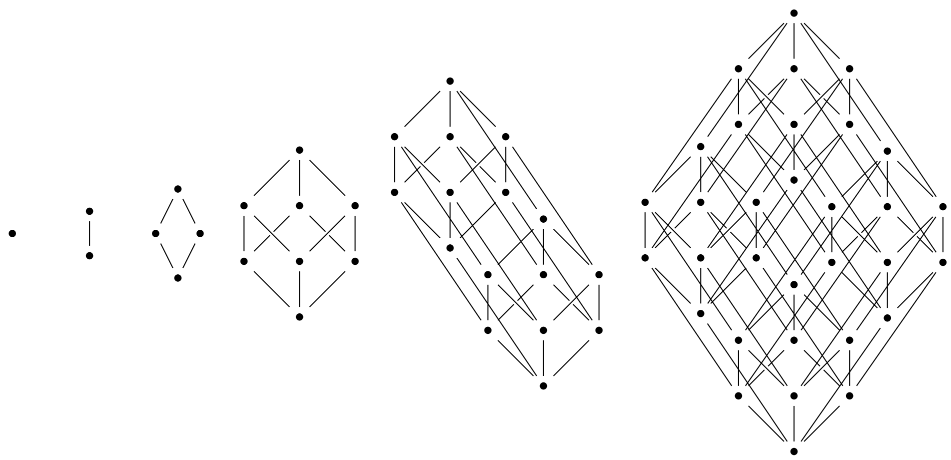

Figure 5.5: n-cube lattices from $n = 0$ (left) to $n = 5$ (right).

Given a vector space M and any subset Ω of linear maps in M, the set of subspaces of M which are invariant under Ω is a lattice denoted as $\mathrm{Sub}_\Omega M$.

Here, two subspaces $A, B \in \operatorname{Sub}_\Omega M$ are Ω-isomorphic if there is a bijective map $\varphi\colon A \to B$ such that $\varphi(f(a)) = f(\varphi(a))$ for any $a \in A$ and any $f \in \Omega$.

Lemma 5.3.3. *Let Ω be a Lie algebra of linear transformations in a vector space M and let $\operatorname{Sub}_\Omega M$ be the lattice of Ω-invariant subspaces of M. Then, the following assertions are equivalent:*

(a) $\operatorname{Sub}_\Omega M$ *is a finite and boolean lattice.*

(b) $\operatorname{Sub}_\Omega M$ *is a finite and complemented lattice.*

(c) Ω *is completely reducible in M and* $\operatorname{Sub}_\Omega M$ *has no irreducible Ω-isomorphic elements.*

(d) $\Omega = \Omega_1 \oplus A$*, where Ω_1 is a semisimple ideal of Ω, A is the centre, the elements of A are semisimple, i.e. the minimum polynomial of any linear map of A is the product of relatively prime irreducible polynomials, and* $\operatorname{Sub}_\Omega M$ *has no irreducible Ω-isomorphic elements.*

If any of these equivalent conditions hold, M has a unique decomposition $M = M_1 \oplus \cdots \oplus M_r$, where $M_j \in \operatorname{Sub}_\Omega M$ are irreducible and no Ω-isomorphic subspaces. In particular,

$$\operatorname{Sub}_\Omega M = \{M_{i_1} \oplus \cdots \oplus M_{i_k} : 1 \leq i_1 < \ldots < i_k \leq r\} \cup \{0\},$$

and $\operatorname{Sub}_\Omega M$ *is isomorphic to the r-cube lattice Q_r.*

Proof. If we omit the condition of finiteness and the property of nonexistence of Ω-isomorphic subspaces, the equivalence of the assertions (b), (c) and (d) follows from [Jacobson, 1979], specifically from Theorem 9 in Section 5 of Chapter II, and Theorem 10 in Section 7 of Chapter III.

Now assume $\operatorname{Sub}_\Omega M$ is a finite and complemented lattice. Then, there are no irreducible Ω-isomorphic subspaces P and Q. Otherwise, there exists a Ω-isomorphism $\varphi\colon P \to Q$, so $\varphi f = f \varphi$ for any $f \in \Omega$. In this case, for any $\alpha \in \mathbb{F}$ we can define the nonzero subspace,

$$R^\alpha = \{a + \alpha\, \varphi(a) \mid a \in P\} \subseteq P \oplus Q.$$

It is clear that $R^\alpha \in \operatorname{Sub}_\Omega M$. Suppose $R^\alpha = R^\beta$, so $a + \alpha\, \varphi(a) = b + \beta\, \varphi(b)$, $a, b \in P \setminus \{0\}$. The last equation implies

$$a - b = \beta\, \varphi(b) - \alpha\, \varphi(a) = \varphi(\alpha a - \beta b) \in P \cap Q = 0,$$

and therefore $\beta = \alpha$. Hence for any scalar α we get distinct Ω-invariant subspaces R^α, a contradiction because the based field $\mathbb{F}$ is infinite and $\operatorname{Sub}_\Omega M$ is a finite set.

Assume finally that Ω is completely reducible in M and M does not contain Ω-isomorphic irreducible subspaces. Then $M = M_1 \oplus \cdots \oplus M_r$, where the different $M_j \in \operatorname{Sub}_\Omega M$ are not Ω-isomorphic. For any irreducible subspace P, it follows that $M = P \oplus P_2 \oplus \cdots \oplus P_k$, where P_i are irreducible. Since there are no Ω-isomorphic irreducible subspaces, applying Krull-Schmidt Theorem of sets of linear transformations, we get that $k+1 = r$ and $P = M_j$ for some $1 \leq j \leq r$. So, apart from the trivial subspace, the Ω-invariant subspaces of M are direct sums of a finite number of subspaces $M_1, \ldots, M_r$. The equivalence of (a) and (c) follows from [Pierce, 1982, Chapter 2, Corollary c, Section 2.4] and therefore Theorem 5.3.2 ends the proof. □

This lemma prepares us to prove one of the main results for lattices.

Theorem 5.3.4. *Finite lattices of ideals of Lie algebras are always distributive. Up to isomorphisms, these lattices are sublattices of n-cube lattices.*

Proof. Let L be a Lie algebra and $\mathcal{L} = \mathcal{L}_{\mathrm{Id}}(L)$ be its lattice of ideals. Assume the result is false. Since $\mathcal{L}$ is a modular and nondistributive lattice, there is a sublattice of type M_3 in $\mathcal{L}$ according to Theorem 5.3.1. So there exist ideals K, R, P_1, P_2 and P_3 such that $P_i \cap P_j = K$, $P_i + P_j = R$, and R covers P_i and P_i covers K for each $i = 1, 2, 3$. Consider now the quotient Lie algebra L/K and note that $Q_i = P_i/K$ are minimal ideals. Since $[P_i, P_j] \subseteq K$ and $P_k \subseteq P_i + P_j = R$, the ideal $T = R/K$ decomposes as the direct sums, $T = Q_1 \oplus Q_2 = Q_1 \oplus Q_3 = Q_2 \oplus Q_3$. Then $[Q_i, Q_j] \subseteq Q_i \cap Q_j = 0$, and therefore each Q_i is an abelian ideal. Let M denote the sum of all minimal abelian ideals of L/K, and consider the Lie algebra of linear maps in M, $\Omega = \operatorname{ad}_M L/K$, $\operatorname{ad} x(m) = [x, m]$ for all $x \in L/K, m \in M$. The elements of the lattice $\operatorname{Sub}_\Omega M$ are just the set of ideals of L/K inside M. This lattice is finite and complemented because of M decomposes as a direct sum of minimal abelian ideals. Using Lemma 5.3.3, there is no Ω-isomorphic ideals of L contained in M. Since Q_i are ideals, every canonical projection $\varphi_{ijk}\colon T = Q_i \oplus Q_j \to Q_k$, $\{(i, j, k) : i < j, k = i, j\}$, satisfies $\varphi_{ijk}([x, a]) = [x, \varphi_{ijk}(a)]$ for all $x \in L/K$. So, in a natural way, the maps φ_{ijk} let us define an Ω-isomorphism $\varphi\colon Q_1 \to Q_2$, a contradiction. The final part follows from Theorem 5.3.2. □

In conclusion, the set of ideals of a Lie algebra is a bounded and modular lattice. If we impose finiteness, it is also distributive. In Figure 5.6 we show all possible distributive lattices up to 8 nodes according to [Erné et al., 2002]. Lie algebras up to 6 ideals have been characterized in [Benito, 1992b] and [Roldán-López, 2017]. Using ideas and techniques included in [Benito, 1992b], [Benito, 1995] and [Roldán-López, 2017], most of the distributive lattices of 7 and 8 nodes can be easily performed as the lattices of ideals of some Lie algebra.

As a direct consequence of the Theorem 5.3.4, we have the next result which describes the variety of Lie algebras whose lattice of ideals is either complemented or boolean, all of them self-dual. We remark that the general assertion about complemented lattices and the equivalence of statements (a) and (e) have been previously stablished in [Benito, 1995, Lemma 2.3].

Corollary 5.3.5. *The Lie algebras with complemented lattice of ideals are of the form $L = S \oplus A$, where S is a semisimple ideal of L and A is an abelian ideal, so $A = Z(L)$. Moreover, the following assertions are equivalents:*

(a) *$\mathcal{L}_{\mathrm{Id}}(L)$ is a boolean lattice.*

(b) *$\mathcal{L}_{\mathrm{Id}}(L)$ is finite and complemented.*

(c) *$\mathcal{L}_{\mathrm{Id}}(L)$ is an n-cube lattice for some $n \geq 0$.*

(d) *The Jacobson radical of L is trivial and $Z(L)$ has dimension at most 1.*

(e) *L is either 0 or one of the following Lie algebras: $\mathbb{F}z$, a semisimple algebra S or a direct sum as ideals of $\mathbb{F}z$ and S.*

In this case, L has 2^n ideals where n is either r or $r+1$ and r is the number of simple components of S ($r = 0$ is also possible).

Proof. Let $\Omega = \operatorname{ad} L = \{\operatorname{ad} x : x \in L\}$, where $\operatorname{ad} x(a) = [x, a]$. Note that $\mathcal{L}_{\mathrm{Id}}(L)$ is the set of Ω-invariant subspaces of L, so $\mathcal{L}_{\mathrm{Id}}(L) = \operatorname{Sub}_\Omega L$. First, we will prove the characterization of complemented lattices. Assume $\mathcal{L}_{\mathrm{Id}}(L)$ is complemented and let A be an ideal such that $L = L^2 \oplus A$. Then $[L, A] \subseteq L^2 \cap A = 0$, thus $A \subset \mathrm{Z}(L)$. On the other hand, the derived ideal L^2 decomposes as a sum of minimal ideals, $L^2 = I_1 \oplus \cdots \oplus I_k$, and $I_j^2 = I_j$ because of $(L^2)^2 = L^2$. Hence $L = L^2 \oplus \mathrm{Z}(L)$ is a direct sum of the semisimple ideal L^2 and the

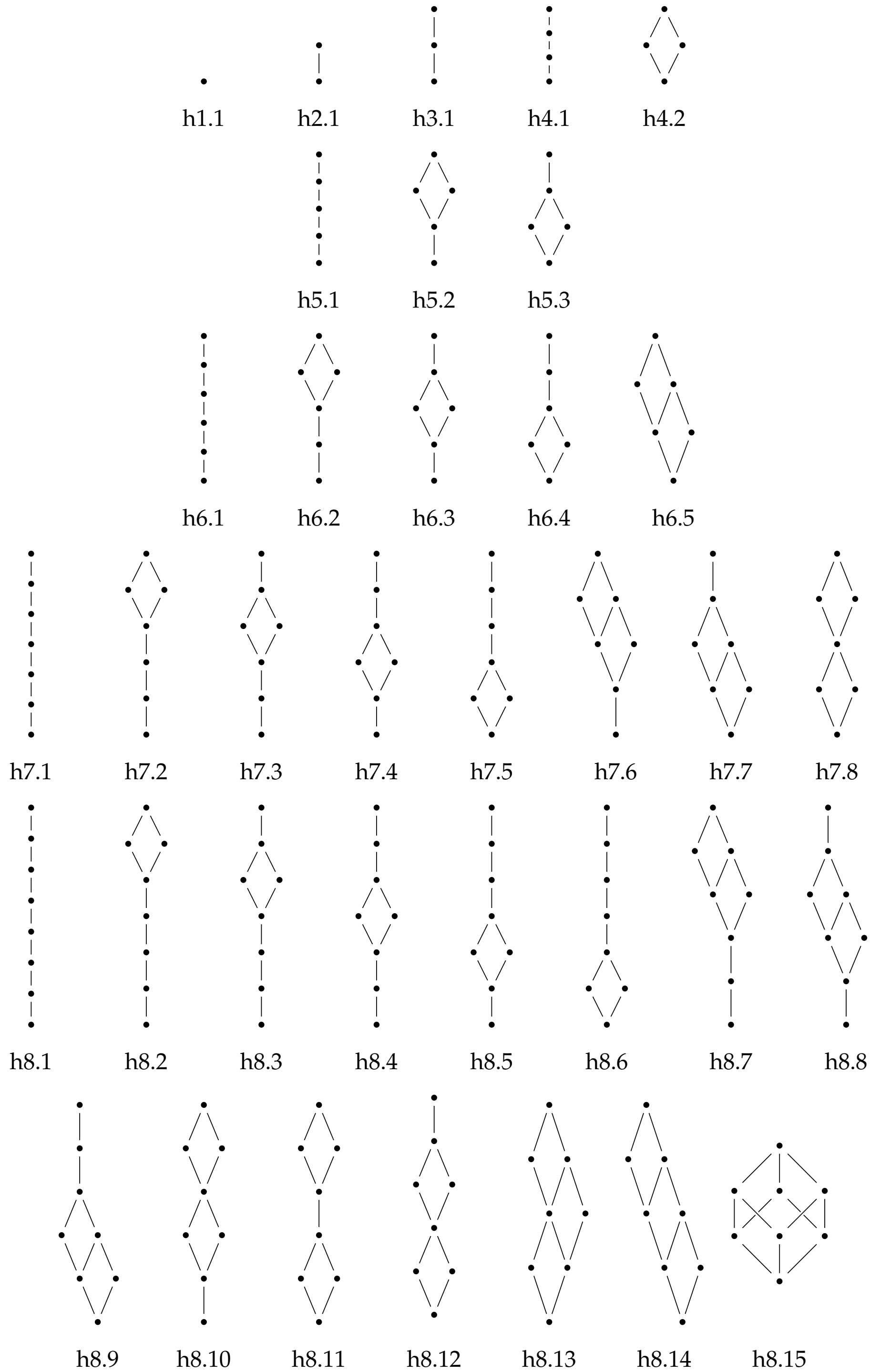

Figure 5.6: All Lie algebras Hasse diagrams up to 8 ideals (nodes).

centre. Conversely, in the case $L = S \oplus Z(L)$, $\Omega = \operatorname{ad} L \cong \operatorname{ad} S$ and $\mathcal{L}_{\mathrm{Id}}(L)$ is complemented by [Jacobson, 1979, Theorem 10, Section 7, Chapter III].

Now we will check the equivalence of the five conditions. Theorem 5.3.4 shows that statement (b) implies (a). The previous paragraph and Example 5.3.3 give us item (e) from item (a). Since the Jacobson radical of L is just $L^2 \cap \operatorname{Rad}(L) = [L, \operatorname{Rad}(L)]$, where $\operatorname{Rad}(L)$ is the solvable radical (see [Jacobson, 1979, Chapter III, Section 9] and [Marshall, 1967]), item (e) implies (d) follows from $[L, \operatorname{Rad}(L)] = [L, \mathbb{F}z] = 0$. In the case $L^2 \cap \operatorname{Rad}(L) = 0$ we have $L = S \oplus \mathrm{Z}(L)$, so $S = L^2$ is semisimple and $\operatorname{ad} L \cong S$. Then the implication from statement (d)to item (b) follows by using $\dim \mathrm{Z}(L) \leq 1$ and Lemma 5.3.3. Finally, Theorem 5.3.2 ensures that items (a) and (c) are equivalent. $\square$

The algebras in this corollary are quadratic Lie algebras as they are reductive. Lattices in Figures 5.2 and 5.5 also belong to quadratic Lie algebras. We can observe all of their Hasse diagrams coincide with themselves when they are turned upside down. This symmetry is a consequence of the mapping Φ in equation (5.8). This function maps between L^2 and $Z(L)$ and in general, the terms in the central descending series to the ascending ones. Since $I \subseteq J$ implies $J^\perp \subseteq I^\perp$, Φ forces quadratic Lie algebras to have a selfdual lattice of ideals. In case of finite lattices, the dualization provides a symmetry in the Hasse diagram respect an horizontal axis. These results are pointed out in [Hofmann and Keith, 1986, Section 1], and appear summarized in the following proposition.

Proposition 5.3.6. *In any Lie algebra L endowed with an non-degenerate symmetric invariant form, the map* $\Phi\colon \mathcal{L}_{\mathrm{Id}}(L) \to \mathcal{L}_{\mathrm{Id}}(L)$ *such that* $\Phi(I) = I^\perp$ *is an involutive anti-automorphism of the lattice of ideals of L. In particular:*

(a) $I \subseteq J$ *if and only if* $\phi(J) \subseteq \phi(I)$.

(b) $\Phi(I) = J$ *if and only if* $\Phi(J) = I$.

(c) $\Phi(I + J) = I \cap J$.

(d) $\Phi(L^k) = Z_{k-1}(L)$, *in particular,* $\Phi(L^2) = Z(L)$.

(e) *If I is minimal,* $\Phi(I)$ *is maximal and vice versa. Thus* $\Phi(\operatorname{soc}(L)) = \mathcal{J}(L)$.

(f) $\dim L = \dim I + \dim \Phi(I)$, *in particular,* $\dim I/J = \dim \Phi(J)/\Phi(I)$.

So, the lattices of ideals of quadratic Lie algebras are self-dual.

It is important to note, this lattice duality is a necessary condition, but not sufficient. Their codimensions must also respect this symmetry due to item (f) in previous proposition. But, there are also Lie algebras with symmetric Hasse diagram and symmetric codimensions in their ideals which are not quadratic as seen in the following example.

Example 5.3.6. Let $\mathbb{F} \cdot \widehat{\mathrm{Id}} \ltimes \mathfrak{h}_3$ be a Lie algebra obtained from the 3-dimensional Heisenberg in a semidirect product with a derivation obtained from extending the identity (check equation (5.3)). This algebra has a self-dual lattice (see Figure 5.7), with symmetric dimensions as each ideal has codimension 1 in its consecutive ideals.

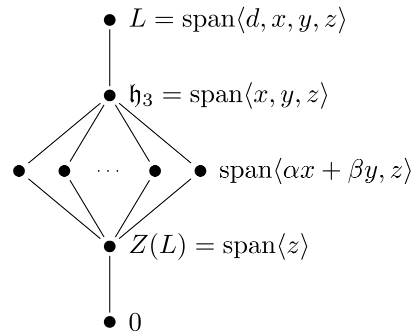

Figure 5.7: Lattice of ideals of $\mathbb{F} \cdot \widehat{\mathrm{Id}} \ltimes \mathfrak{h}_3$. Note the interval sublattice $[\mathfrak{h}_3 : Z(L)]$ is a $\mathrm{Sub}\,\mathbb{F}^2$ lattice.

The non-null products in this algebra, taking basis $\{d, x, y, z\}$ are

$$[d, x] = x, \qquad [d, y] = y, \qquad [d, z] = 2z, \qquad [x, y] = z.$$

But it is not quadratic because respect to $\mathrm{ad}\, d$ both x, y and z are eigenvectors of non-opposite eigenvalues. Therefore, as

$$\varphi(x, z) = \varphi([d, x], z) = -\varphi(x, [d, z]) = -2\varphi(x, z),$$

$\varphi(x, z)$ must be zero. Analogously for every pair in $\{x, y, z\}$, we obtain $\varphi|_{\mathfrak{h}_3} = 0$ which prevents φ from being non-degenerate. Despite being similar, this algebra differs in $\mathrm{ad}\, d$ with respect the oscillator, which it was quadratic.

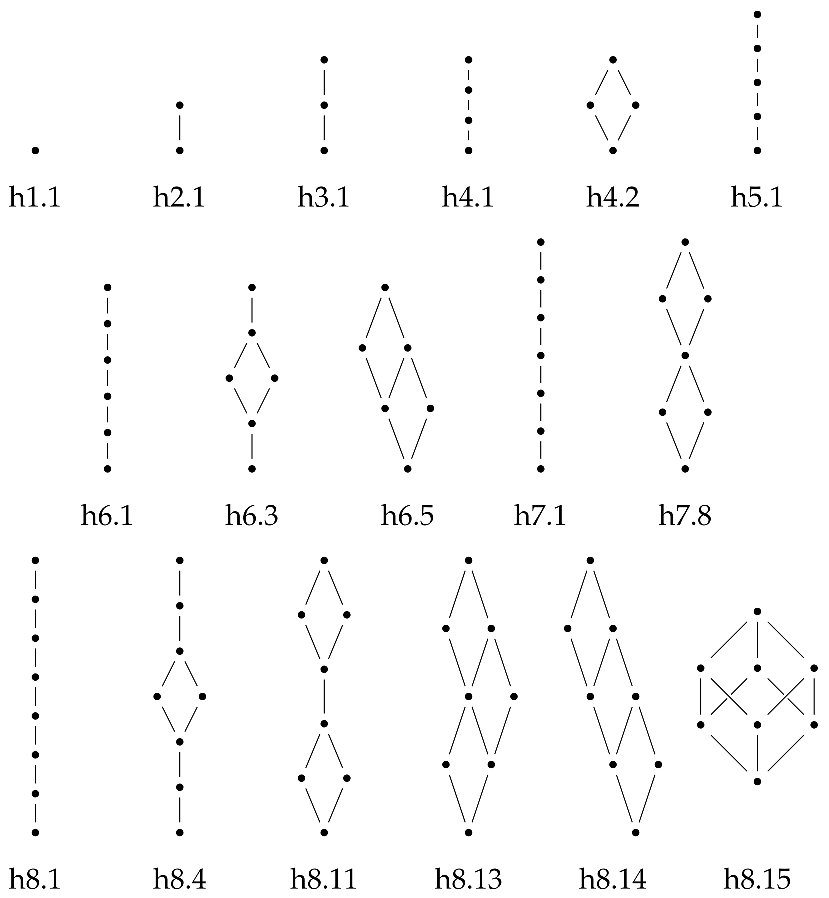

Figure 5.8: All Lie algebras self-dual Hasse diagrams up to 8 ideals (nodes).

Among all lattices of ideals up to 8 nodes seen in Figure 5.6, only few of them are selfdual. We can see which of them are in Figure 5.8. We can easily obtain some quadratic Lie algebras from those self-dual lattices by using our deconstructing results on Chapter 3, structural patterns and some classical constructions from Chapter 2:

- Lattices h3.1, h5.1, h7.1, h7.8 and h8.13 are T^*-extensions using Theorem 2.2.23 because they contain a lagrangian ideal. As an example, h3.1 is the lattice of $T_0^*(S)$ for any S simple Lie algebra.
- h2.1, h4.2 and h8.15 correspond, according to Corollary 5.3.5, to reductive Lie algebras with $\dim Z(L) \leq 1$ containing exactly one, two and three factors respectively.

- h6.5 comes from an algebra, whose lattice is h3.1, in direct sum with a one-dimensional or simple ideal. In the same way h8.14 follows from h4.1.

- hi.1 for $i = 1, \ldots, 8$ are chains. We have seen some quadratic algebras, as the real 4-dimensional oscillator one, whose lattices are of this type. For $i \geq 3$, they are local algebras according to Lemma 5.2.1, which also says they can be decomposed as its nilradical plus a simple or 1-dimensional subalgebra. Thus, using Theorem 3.1.8, we can obtain them as double extensions of $N/N^{\perp}$. In these chains, the nilradical occupies the second ideal starting from above, and its orthogonal appears in the penultimate position.

Apart from finite lattices, some of the algebras we are studying present infinite lattices. In those one, we can still clearly observe the self-dual property as seen in the two first lattices in Figure 5.9. The third one shows the existence of quadratic Lie algebras whose lattice of ideals is a n-chain for any $n \in \mathbb{N}$ as long as you want. This, in combination with lattices hi.1 for $i = 1, \ldots, 8$ from Figure 5.8, opens up our next section.

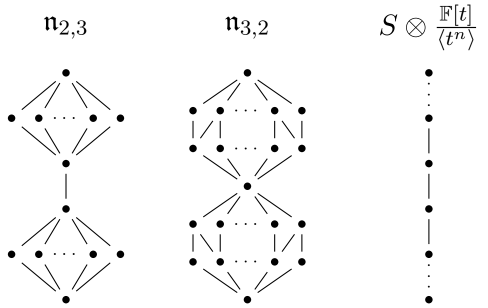

Figure 5.9: Infinite quadratic lattices and n-chains for S simple. $[n_{2,3} : n_{2,3}^2]$ and $[n_{3,2} : n_{3,2}^2]$ are respectively $\operatorname{Sub} \mathbb{F}^2$ and $\operatorname{Sub} \mathbb{F}^3$.

5.3.2 Chain lattices of ideals

From Dilworth's Chain Decomposition Theorem [Dilworth, 1950], any finite lattice decomposes as a disjoint union of chains. This fact highlights n-element chain lattices as basic blocks for embedding or decomposing finite lattices. As describe in the characterization of these chains in [Benito, 1992a], for dimension greater or equal than 2, solvable n-chains Lie algebras are 1-dimensional extensions of a nilpotent Lie algebra $\mathfrak{n}$. This algebra $\mathfrak{n}$ is the nilradical and it can be a GHA, or a *filiform*[2] with nilpotency index greater or equal than 3, or a finite *thin algebra* with two diamonds (both the centre and $\mathfrak{n}/\mathfrak{n}^2$ have dimension two and $\frac{\mathfrak{n}^i}{\mathfrak{n}^{i+1}}$ is one-dimensional otherwise). The common pattern to all $\mathfrak{n}$ is that they are *naturally graded* and generated by a subspace. That is, they are *quasi-cyclic* Lie algebras.

In the non-solvable case, there is a large number of different types of mixed algebras with chain lattices of ideals. The easiest example is the 3-chain given by any split extension of a simple Lie algebra and a nontrivial irreducible module. Mixed n-chain algebras are formed from a nilpotent radical and a simple Levi factor. As in the solvable case, the chain ideal condition also imposes positive gradings on the nilradical. Examples are given at the end of this section.

5.3.2.1 Examples and theoretical support

Using free nilpotent models in equation (2.7), the UMP and description of the derivations in Proposition 5.1.1, Corollary 5.1.2 and the remark below, any linear map $f\colon \mathfrak{u} \to \mathfrak{u}$ extends to a derivation d_f of $\mathfrak{n}_{d,t}$. The set of such extension maps that have zero trace is just the Levi subalgebra of $\operatorname{Der}\mathfrak{n}_{d,t}$ that we also denote as $\mathfrak{sl}(\mathfrak{u},\mathbb{F})$ (for short $\mathfrak{sl}_d(\mathbb{F})$ or $\mathfrak{sl}(\mathfrak{u})$). Seen as $\mathfrak{sl}(\mathfrak{u})$-modules[3] for $d \geq 3$, we have $\mathfrak{u} = V(\lambda_1)$, $\Lambda^2\mathfrak{u} = V(\lambda_2)$ and $\Lambda^3\mathfrak{u} = \mathbb{F}$ if $d = 3$ and $\Lambda^3\mathfrak{u} = V(\lambda_3)$ if $d \geq 4$. In addition,

$$\frac{\mathfrak{u}\otimes\Lambda^2\mathfrak{u}}{\Lambda^3\mathfrak{u}} = V(\lambda_1+\lambda_2).$$

Denoting by ρ_1 and ρ_2 the natural representations of $\mathfrak{sl}(\mathfrak{u})$ on $\mathfrak{n}_{d,2}$ and $\mathfrak{n}_{d,3}$, we arrive at the series of mixed Lie algebras with 4-chain and 5-chain ideals:

$$\mathfrak{sl}(\mathfrak{u}) \oplus_{\rho_1} \mathfrak{n}_{d,2} \quad \text{and} \quad \mathfrak{sl}(\mathfrak{u}) \oplus_{\rho_2} \mathfrak{n}_{d,3}. \tag{5.10}$$

Example 5.3.7. The smallest Lie algebras in equation (5.10) correspond to a vector space $\mathfrak{u}$ of dimension 2. In this case, we get algebras of dimension 6 and 8 with $\mathfrak{sl}_2(\mathbb{F})$-irreducible decomposition $V_2\oplus V_1\oplus V_0$ and $V_2\oplus V_1\oplus V_0\oplus V_1$. Here

[2] See for example [Bordemann, 1997] for a definition.

[3] For any simple Lie algebra, following [Humphreys, 1997], any irreducible module is of the form $V(\lambda)$ where λ is a dominant weight.

V_n is the $(n+1)$-dimensional irreducible module of $\mathfrak{sl}_2(\mathbb{F})$ and V_2 is just the adjoint module of $\mathfrak{sl}_2(\mathbb{F})$. Equation (5.12) describes the modules, and later, through this section, we will give a complete description of this algebra using basis and bracket product.

Example 5.3.8. From the set of skew-maps of a fixed vector space $\mathfrak{u}$ relative to a bilinear and non-degenerate form φ, either symmetric or skew-symmetric, we get classical simple Lie algebras $\mathfrak{so}(\mathfrak{u},\varphi)$ and $\mathfrak{sp}(\mathfrak{u},\varphi)$. The natural module $\mathfrak{u} = V(\lambda_1)$ of $\mathfrak{so}(\mathfrak{u},\varphi)$ ($\dim \mathfrak{u} \geq 7$ is required for tensor decompositions) provides the representation ρ_i, (it is the restricted representation of the one given in equation (5.10)). Then, $\Lambda^2\mathfrak{u} = V(\lambda_2)$,

$$Z(\mathfrak{n}_{d,3}) = \frac{\mathfrak{u}\otimes\Lambda^2\mathfrak{u}}{\Lambda^3\mathfrak{u}} = V(\lambda_1+\lambda_2)\oplus V(\lambda_1),$$

and we get the series of Lie algebras of seven ideals

$$\mathfrak{L}(\mathfrak{u},\varphi) = \mathfrak{so}(\mathfrak{u},\varphi)\oplus_{\rho_2}\mathfrak{n}_{d,3}.$$

The lattice of ideals of $\mathfrak{L}(\mathfrak{u},\varphi)$ is a 4-element chain connected by the rhombus ideal $Z(\mathfrak{n}_{d,3})$ at the bottom. But the quotient Lie algebras by minimal ideals inside $Z(\mathfrak{n}_{d,3})$,

$$\frac{\mathfrak{L}(\mathfrak{u},\varphi)}{V(\lambda_1)} \quad \text{and} \quad \frac{\mathfrak{L}(\mathfrak{u},\varphi)}{V(\lambda_1+\lambda_2)},$$

are 5-chain mixed Lie algebras. In both cases we get the 5-chain by removing a minimal node in the complete lattice of ideals of $\mathfrak{L}(\mathfrak{u},\varphi)$.

Example 5.3.9. GHA $\mathfrak{h}_{2n+1}$, in their standard basis, endow the vector space $W_{2n}$with the non-degenerate skew-form $[a,b]=\varphi(a,b)z$. According to equation (5.2), the Levi factor of $\mathrm{Der}\,\mathfrak{h}_{2n+1}$ is the Lie algebra $\mathfrak{s}$ of extended maps d_f where and $d_f(z)=0$ and $d_f|_{\mathfrak{u}} = f$ for every $f\in\mathfrak{sp}(\mathfrak{u},\varphi)$. So $\mathfrak{s}\cong\mathfrak{sp}(\mathfrak{u},\varphi)$ and $\mathfrak{h}_{2n+1}$ decomposes as the natural $\mathfrak{sp}(\mathfrak{u},\varphi)$-module $\mathfrak{u}=V(\lambda_1)$, and the trivial one-dimensional module $Z(\mathfrak{h}_{2n+1}) = \mathbb{F}\cdot z$. Clearly, the ideals of the Lie algebra $\mathfrak{s}\oplus_{id}\mathfrak{h}_{2n+1}$ (a mixed subalgebra of the mixed algebra $\mathrm{Der}\,\mathfrak{h}_{2n+1}\oplus_{id}\mathfrak{h}_{2n+1}$) form a 4-chain. For $n=1$, since $\mathfrak{sp}_2(\mathbb{F})\cong\mathfrak{sl}_2(\mathbb{F})$, previous 4-chain Lie algebra is encoded in a Lie structure $V_2\oplus V_1\oplus V_0$ in Example 5.3.7. Here $V_1\oplus V_0$ is just $\mathfrak{h}_3$.

Example 5.3.10. The tensor product $S\otimes A$ of a Lie algebra S by a commutative and associative algebra A produces a Lie algebra named in the literature *current Lie algebra of S by A*. *Poisson structures* and invariant bilinear forms on current Lie algebras are treated in [Zusmanovich, 2014, Theorem 2, Corrollary 2.2 and Lemma 2.3]. If A has unit, a copy of S appears as subalgebra of $S \otimes A$. If S is simple, the ideals of $S \otimes A$ are of the form $S \otimes I$ where I is an ideal of A. Since the Killing form is an invariant and non-degenerate form of any simple Lie algebra, from Lemma 2.3 in [Zusmanovich, 2014], the current Lie algebra $S \otimes A$ can also be endowed with an invariant symmetric and non-degenerate bilinear form. In this way we get a *metric Lie structure* (*quadratic or metrizable Lie algebra*). Consider now the series of current algebras introduced in Example 2.2.6

$$\mathfrak{g}_n(S) = S \otimes \frac{\mathbb{F}[t]}{\operatorname{span}\langle t^n \rangle},$$

for S a simple Lie algebra. Note, the block $\mathfrak{s} = S \otimes 1$ is a Levi subalgebra of $\mathfrak{g}_n$, and the solvable radical, $\mathfrak{r}(\mathfrak{g}_n) = \mathfrak{n}(\mathfrak{g}_n) = \oplus_{i=1}^{n-1} S \otimes x^i$, is a *positive naturally graded* Lie algebra (here x is the class of the element t^i mod $\operatorname{span}\langle t^n \rangle$) generated by $S \otimes x$. So, $\mathfrak{n}(\mathfrak{g}_n)$ is Carnot and the whole algebra $\mathfrak{g}_n$ is also naturally graded. As $\mathfrak{s}$-module, $\mathfrak{g}_n$ decomposes as the direct sum of n copies of the adjoint module of S and its lattice of ideals is a $(n+1)$-element chain. In addition, $\mathfrak{g}_n$ is a *quadratic Lie algebra*. The smallest algebras appear by taking $S = \mathfrak{sl}_2(\mathbb{F})$. The $\mathfrak{sl}_2(\mathbb{F})$-module decomposition of $\mathfrak{g}_n(\mathfrak{sl}_2(\mathbb{F}))$ is $V_2 \oplus \cdots \oplus V_2$ (n summands).

From all these examples, the only ones which provide quadratic Lie algebras comes from Example 5.3.10. The rest of them do not provide symmetric codimension in their lattice of ideals. Now, we will try to build chains in a general way using $\mathfrak{sl}_2$ and the theoretical results presented below.

The anticonmutivity and Jacobi identity are the identities that determine any Lie algebra $\mathfrak{g}$. The first one is equivalent to saying that the product $[x, y]$ on $\mathfrak{g}$ in is given by a bilinear map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$; while the latest is equivalent to stating that the right multiplication $\operatorname{ad} x$ is a derivation of $\mathfrak{g}$, for every $x \in \mathfrak{g}$. If $\mathfrak{g}$ is simple, it is irreducible as adjoint module and a copy of $\mathfrak{g}$ is inside $\Lambda^2\mathfrak{g}$. Reversing and generalizing this argument, for an irreducible representation ρ of a semisimple Lie algebra $\mathfrak{s}$ over a vector space V, the existence of a copy of V inside $\Lambda^2 V$ let us define an skew-product $\star\colon V \otimes V \to V$ such that $\rho(\mathfrak{s}) \subseteq \operatorname{Der}(V, \star)$. This induces naturally a Lie structure on the vector space $\mathfrak{s} \oplus_\rho V$. Along this section, we follow this idea in order to get Lie algebras with chain ideal lattices.

Our algorithms to give the desired Lie structure are based on the representation theory of $\mathfrak{sl}_2(\mathbb{F})$ and the use of transvections to express skew-products, and the structure results given in [Benito, 1992a, Theorem 2.2] and [Šnobl, 2010, Theorem 2]. Both theorems can be found below.

Theorem 5.3.7 (see [Benito, 1992b]). *Let $\mathfrak{g}$ be a mixed Lie algebra. Then, the ideals of $\mathfrak{g}$ are in chain if and only if $\mathfrak{g}$ is a simple Lie algebra or a direct sum of a nonzero nilpotent ideal $\mathfrak{n}$ and a simple algebra $\mathfrak{s}$ such that $\mathfrak{n}/\mathfrak{n}^2$ is a faithful $\mathfrak{s}$-module and $\mathfrak{n}^j/\mathfrak{n}^{j+1}$ are irreducible $\mathfrak{s}$-modules for $j \geq 1$. In that case, if t is the nilindex of $\mathfrak{n}$, the ideals of $\mathfrak{g}$ are the $(t+1)$-element chain $0 = \mathfrak{n}^t \subsetneq \mathfrak{n}^{t-1} \subsetneq \cdots \subsetneq \mathfrak{n}^i \subsetneq \cdots \subsetneq \mathfrak{n} \subsetneq \mathfrak{g}$.*

In order to obtain a Lie algebra, as described in previous theorem, we need a triad $(\mathfrak{s}, \mathfrak{n}, \rho)$ where $\mathfrak{s}$ is simple, $\mathfrak{n}$ nilpotent and $\rho\colon \mathfrak{s} \to \operatorname{Der}\mathfrak{n}$. So $\mathfrak{n} = m_1 \oplus m_2 \oplus \cdots \oplus m_t$ is a direct sum of irreducible $\mathfrak{s}$-modules $m_i \cong \mathfrak{n}^i/\mathfrak{n}^{i+1}$. The nilpotency of $\mathfrak{n}$ makes the construction easier because the terms in the lower central series are characteristic ideals (i.e., $\mathfrak{n}^i$ is $\operatorname{Der}\mathfrak{n}$-invariant for all i) and $\mathfrak{n}$ is generated by any subspace V such that $\mathfrak{n} = V \oplus \mathfrak{n}^2$. We also note that $\mathfrak{g} = \mathfrak{s} \oplus_\rho \mathfrak{n}$ is indecomposable, so ρ is faithful. Even more:

Theorem 5.3.8 (see [Šnobl, 2010]). *Let $\mathfrak{g}$ be an indecomposable Lie algebra with product $[x, y]$, nilpotent radical $\mathfrak{n}$ of $(t+1)$-nilindex and nontrivial Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ for some semisimple Lie algebra $\mathfrak{s}$. Then, there exists a decomposition of $\mathfrak{n}$ into a direct sum of $\mathfrak{s}$-modules.*

$$\mathfrak{n} = m_1 \oplus m_2 \oplus \cdots \oplus m_t,$$

where $\mathfrak{n}^j = m_j \oplus \mathfrak{n}^{j+1}$, $m_j \subseteq [m_1, m_{j-1}]$ such that m_1 is a faithful $\mathfrak{s}$-module and for $2 \leq j \leq t$, m_j decomposes into a sum of some subset of irreducible components of the tensor representation $m_1 \otimes m_{j-1}$.

From Theorems 5.3.7 and 5.3.8, we get the following general construction of mixed algebras with chained lattices of ideals

Theorem 5.3.9. *Let $\mathfrak{s}$ be a simple Lie algebra and m_1, m_2, …, m_t irreducible $\mathfrak{s}$-modules with representations $\rho_i\colon \mathfrak{s} \to \mathfrak{gl}(m_i)$ for $i = 1, \ldots, t$ being ρ_1 faithful. Also, we have $\mathfrak{s}$-module homomorphisms*

$$p_{ijk}\colon m_i \otimes m_j \to m_k$$

where $1 \leq i \leq j \leq k \leq t$ and $i + j \leq k$ such that

- p_{ijk} *is skew-symmetric when* $i = j$,
- p_{ijk} *is not null when* $i = 1$ *and* $k = 1 + j$

which also verify the identity

$$\sum_{l=j+k}^{t-i} \sum_{r=i+l}^{t} p_{ilr}(u, p_{jkl}(v, w)) - \sum_{l=i+k}^{t-j} \sum_{r=j+l}^{t} p_{jlr}(v, p_{ikl}(u, w)) \\ + \sum_{l=i+j}^{t-k} \sum_{r=k+l}^{t} \hat{p}_{klr}(w, p_{ijl}(u, v)) = 0, \quad (5.11)$$

where $u \in m_i$, $v \in m_j$ *and* $w \in m_k$ *for* $i+j+k \leq t$ *and* $i \leq j \leq k$. *Here* $\hat{p}_{klr} = p_{klr}$ *if* $k \leq l$, *or* $\hat{p}_{klr} = -p_{lkr}$ *otherwise.*

The vector space $\mathfrak{g} = \mathfrak{s} \oplus m_1 \oplus m_2 \oplus \cdots \oplus m_t$ *with product*

$$\begin{aligned} [s, s']_{\mathfrak{g}} &= [s, s']_S, \\ [s, u]_{\mathfrak{g}} &= \rho_i(s)(u), \\ [u, v]_{\mathfrak{g}} &= \sum_{k=i+j}^{t} p_{ijk}(u, v), \end{aligned}$$

for $s, s' \in \mathfrak{s}$, $u \in m_i$, $v \in m_j$ *and* $i \leq j$ *gives a Lie algebra with a* $(t+2)$*-chain lattice of ideals*

$$0 < m_t < m_{t-1} \oplus m_t < \ldots < m_1 \oplus m_2 \oplus \cdots \oplus m_t < \mathfrak{g}.$$

Moreover, every mixed Lie algebra with $(t+2)$*-chain lattice of ideals has this form.*

Proof. By definition $\mathfrak{g}$ is skew-symmetric and satisfies Jacobi identity, as it involves the usual product in $\mathfrak{s}$, some of its representations ρ_i, or it is imposed by condition in equation (5.11), which is effectively Jacobi inside $\mathfrak{n} = m_1 \oplus \cdots \oplus m_t$. Since m_k is irreducible, any (nonzero) map p_{1jk} is surjective when $k = j + 1$. Then, from $[\mathfrak{s}, m_i]_{\mathfrak{g}} \subseteq m_i$ and $[m_i, m_j]_{\mathfrak{g}} = 0$ for $i + j \geq t + 1$, it is straightforward to check that $\mathfrak{n}$ is a nilpotent ideal with k^{th} lower central term

$$\mathfrak{n}^k = \bigoplus_{s \geq k} m_s.$$

In fact $\mathfrak{n}$ is the only maximal ideal of $\mathfrak{g}$ because of ρ_1 is faithful and irreducible, and $\mathfrak{s}$ is a simple Lie algebra. The ideals we obtain, and the reason why every non-solvable chain has this form is obtained from Theorem 5.3.7. □

Remark 5.3.10. Lie algebras $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ described in Theorem 5.3.9 are perfect algebras ($\mathfrak{g} = \mathfrak{g}^2$) with nilpotent solvable radical, $\mathfrak{n} = m_1 \oplus m_2 \oplus \cdots \oplus m_t$ with LCS terms $\mathfrak{n}^k = m_k \oplus \ldots m_t$ and $[\mathfrak{n}^i, \mathfrak{n}^j] \subseteq \mathfrak{n}^{i+j}$. In addition, the summands m_i are irreducible and either $\dim m_i = 1$, so $[\mathfrak{s}, m_i] = 0$, or $[\mathfrak{s}, m_i] = m_i$, and $m_{j+1} \subseteq [m_1, m_j]$. In particular, the module m_1 generates $\mathfrak{n}$ as a subalgebra and $m_2 \subseteq \Lambda^2 m_1$ by skew commutativity.

Example 5.3.8 and equation (5.10) in Section 5.3.2.1 follow the rules of the decompositions given in Theorem 5.3.9 by using the simple Lie algebras $\mathfrak{sl}(\mathfrak{u})$ and $\mathfrak{so}(\mathfrak{u})$ and taking $m_1 = \mathfrak{u}$ the irreducible natural module and irreducible quotients of $\Lambda^2\mathfrak{u}$ and $\mathfrak{u} \otimes \Lambda^2\mathfrak{u}$ (here $\Lambda^3\mathfrak{u}$ must be removed). In each example, the homomorphisms p_{ijk} are given by the projections inside tensor product modules. These examples are particular cases of a more general situation. According to [Benito and de-la-Concepción, 2013, Theorem 3.5] and Theorem 5.3.7, the mixed Lie algebras with nilradical of type d and nilindex $t+1$ in which the lattice of ideals is a n-element chain are of the form

$$\mathfrak{s} \oplus_{\mathrm{Id}} \frac{\mathfrak{n}_{d,t}}{\mathfrak{i}},$$

where $\mathfrak{s}$ is a simple subalgebra of the Levi subalgebra of derivations of the free nilpotent $\mathfrak{n}_{d,t}$ (see [Benito and de-la-Concepción, 2013, Section 3] for a complete description of $\mathrm{Der}\,\mathfrak{n}_{d,t}$) such that $\mathfrak{n}_{d,t}/\mathfrak{n}_{d,t}^2$ is a $\mathfrak{s}$-irreducible and faithful module, $\mathfrak{i}$ is an ideal, and also a $\mathfrak{s}$-submodule of $\mathfrak{n}_{d,t}^2$, that properly contains $\mathfrak{n}_{d,t}^t$. And each $\mathfrak{s}$-quotient module

$$\frac{\mathfrak{n}_{d,t}^k + \mathfrak{i}}{\mathfrak{n}_{d,t}^{k+1} + \mathfrak{i}},$$

for $2 \leq k \leq t$, is $\mathfrak{s}$-irreducible. Explicit expressions for the irreducible blocks m_i or the general product in Theorem 5.3.9 for the Lie algebra $\mathcal{L}_t(\mathfrak{s}, \mathfrak{u}, \mathfrak{i})$ are not easy to get, even in low nilindex. In the case of the 3-dimensional split simple Lie algebra, where the irreducible modules can be described in terms of differential operators and the maps p_{ijk} are given by using partial differentiation of polynomials, computational algorithms with a detailed description (including bases and bracket products) of the algebras can be implemented.

$\mathfrak{sl}_2(\mathbb{F})$-modules and transvections As our final aim is constructing chains like the ones in Theorem 5.3.9 where $\mathfrak{s} = \mathfrak{sl}_2(\mathbb{F})$, we are going to see some arithmetic particularities of this simple algebra. They will play a significant role in obtaining these algebras, automatically and theoretically. We follow ideas and tools from [Dixmier, 1984] and [Bremner and Hentzel, 2004].

First, we work with the classic basis in $\mathfrak{sl}_2$: $\{e, f, h\}$ with products $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. Let $\mathbb{F}[x, y]$ be the ring of polynomials in the variables x and y. For every d greater or equal than 0, we denote as

$$V_d = V(d) = \operatorname{span}\langle x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\rangle \tag{5.12}$$

the set of homogeneous polynomials of degree d. Abusing notation, we will write $\deg V_d = d$. Then, V_d are vector spaces of dimension $d+1$, with $V_0 = \mathbb{F}\cdot 1$. The set V_d can also be viewed as a $\mathfrak{sl}_2(\mathbb{F})$-module in a natural way once $\mathfrak{sl}_2(\mathbb{F})$ is identified, into the Lie algebra $\mathfrak{gl}(\mathbb{F}[x, y])$, as the Lie subalgebra of partial derivations

$$\operatorname{span}\left\langle e = x\,\frac{\partial}{\partial y},\;\; f = y\,\frac{\partial}{\partial x},\;\; h = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}\right\rangle. \tag{5.13}$$

This action turns V_d into an irreducible module as seen in Figure 5.10. Even more, any finite-dimensional irreducible module of $\mathfrak{sl}_2(\mathbb{F})$ can be viewed in this way, being V_0 the trivial module.

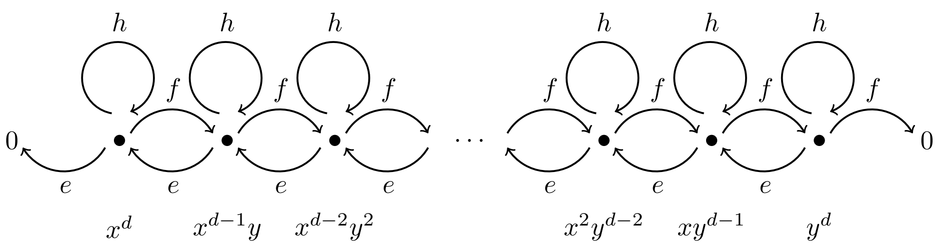

Figure 5.10: Diagram representing the $\mathfrak{sl}_2$-action over module V_d.

The *Clebsch-Gordan's formula* gives the following decomposition of the tensor product of two $\mathfrak{sl}_2(\mathbb{F})$-irreducible modules. For $n \geq m$ we have

$$V_n \otimes V_m \cong V_m \otimes V_n \cong \bigoplus_{k=0}^{m} V_{n+m-2k} \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}. \tag{5.14}$$

While, when $n = m$ we can decompose

$$\Lambda^2 V_n \cong \bigoplus_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} V_{2n-4k-2} \cong V_{2n-2} \oplus V_{2n-6} \oplus V_{2n-10} \oplus \ldots,$$

which is simply taking the odd k-summands in equation (5.14).

Now, for $0 \leq k \leq \min(n, m)$, let us consider the bilinear *transvection map* introduced in [Dixmier, 1984] as $(\cdot,\cdot)_k \colon V_n \times V_m \to V_{n+m-2k}$ where

$$(f, g)_k = \frac{(m-k)!}{m!}\,\frac{(n-k)!}{n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k f}{\partial x^{k-i} \partial y^i} \frac{\partial^k g}{\partial x^i \partial y^{k-i}}.$$

We will use these transvections to define $\mathfrak{sl}_2(\mathbb{F})$-invariant products as it is explained in [Dixmier, 1984]. From Schur's Lemma and Clebs-Gordan's formula, it is easy to prove the following result:

Lemma 5.3.11. *Any bilinear $\mathfrak{sl}_2(\mathbb{F})$-invariant product $P_{n,m,p} \colon V_n \otimes V_m \to V_p$, satisfies:*

- *$P_{n,m,p} = \alpha \cdot (f, g)_k$ for some $\alpha \in \mathbb{F}$ when $p = n + m - 2k$ and $0 \leq k \leq \min\{n, m\}$. Here $P_{n,m,p}(b, a) = (-1)^k P_{n,m,p}(a, b)$.*
- *$P_{n,m,p} = 0$ otherwise.*

So the product $P_{n,m,p}$ is either symmetric or skew-symmetric.

Proof. Note that the set of $\mathfrak{sl}_2(\mathbb{F})$-invariant products $P_{m,n,p}$ are just the vector space of module homomorphisms $Hom_{\mathfrak{sl}_2(\mathbb{F})}(V_n \otimes V_m, V_p)$. The dimension of this set is equal to the number of copies of the irreducible V_p inside $V_n \otimes V_m$. According to Clebs-Gordan's formula, the dimension is at most 1. $\square$

This lemma plays an important role in the construction of Lie algebras in which their Levi factor is, up to isomorphism, $\mathfrak{sl}_2(\mathbb{F})$.

5.3.2.2 Algorithms

Now we have all the tools to start constructing Lie algebras whose ideals are in a chain. First, note we will go back to notation $p_{ijk} \colon m_i \otimes m_j \to m_k$, instead of the one Lemma 5.3.11, as it will be more convenient. Aside, as every $\mathfrak{sl}_2$-module m_i can be identified by an integer, our algorithm will receive integers. But, instead of integers referring to the dimension or degree of each module we will set the integers in the following way: $n_1, n_2, \ldots, n_t$ will define modules $m_1, m_2, \ldots, m_t$ where

$$m_i = V_{i \cdot n_1 - 2\sum_{j=2}^{i} n_j} = V_{i \cdot n_1 - 2n_2 - \ldots - 2n_i} \tag{5.15}$$

So $m_1 = V_{n_1}$, $m_2 = V_{2n_1-2n_2}$, $m_3 = V_{3n_1-2n_2-2n_3}$ and so on. Here

$$\begin{aligned}\dim m_i &= \dim m_1 + \dim m_{i-1} - 2n_i - 1 \\ &= i \cdot n_1 - 2\sum_{j=2}^{i} n_j + 1 = i \cdot n_1 - 2n_2 - \ldots - 2n_i + 1.\end{aligned}$$

As this type of algebras would appear constantly till the end of the article, we would introduce the following definition:

Definition 5.3.7. We say a Lie algebra $\mathfrak{g}$ is a $\mathfrak{sl}_2$-*chained Lie algebra* of length $t+2$ when $\mathfrak{g}$ has Levi decomposition $\mathfrak{s} \oplus \mathfrak{n}$ with $\mathfrak{s} \cong \mathfrak{sl}_2$ and it is formed as in Theorem 5.3.9.

We will denote these $\mathfrak{sl}_2$-chained Lie algebras as

$$\mathfrak{C}(\{m_i\}_{i=1,\ldots,t}, \{\alpha_{ijk}\}_{i=1,\ldots,\lfloor \frac{t}{2} \rfloor;\, j=i,\ldots,t-1;\, k=i+j,\ldots,t}),$$

where m_i will be the $\mathfrak{sl}_2$-modules of the form V_{k_i} for some k_i and

$$p_{ijk} = \alpha_{ijk}(\cdot,\cdot)_{c_{ijk}},$$

where

$$\begin{aligned}c_{ijk} &= \frac{\dim m_i + \dim m_j - \dim m_k - 1}{2} \\ &= \frac{\deg m_i + \deg m_j - \deg m_k}{2} \\ &= \left(\frac{i+j-k}{2}\right) n_1 - \sum_{l=2}^{i} n_l + \sum_{l=j+1}^{k} n_l \\ &= \left(\frac{i+j-k}{2}\right) n_1 - n_2 - \cdots - n_i + n_{j+1} + \cdots + n_k.\end{aligned}$$

Therefore, our chains will be of the form

$$\mathfrak{g} = \mathfrak{sl}_2 \oplus m_1 \oplus m_2 \oplus \cdots \oplus m_t,$$

with $t+2$ ideals in a chain: 0, $\bigoplus_{i=k}^{t} m_i$ for $k = 1, \ldots, t$; and $\mathfrak{g}$. They will also have the following Lie bracket definition ($s, s' \in \mathfrak{sl}_2$, $u \in m_i$):

- Inside $\mathfrak{sl}_2$, where the elements are viewed as partial differentiation maps according to equation (5.13), the definition is given by the using usual special linear Lie bracket $[s, s'] = ss' - s's$.

- The product $[\mathfrak{sl}_2, m_i]$ is defined using the representation ρ_i, as

$$[s,u] = \rho_i(s)(u).$$

This way, for the standard basis in $\mathfrak{sl}_2$, it is defined as

$$\rho(e) = x\,\frac{\partial}{\partial y},$$
$$\rho(f) = y\,\frac{\partial}{\partial x},$$
$$\rho(h) = x\,\frac{\partial}{\partial x} - y\,\frac{\partial}{\partial y}.$$

- The product between the modules satisfies a couple of conditions:

$$[m_1, m_i] \supseteq m_{i+1},$$
$$[m_i, m_j] \subseteq \sum_{k=i+j}^{t} m_k = m_{i+j} + m_{i+j+1} + \cdots + m_t.$$

Given $u \in m_i$ and $v \in m_j$ where $i \leq j$ the Lie product is

$$[u,v] = \sum_{k=i+j}^{t} p_{ijk}(u,v),$$

where $p_{ijk} \colon m_i \otimes m_j \to m_k$ such that

$$p_{ijk}(u,v) = \alpha_{ijk} \cdot (u,v)_{c_{ijk}}$$

If $c_{ijk} \notin \mathbb{Z}_{\geq 0}$ then, as stated in Lemma 5.3.11, $\alpha_{ijk} = 0$. Moreover

  - $\alpha_{ijk} \neq 0$ for $i = 1$ and $k = j+1$,
  - As p_{iik} must be skew-symmetric then c_{iik} must be odd or $\alpha_{iik} = 0$.

So, when $t = 3$ we will have chains $\mathfrak{C}(\{m_1, m_2, m_3\}, \{\alpha_{112}, \alpha_{113}, \alpha_{123}\})$, and for $t = 4$ chains will be

$$\mathfrak{C}(\{m_1, m_2, m_3, m_4\}, \{\alpha_{112}, \alpha_{113}, \alpha_{114}, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{224}\}).$$

Now, all these tools and notation, in combination with Theorem 5.3.9, open up the idea to develop Algorithm 5.3.1 to find all these algebras given the already mentioned list of integers referring to the irreducible modules.

Algorithm 5.3.1. Checks if a succession of t integers referring to t modules is compatible with being a Lie algebra whose ideals are in chain of $t+2$ ideals.

Input: The algorithm receives t integers $n_1, n_2, \dots, n_t$ referring to $\mathfrak{sl}_2$-modules as defined by equation (5.15).

Output: A boolean value (true or false) indicating if there is a Lie algebra $\mathfrak{C}(\{m_1, m_2, \dots, m_t\}, \{\alpha_{ijk}\}_{i=1,\dots,\lfloor \frac{t}{2} \rfloor;\, j=i,\dots,t-1;\, k=i+j,\dots,t})$ for some α_{ijk}. And the list those valid α_{ijk} such the chained algebra exists.

In case the validity of the algebra is subject to only some values of the parameters α_{ijk} the algorithm also gives them.

Steps: The algorithm is divided in two main steps:

(a) Check integers input: this is equivalent to checking if

- $m_2 \subseteq \Lambda^2 m_1$
- $m_{i+1} \subseteq m_1 \otimes m_i$ for every $i = 2, \dots, t$

In terms of integers, this translates into n_2 being an odd number, and for $i = 1, \dots, t-1$,

$$0 \leq n_{i+1} \leq \min(n_1, \dim m_i - 1) = \min\left(n_1, i \cdot n_1 - 2\sum_{j=2}^{i} n_j\right).$$

(b) Check Jacobi identities inside $m_1 \oplus m_2 \oplus \cdots \oplus m_t$: We need to study $J(u, v, w) = 0$ for $u \in m_i$, $v \in m_j$ and $w \in m_k$ such $i + j + k \leq t$ and $i \leq j \leq k$. As seen in Theorem 5.3.9, here

$$\begin{aligned} J(u, v, w) &= [u, [v, w]] + [v, [w, u]] + [w, [u, v]] \\ &= \sum_{l=j+k}^{t-i} \sum_{r=i+l}^{t} p_{ilr}(u, p_{jkl}(v, w)) - \sum_{l=i+k}^{t-j} \sum_{r=j+l}^{t} p_{jlr}(v, p_{ikl}(u, w)) \\ &\qquad + \sum_{l=i+j}^{t-k} \sum_{r=k+l}^{t} \hat{p}_{klr}(w, p_{ijl}(u, v)) = 0, \end{aligned} \tag{5.16}$$

where $\hat{p}_{klr} = p_{klr}$ if $k \leq l$, or $\hat{p}_{klr} = -p_{lkr}$ otherwise.

In case we want to find every tuple $(n_1, n_2, \dots, n_t)$ that gives a chain we should call Algorithm 5.3.1 using at least all integers that satisfy step 1 in Algorithm 5.3.1.

Note, Algorithm 5.3.1 only works for every chain of 3 or more ideals (so $t \geq 1$). But we do not need more in case we want to study smaller cases. A 1-chain is simply the zero-dimensional Lie algebra, the only 2-chain of this form is the simple Lie algebra $\mathfrak{sl}_2$ with no modules. And, even for cases as 3-chains of 4-chains ($t = 1, 2$) the algorithm is unnecessary as the algebras are $\mathfrak{sl}_2 \oplus V_n$ for every $n \geq 0$, and $\mathfrak{sl}_2 \oplus V_n \oplus V_{2n-2k}$ for $1 \leq k \leq n$ and k being an odd number. In all those algebras, the skew-symmetry is guaranteed by transvection properties, while the Jacobi identity is trivially null in the modules. It is only in larger cases when the use of the algorithm becomes relevant for finding valid algebras. That is why, for cases $t = 3$ and $t = 4$ we can find a detailed implementation in Algorithm 5.3.2 and Algorithm 5.3.3 respectively.

Algorithm 5.3.2. Detailed implementation of Algorithm 5.3.1 in case $t = 3$.

Input: Three integers n_1, n_2 and n_3 referring to modules

- $m_1 = V_{n_1}$,
- $m_2 = V_{2n_1-2n_2}$,
- $m_3 = V_{3n_1-2n_2-2n_3}$.

Output: A boolean value (true or false) indicating if there is a Lie algebra $\mathfrak{C}(\{m_1, m_2, m_3\}, \{\alpha_{112}, \alpha_{113}, \alpha_{123}\})$ for any α_{ijk} such that $\alpha_{112} \cdot \alpha_{123} \neq 0$.

Steps: The algorithm is divided in two main steps:

1. Check integers input:
   - $1 \leq n_2 \leq n_1$ and n_2 is odd,
   - $0 \leq n_3 \leq \min(n_1, \dim m_2 - 1) = \min(n_1, 2n_1 - 2n_2)$.
2. Check Jacobi identity inside the nilradical $N = m_1 \oplus m_2 \oplus m_3$. As we have $t = 3$, equation (5.16) can be simplified as we have only one possibility: $i = j = k = 1$. Therefore, the only identity to check is

   $$p_{123}(u, p_{112}(v, w)) + p_{123}(v, p_{112}(w, u)) + p_{123}(w, p_{112}(u, v)) = 0.$$

   for every $u, v, w \in m_1$. Every term of this equality has both coefficients α_{123} and α_{112}, and, as they are not zero, we can simplify obtaining

   $$(u, (v, w)_{n_2})_{n_3} + (v, (w, u)_{n_2})_{n_3} + (w, (u, v)_{n_2})_{n_3} = 0. \tag{5.17}$$

If this is true $\mathfrak{C}(\{m_1, m_2, m_3\}, \{\alpha_{112}, \alpha_{113}, \alpha_{123}\})$ is a Lie algebra for any α_{ijk} such that $\alpha_{112} \cdot \alpha_{123} \neq 0$.

Applying Algorithm 5.3.2, we can obtain every possible 5-chain for $n_1 \leq 32$. This way we observe two families: the first is formed by 4 chains that exist for every n_1 and are described in Table 5.1. The second family is formed by algebras which repeat every four n_1 values. This last family appears in Table 5.2. And, up to[4] $n_1 = 32$ these are the only chains of 5 ideals. Therefore, it is quite probable that this extends for every n_1.

n_1	n_2	n_3	$\mathfrak{sl}_2$-modules	Condition
n	1	0	$V_n \oplus V_{2n-2} \oplus V_{3n-2}$	$n \geq 1$
n	1	1	$V_n \oplus V_{2n-2} \oplus V_{3n-4}$	$n \geq 2$
n	1	3	$V_n \oplus V_{2n-2} \oplus V_{3n-8}$	$n \geq 3$
n	3	1	$V_n \oplus V_{2n-6} \oplus V_{3n-8}$	$n \geq 4$

Table 5.1: Chains of 5 ideals for $n_1 \leq 32$ which repeat for every n_1.

n_1	n_2	n_3	$\mathfrak{sl}_2$-modules	Condition
$4n$	$2n+1$	$4n-3$	$V_{4n} \oplus V_{4n-2} \oplus V_4$	$n \geq 2$
$4n+1$	$2n+1$	$4n$	$V_{4n+1} \oplus V_{4n} \oplus V_1$	$n \geq 0$
$4n+2$	$2n+1$	$4n+1$	$V_{4n+2} \oplus V_{4n+2} \oplus V_2$	$n \geq 0$
$4n+3$	$2n+1$	$4n+3$	$V_{4n+3} \oplus V_{4n+4} \oplus V_1$	$n \geq 0$
$4n+4$	$2n+1$	$4n+3$	$V_{4n+4} \oplus V_{4n+6} \oplus V_4$	$n \geq 0$

Table 5.2: Chains of 5 ideals for $n_1 \leq 32$ which repeat every four n_1.

Algorithm 5.3.3. Detailed implementation of Algorithm 5.3.1 in case $t = 4$.

Input: Four integers n_1, n_2, n_3 and n_4 referring to modules

- $m_1 = V_{n_1}$,
- $m_2 = V_{2n_1-2n_2}$,
- $m_3 = V_{3n_1-2n_2-2n_3}$,

[4]The $n_1 = 32$ cap is set arbitrarily to limit the computational cost, and it is not based on some property that does not work for greater values.

- $m_4 = V_{4n_1-2n_2-2n_3-2n_4}$.

Output: A boolean value (true or false) indicating if there is a Lie algebra $\mathfrak{C}(\{m_1, m_2, m_3, m_4\}, \{\alpha_{ijk}\}_{i=1,2;\, j=i,\ldots,3;\, k=i+j,\ldots,4})$ for some α_{ijk}. It also returns a value α which gives the following scalar restriction over α_{224}:

$$\alpha = \frac{\alpha_{224} \cdot \alpha_{112}}{\alpha_{123} \cdot \alpha_{134}}. \tag{5.18}$$

Any values for $\alpha_{112}, \alpha_{123}, \alpha_{134}$ different from zero would give an algebra, at least when considering $\alpha_{124} = \alpha_{113} = 0$.

Steps: The algorithm is divided in two main steps:

1. Check integers input:
   - $1 \leq n_2 \leq n_1$ and n_2 is odd,
   - $0 \leq n_3 \leq \min(n_1, \dim m_2 - 1) = \min(n_1, 2n_1 - 2n_2)$,
   - $0 \leq n_4 \leq \min(n_1, \dim m_3 - 1) = \min(n_1, 3n_1 - 2n_2 - 2n_3)$.

2. Check Jacobi identity inside the nilradical $N = m_1 \oplus m_2 \oplus m_3 \oplus m_4$. Here $t = 4$, so equation (5.16), taking into account p_{114} and p_{124} go to the centre, appears in two scenarios:

   (a) Three elements $u, v, w \in m_1$.

   $$\begin{aligned} &p_{123}(u, p_{112}(v, w)) + p_{123}(v, p_{112}(w, u)) + p_{123}(w, p_{112}(u, v)) \\ &\quad + p_{124}(u, p_{112}(v, w)) + p_{124}(v, p_{112}(w, u)) + p_{124}(w, p_{112}(u, v)) \\ &+ p_{134}(u, p_{113}(v, w)) + p_{134}(v, p_{113}(w, u)) + p_{134}(w, p_{113}(u, v)) = 0. \end{aligned}$$

   Taking projections, we can separate the first three addends from the rest into two null equations. The first one, following the same procedure as in Algorithm 5.3.2, turns again into equation (5.17). The second part could be omitted taking $\alpha_{124} = 0$ and $\alpha_{113} = 0$.

   (b) Two elements $u, v \in m_1$ and another $w \in m_2$.

   $$p_{134}(u, p_{123}(v, w)) - p_{134}(v, p_{123}(u, w)) + p_{224}(w, p_{112}(u, v)) = 0. \tag{5.19}$$

   For this equation we have two options:

- If $n_3 + n_4 - n_2$ is even or negative, or $n_2 + n_3 + n_4 > 2n_1$ then $\alpha_{224} = 0$, and equation (5.19) turns into

$$(u, (v, w)_{n_3})_{n_4} - (v, (u, w)_{n_3})_{n_4} = 0$$

- If not, equation (5.19) turns into

$$(u, (v, w)_{n_3})_{n_4} - (v, (u, w)_{n_3})_{n_4} + \alpha \cdot (w, (u, v)_{n_2})_{n_3+n_4-n_2} = 0$$

for $\alpha = \frac{\alpha_{224}\alpha_{112}}{\alpha_{123}\alpha_{134}}$. Note this α is unique as in this case we have $(w, (u, v)_{n_2})_{n_3+n_4-n_2} \neq 0$.

On the same way, applying Algorithm 5.3.3, we can try to obtain every possible module combination of 6-chains for $n_1 \leq 32$. Again, we can distinguish two groups. The general one that repeats for every n_1 which appears in Table 5.3. But, in contrast to what happens with five ideals, in this case there are some chains that only work for some n_1 values. These particular cases are listed in Table 5.4.

n_1	n_2	n_3	n_4	$\mathfrak{sl}_2$-modules	α	Condition
n	1	0	0	$V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-2}$	0	$n \geq 1$
n	1	0	2	$V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-6}$	$\frac{4(4n-3)}{3(3n-2)}$	$n \geq 2$
n	1	1	1	$V_n \oplus V_{2n-2} \oplus V_{3n-4} \oplus V_{4n-6}$	$\frac{2n-2}{3n-4}$	$n \geq 2$

Table 5.3: Chains of 6 ideals for $n_1 \leq 32$ which repeat for every n_1. Parameter α is defined in equation (5.18).

Remark 5.3.12. In case we want to study what happens in $(t + 2)$-chains for $t \geq 5$ we can use the general Algorithm 5.3.1. It is important to note, that in these cases the complexity increases rapidly. For instance, when $t = 5$, which is the simplest case, checking Jacobi identity following equation (5.16) produces up to 4 cases to study:

1. Three elements in m_1
2. Two elements in m_1 and other in m_2
3. Two elements in m_1 and other in m_3

n_1	n_2	n_3	n_4	$\mathfrak{sl}_2$-modules	α
3	1	1	3	$V_3 \oplus V_4 \oplus V_5 \oplus V_2$	7/5
3	1	3	1	$V_3 \oplus V_4 \oplus V_1 \oplus V_2$	-2
4	1	3	3	$V_4 \oplus V_6 \oplus V_4 \oplus V_2$	3/2
4	3	1	3	$V_4 \oplus V_2 \oplus V_4 \oplus V_2$	1/2
5	1	3	5	$V_5 \oplus V_8 \oplus V_7 \oplus V_2$	22/21
5	3	1	5	$V_5 \oplus V_4 \oplus V_7 \oplus V_2$	12/7
6	1	3	5	$V_6 \oplus V_{10} \oplus V_{10} \oplus V_6$	1

Table 5.4: Chains of 6 ideals for $n_1 \leq 32$ which do not repeat for n_1. Parameter α is defined in equation (5.18).

4. Two elements in m_2 and other in m_1

As seen in Algorithm 5.3.2 and Algorithm 5.3.3, many α_{ijk} could be considered null and we could simplify them. But, as α_{224} could be different from zero many more subcases appear making it much harder to solve.

5.3.2.3 Lie algebra structure existence

Although our algorithms are useful for finding chains, they do not let us generalize and prove the existence of 5 or 6-chains whose dimension is as big as we want. But, at least, seeing their results we know approximately where we should look.

Before proving some results of existence, we need to introduce *Gordan identities*, which appear in [Dixmier, 1984] (see also [Bremner and Hentzel, 2004] for further information). These are some relationships that transvections fulfil which will be helpful during proofs.

Definition 5.3.8 (Gordan's Identity)**.** Let $f \in V_n$, $g \in V_m$ and $h \in V_p$, and let α_1, α_2 and α_3 be non-negative integers such that $\alpha_1 + \alpha_2 \leq p$, $\alpha_2 + \alpha_3 \leq m$, $\alpha_3 + \alpha_1 \leq n$, with $\alpha_1 = 0$ o $\alpha_2 + \alpha_3 = m$. Then

$$\begin{bmatrix} f & g & h \\ m & n & p \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = 0,$$

where

$$\begin{bmatrix} f & g & h \\ m & n & p \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix} = \sum_{i\geq 0} \frac{\binom{n-\alpha_1-\alpha_3}{i}\binom{\alpha_2}{i}}{\binom{m+n-2\alpha_3-i+1}{i}}((f,g)_{\alpha_3+i},h)_{\alpha_1+\alpha_2-i}$$

$$+ (-1)^{\alpha_1+1}\sum_{i\geq 0} \frac{\binom{p-\alpha_1-\alpha_2}{i}\binom{\alpha_3}{i}}{\binom{m+p-2\alpha_2-i+1}{i}}((f,h)_{\alpha_2+i},g)_{\alpha_1+\alpha_3-i}.$$

Chains with five ideals Now, we have all the necessary tools to prove all chains in Tables 5.1 and 5.2 can be extended for any n and not only up to 32. But before proving the results we introduce the following simplified notation for Gordan's Identity. When writing $[f,g,h,n,\alpha_1,\alpha_2,\alpha_3]$ we refer to (for $f,g,h\in V_n$)

$$\begin{bmatrix} f & g & h \\ n & n & n \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}$$

and

$$[f,g,h,n,\alpha_1,\alpha_2,\alpha_3]^* = \sum_{\substack{\circlearrowleft \\ f,g,h}} [f,g,h,n,\alpha_1,\alpha_2,\alpha_3] - \sum_{\substack{\circlearrowleft \\ g,f,h}} [g,f,h,n,\alpha_1,\alpha_2,\alpha_3].$$

From now on $\rho_{(n_1,\dots,n_t)} = \rho_{n_1}\oplus\cdots\oplus\rho_{n_t}$ will denote the direct sum representation of $\mathfrak{sl}_2(\mathbb{F})$ on the vector space module

$$W(n_1,\dots,n_t) = V_{n_1}\oplus V_{2n_1-2n_2}\oplus\cdots\oplus V_{tn_1-2\sum_{i=2}^t n_i}.$$

According to Section 5.3.2.1, the action ρ_{n_j} on the module $V_{jn_1-2\sum_{k=2}^j n_k}$ of homogeneous polynomials of degree $jn_1-2\sum_{k=2}^j n_k$, is given in terms of differential operators. The Lie bracket that makes $W(n_1,\dots,n_t)$ into a nilpotent Lie algebra is defined rescaling by $\alpha_{ijk}\in\mathbb{F}$ the c_{ijk}-transvection $(f,g)_{c_{ijk}}\in V_{kn_1-2\sum_{s=2}^k n_s}$ of $f\in V_{in_1-2\sum_{r=2}^i n_r}$ and $g\in V_{jn_1-2\sum_{q=2}^j n_q}$. Here $i=1,\dots,\lfloor\frac{t}{2}\rfloor$, $j=i,\dots,t-1$ and $k=i+j,\dots,t$. In the particular case where $i=1$, $j=p$ and $k=p+1$

$$[f,g]_W = \alpha_{ijk}(f,g)_{n_k} \text{ and } \alpha_{ijk}\neq 0.$$

We will also denote the tuple $\lambda_{(n_1,\dots,n_t)} = (\alpha_{ijk})_{ijk}$. This encodes the structure constants of the Lie algebra $W(n_1,\dots,n_t)$. Here, α_{ijk} is nonzero for all $i=$

$1, j = p = 1, \dots t-1$ and $k = p+1$. We shall refer to a fold $(\alpha_{ijk})_{ijk}$ structure constants fold of the Lie algebra

$$\mathfrak{g}^{\rho,\lambda}_{(n_1,\dots,n_t)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho W(n_1,\dots,n_t)_\lambda \quad \text{and} \quad \begin{cases} \rho = \rho_{(n_1,\dots,n_t)}, \\ \lambda = \lambda_{(n_1,\dots,n_t)}. \end{cases}$$

Proposition 5.3.13. *The* 3*-tuples* $(n,1,0)$*,* $(n,1,1)$*,* $(n,1,3)$ *and* $(n,3,1)$ *generate the following* λ*-parametric families of* $\mathfrak{sl}_2$*-chained Lie algebras:*

(a) $\mathfrak{g}^{\rho,\lambda}_{(n,1,0)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-2})_\lambda$, *for* $n \geq 1$,

(b) $\mathfrak{g}^{\rho,\lambda}_{(n,1,1)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-4})_\lambda$, *for* $n \geq 2$,

(c) $\mathfrak{g}^{\rho,\lambda}_{(n,1,3)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-8})_\lambda$, *for* $n \geq 3$,

(d) $\mathfrak{g}^{\rho,\lambda}_{(n,3,1)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-6} \oplus V_{3n-8})_\lambda$, *for* $n \geq 4$,

where scalar threefolds $\lambda_{(n_1,n_2,n_3)} = (\alpha_{112}, \alpha_{113}, \alpha_{123})$ *determine the product in nilradical by* $\alpha_{112}(\cdot,\cdot)_{n_2}$*,* $\alpha_{113}(\cdot,\cdot)_{\frac{2n_2+2n_3-n_1}{2}}$ *and* $\alpha_{123}(\cdot,\cdot)_{n_3}$*. And they take values:*

$$\begin{aligned}
&\lambda_{(n,1,0)} = (\alpha_{112}, 0, \alpha_{123}) \\
&\lambda_{(n,1,1)} = (\alpha_{112}, 0, \alpha_{123}), \textit{for } n \neq 2, \qquad \lambda_{(2,1,1)} = (\alpha_{112}, \alpha_{113}, \alpha_{123}) \\
&\lambda_{(n,1,3)} = (\alpha_{112}, 0, \alpha_{123}), \textit{for } n \neq 6, \qquad \lambda_{(6,1,3)} = (\alpha_{112}, \alpha_{113}, \alpha_{123}) \\
&\lambda_{(n,3,1)} = (\alpha_{112}, 0, \alpha_{123}), \textit{for } n \neq 6, \qquad \lambda_{(6,3,1)} = (\alpha_{112}, \alpha_{113}, \alpha_{123})
\end{aligned}$$

Proof. We denote the different tuples in the general form (n_1, n_2, n_3), so $m_1 = V_{n_1}$, $m_2 = V_{2n_1-2n_2}$ and $m_3 = V_{3n_1-2n_2-2n_3}$. It is a straightforward computation that the module summand $W(n_1,n_2,n_3) = m_1 \oplus m_2 \oplus m_3$ is as described in all four items. Using Clebsch-Gordan's formula from equation (5.14) we can observe m_2 appears in $\Lambda^2 m_1$ and m_3 appears in $m_1 \otimes m_2$. Therefore, by construction, we only need to check Jacobi identity from equation (5.17), for every $f, g, h \in m_1$. And this equality can be proved using Gordan's Identity from Definition 5.3.8 on expressions:

$$
\begin{aligned}
[f,g,h,n,0,0,1] &= ((f,g)_1,h)_0 - ((f,h)_0,g)_1 - \frac{1}{2}((f,h)_1,g)_0,\\
[f,g,h,n,0,1,1] &= ((f,g)_1,h)_1 + \frac{1}{2}((f,g)_2,h)_0 - ((f,h)_1,g)_1 - \frac{1}{2}((f,h)_2,g)_0,\\
[f,g,h,n,0,2,2] &= ((f,g)_2,h)_2 + ((f,g)_3,h)_1 + \frac{(n-2)(n-3)}{(2n-5)(2n-6)}((f,g)_4,h)_0\\
&\quad - ((f,h)_1,g)_1 - ((f,h)_3,g)_1 - \frac{(n-2)(n-3)}{(2n-5)(2n-6)}((f,h)_4,g)_0,\\
[f,g,h,n,0,1,3] &= ((f,g)_3,h)_1 + \frac{1}{2}((f,g)_4,h)_0 - ((f,g)_1,h)_3 - \frac{3}{2}((f,h)_2,g)_2\\
&\quad - \frac{3(n-1)}{2(2n-3)}((f,h)_3,g)_1 - \frac{(n-1)}{4(2n-5)}((f,h)_4,g)_0.
\end{aligned}
$$

Depending on the different tuples, equation (5.17) is equivalent to the following identities (note that $(a,b)_k = (-1)^k(b,a)_k$ according to Lemma 5.3.11):

- $(n_1,n_2,n_3) = (n,1,0)$ for $n \geq 1$: equation (5.17) follows from,

  $$[f,g,h,n,0,0,1] - [h,g,f,n,0,0,1] = 0.$$

  We note that $\alpha_{113} = 0$ because $m_3 = V_{3n-2}$ is not contained in $\Lambda^2 m_1$.

- $(n_1,n_2,n_3) = (n,1,1)$ for $n \geq 2$: equation (5.17) follows from,

  $$[f,g,h,n,0,1,1] + [g,f,h,n,0,1,1] + [h,g,f,n,0,1,1] = 0.$$

  In this case, $m_3 = V_{3n_1-4} \subset \Lambda^2 m_1$ implies $3n_1 - 4 = 2n_1 - 2n_2$ with n_2 odd. Then $2n_2 = 4 - n_1 \geq 0$ and therefore $\alpha_{113} = 0$ if $n_1 = n \geq 3$. And $n_1 = n = 2$ implies $n_2 = 1$ and any α_{113} is valid.

- $(n_1,n_2,n_3) = (n,1,3)$ for $n \geq 4$: equation (5.17) follows from,

  $$
  \begin{aligned}
  [f,g,h,n,0,1,3]^* - \frac{7n-9}{4n-6}\Big([f,g,h,n,0,2,2]&\\
  - [g,f,h,n,0,2,2] - [h,g,f,n,0,2,2]\Big) &= 0.
  \end{aligned}
  $$

  And it is equivalent to $[f,g,h,3,1,1,2]^* = 0$ if $n = 3$. Here, $\alpha_{113} = 0$ if $n \neq 6$ and $n = 6$ implies $n_2 = 1$ and any α_{113} is valid.

- $(n_1,n_2,n_3) = (n,3,1)$ for $n \geq 4$: here equation (5.17) is equivalent to $[f,g,h,n,0,2,2] + [g,f,h,n,0,2,2] + [h,g,f,n,0,2,2] = 0.$ □

Most of previous information on the Proposition 5.3.13 can be originally found distributed in several lemmas in [Pérez-Aradros, 2016, Sección 2.3.1], work that has been revisited, sorted and extended to produce the mentioned proposition. From the algorithms, we also reach the following series of Lie algebras.

Proposition 5.3.14. *The* 3*-tuples* $(4n, 2n+1, 4n-3)$, $(4n+1, 2n+1, 4n)$, $(4n+2, 2n+1, 4n+1)$, $(4n+3, 2n+1, 4n+3)$ *and* $(4n+4, 2n+1, 4n+3)$ *generate only algebras with* $\mathbb{N}^+$*-graded nilradical. So, the valid scalar threefolds are* $\lambda_{(n_1,n_2,n_3)} = (\alpha_{112}, 0, \alpha_{123})$ *for all of them except* $\lambda_{(4n+2,2n+1,4n+1)} = (\alpha_{112}, \alpha_{113}, \alpha_{123})$*. The resulting* λ*-parametric families of* $\mathfrak{sl}_2$*-chained Lie algebras are:*

(a) $\mathfrak{g}^{\rho,\lambda}_{(4n,2n+1,4n-3)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_{4n} \oplus V_{4n-2} \oplus V_4)_\lambda$ *for* $n \geq 2$.

(b) $\mathfrak{g}^{\rho,\lambda}_{(4n+1,2n+1,4n)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_{4n+1} \oplus V_{4n} \oplus V_1)_\lambda$ *for* $n \geq 0$.

(c) $\mathfrak{g}^{\rho,\lambda}_{(4n+2,2n+1,4n+1)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_{4n+2} \oplus V_{4n+2} \oplus V_2)_\lambda$ *for* $n \geq 0$.

(d) $\mathfrak{g}^{\rho,\lambda}_{(4n+3,2n+1,4n+3)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_{4n+3} \oplus V_{4n+4} \oplus V_1)_\lambda$ *for* $n \geq 0$.

(e) $\mathfrak{g}^{\rho,\lambda}_{(4n+3,2n+1,4n+3)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_{4n+4} \oplus V_{4n+6} \oplus V_4)_\lambda$ *for* $n \geq 0$.

And products in the nilradicals are given by $\alpha_{112}(\cdot\,,\cdot)_{n_2}$, $\alpha_{123}(\cdot\,,\cdot)_{n_3}$ *and* $\alpha_{113} = (\cdot\,,\cdot)_{\frac{2n_2+2n_3-n_1}{2}}$ *when not zero.*

Proof. We follow the notation introduced in the proof of Proposition 5.3.13. It is a straightforward computation that the module summand $W(n_1, n_2, n_3) = m_1 \oplus m_2 \oplus m_3$ is as described in all items in the list. Using Clebsch-Gordan's formula from equation (5.14) we check that $\alpha_{113} = 0$ in all the cases, m_2 appears in $\Lambda^2 m_1$ and m_3 appears in $m_1 \otimes m_2$. So, to establish the result, it only remains to prove Jacobi identity from equation (5.17) for every $f, g, h \in m_1$. As in Proposition 5.3.13, we proceed to check using Gordan identities:

- $(n_1, n_2, n_3) = (4n, 2n+1, 4n-3)$, for $n \geq 2$: equation (5.17) is just

$$[f, g, h, 4n, 2n-2, 2n+2, 2n-2]^* - 2[f, g, h, 4n, 2n-2, 2n+1, 2n-1]^* = 0$$

For the other cases $n \geq 0$ is fixed and,

- $(n_1, n_2, n_3) = (4n+1, 2n+1, 4n)$: equation (5.17) follows from the identity $[f, g, h, 4n + 1, 2n, 2n + 1, 2n]^* = 0$.

- $(n_1, n_2, n_3) = (4n + 2, 2n + 1, 4n + 1)$: equation (5.17) is equivalent to $[f, g, h, 4n + 2, 2n, 2n + 1, 2n + 1]^* = 0$.

- $(n_1, n_2, n_3) = (4n + 3, 2n + 1, 4n + 3)$: equation (5.17) is just

$$[f, g, h, 4n + 3, 2n - 1, 2n, 2n + 3]^* + \frac{4n+3}{2n}[f, g, h, 4n + 3, 2n - 1, 2n + 3, 2n]^* = 0$$

  for $n \geq 1$. While case $n = 0$ is just item (c) for $n = 3$ in Proposition 5.3.13.

- $(n_1, n_2, n_3) = (4n + 4, 2n + 1, 4n + 3)$: equation (5.17) is obtained from

$$\begin{aligned}&[f, g, h, 4n+4, 2n-2, 2n+4, 2n]^* - [f, g, h, 4n+4, 2n-2, 2n, 2n+4]^* \\ &- \frac{3(n+2)(2n+3)(7n+10)}{4n(4n+1)(6n+7)}\Big([f, g, h, 4n+4, 2n-2, 2n+4, 2n]^* \\ &\qquad - \frac{5(n+2)}{7n+10}[f, g, h, 4n+4, 2n-2, 2n+3, 2n+1]^*\Big) = 0\end{aligned}$$

  for $n \geq 1$. While case $n = 0$ is item (c) for $n = 4$ in Proposition 5.3.13.

Observe, c_{113} is only an odd integer in the third case, thus, all other α_{113} factors are zero. □

Remark 5.3.15. As seen in some proofs, some algebras are repeated in Propositions 5.3.13 and 5.3.14. The cases $n = 0$ from Proposition 5.3.14 in items (b), (c), (d) and (e) coincide with the cases from Proposition 5.3.13 in item (a) for $n = 1$, item (b) for $n = 2$, and item (c) for $n = 3, 4$.

Remark 5.3.16. Other 3-tuples that do not produce chain ideal Lie algebras are:

$$\begin{array}{cccc} (n,1,2)_{n\geq 2}, & (n,1,4)_{n\geq 4}, & (n,1,5)_{n\geq 5}, & (n,1,6)_{n\geq 6}, \\ (n,3,0)_{n\geq 3}, & (2n,2n-1,1)_{n\geq 3}, & (2n,2n-1,0)_{n\geq 2}, & (n,3,2)_{n\geq 3}, \\ & (2n+1,2n+1,0)_{n\geq 1}, & (2n,2n-1,2)_{n\geq 3}. & \end{array}$$

We can check why and where they do not work as chains in [Pérez-Aradros, 2016, Sección 2.3.1].

Example 5.3.11. The 3-tuples $(n,1,2)_{n\geq 2}$ or $(2n+1,2n+1,0)_{n\geq 1}$ do not produce Lie algebra structures with chain ideal lattice. In the first case, the Lie product must be induced on $V_n \oplus V_{2n-2} \oplus V_{3n-6}$ by using $\lambda = (\alpha_{112}, 0, \alpha_{123})$. But Jacobi identity fails (unless $\alpha_{112}\alpha_{123} = 0$):

$$J(x^n, yx^{n-1}, y^2x^{n-2}) = \sum_{cyclic} ((x^n, yx^{n-1})_1, y^2x^{n-2})_2 = \frac{9n-12}{n^2(2n-3)(n-1)} x^{3n-6}$$

For the second tuple, the vector space is $V_{2n+1} \oplus V_0 \oplus V_{2n+1}$ and Jacobi identity also fails:

$$J(yx^{2n}, y^2x^{2n-1}, y^{2n}x) = \sum_{cyclic} ((yx^{2n}, y^2x^{2n-1})_{2n+1}, y^{2n}x)_0 = \frac{1}{2n+1} y^2x^{2n-1}.$$

Chains with 6 ideals Now, we have to prove the existence results inspired by Table 5.3. Before proving them, we are going to introduce simplified notations as in all the following results $m_1 = V_n$ and $m_2 = V_{2n-2}$. This way, we will write $[h,f,g,n,\alpha_1,\alpha_2,\alpha_3]_1$, $[f,h,g,n,\alpha_1,\alpha_2,\alpha_3]_2$ and $[f,g,h,n,\alpha_1,\alpha_2,\alpha_3]_3$ instead of

$$\begin{bmatrix} h & f & g \\ 2n-2 & n & n \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}, \quad \begin{bmatrix} f & h & g \\ n & 2n-2 & n \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}, \quad \begin{bmatrix} f & g & h \\ n & n & 2n-2 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix},$$

respectively, and

$$\begin{aligned} [f,g,h,n,\alpha_1,\alpha_2,\alpha_3]_\star = {} & [f,g,h,n,\alpha_1,\alpha_2,\alpha_3]_3 - [g,f,h,n,\alpha_1,\alpha_2,\alpha_3]_3 \\ & + [g,h,f,n,\alpha_1,\alpha_2,\alpha_3]_2 - [h,g,f,n,\alpha_1,\alpha_2,\alpha_3]_1 \\ & + [h,f,g,n,\alpha_1,\alpha_2,\alpha_3]_1 - [f,h,g,n,\alpha_1,\alpha_2,\alpha_3]_2, \end{aligned}$$

for $f,g \in V_n$ and $h \in V_{2n-2}$.

Note, as first seen in Algorithm 5.3.3, every chained Lie algebra of length 6 depends on an α parameter which imposes restrictions over α_{224} for every not null α_{112}, α_{123}, α_{134} as seen in equation (5.18).

Proposition 5.3.17. *The* 4*-tuples* $(n,1,0,0)$, $(n,1,0,2)$ *and* $(n,1,1,1)$ *generate the following parametric families of* $\mathfrak{sl}_2$*-chained Lie algebras:*

(a) $\mathfrak{g}^{\rho,\lambda}_{(n,1,0,0)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-2})_\lambda$ *for* $n \geq 1$,

(b) $\mathfrak{g}^{\rho,\lambda}_{(n,1,0,2)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-6})_\lambda$ *for* $n \geq 2$,

(c) $\mathfrak{g}^{\rho,\lambda}_{(n,1,1,1)} = \mathfrak{sl}_2(\mathbb{F}) \oplus_\rho (V_n \oplus V_{2n-2} \oplus V_{3n-4} \oplus V_{4n-6})_\lambda$ *for* $n \geq 2$.

Along here, $\lambda_{(n_1,n_2,n_3,n_4)} = (\alpha_{112}, 0, 0, \alpha_{123}, 0, \alpha_{134}, \alpha_{224})$, *where*

$$\begin{aligned}
\alpha_{224} &= 0 && \textit{when } \lambda = \lambda_{(n,1,0,0)},\\
\alpha_{224} &= \frac{4(4n-3)\,\alpha_{123}\,\alpha_{134}}{3(3n-2)\,\alpha_{112}} && \textit{when } \lambda = \lambda_{(n,1,0,2)},\\
\alpha_{224} &= \frac{(2n-2)\,\alpha_{123}\,\alpha_{134}}{(3n-4)\,\alpha_{112}} && \textit{when } \lambda = \lambda_{(n,1,1,1)},
\end{aligned}$$

defines the product in the $\mathbb{N}$*-graded nilradical given by* $\alpha_{112}(\,\cdot\,,\,\cdot\,)_{n_2}$, $\alpha_{123}(\,\cdot\,,\,\cdot\,)_{n_3}$, $\alpha_{134}(\,\cdot\,,\,\cdot\,)_{n_4}$ *and* $\alpha_{224}(\,\cdot\,,\,\cdot\,)_{n_3+n_4-n_2}$. *We also have the not necessarily graded particular cases, where* α_{113}, α_{114}, α_{124} *are not zero, given by the sevenfolds* $\lambda_{(n,1,k,j)}$,

$$\begin{aligned}
\lambda_{(2,1,0,2)} &= (\alpha_{112}, 0, \alpha_{114}, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{224}),\\
\lambda_{(4,1,0,2)} &= (\alpha_{112}, 0, 0, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{224}),\\
\lambda_{(2,1,1,1)} &= (\alpha_{112}, \alpha_{113}, \alpha_{114}, \alpha_{123}, \alpha_{124}, \alpha_{134}),\\
\lambda_{(4,1,1,1)} &= (\alpha_{112}, 0, 0, \alpha_{123}, \alpha_{124}, \alpha_{134}, \alpha_{224}).
\end{aligned}$$

For these four last parametric families, $\mathfrak{g}^{\rho,\lambda}_{(2,1,k,j)}$ *and* $\mathfrak{g}^{\rho,\lambda}_{(4,1,k,j)}$, *the entry* α_{224} *and the irreducible decomposition are described as in the graded case, and the product in the nilradical is given by* $\alpha_{112}(\cdot,\cdot)_1$, $\alpha_{11p}(\cdot,\cdot)_1$ *for*[5] $p = 3, 4$, $\alpha_{123}(\cdot,\cdot)_k$, $\alpha_{124}(\cdot,\cdot)_{\frac{4-n}{2}}$, $\alpha_{134}(\cdot,\cdot)_j$, $\alpha_{224}(\cdot,\cdot)_1$ *with* $n = 2$ *and* 4 *respectively.*

Proof. According to Proposition 5.3.13 and the notation in its proof, it is easily check that the module summand $W(n_1, n_2, n_3, n_4) = m_1 \oplus m_2 \oplus m_3 \oplus m_4$ is as described in the three items.

Assume first $(n_1, n_2, n_3, n_4) = (n, 1, 0, 0)$ and look at scalar entries α_{113}, α_{114}, α_{124} and α_{224} of $\lambda_{(n,1,0,0)}$. Using Clebsch-Gordan's formula from equation (5.14) we check that $\alpha_{113} = \alpha_{114} = \alpha_{124} = \alpha_{224} = 0$ because modules $m_3 = V_{3n-2}$ and $m_4 = V_{4n-2}$ are not contained in $\Lambda^2 m_1$ and m_4 is not contained in either $m_1 \otimes m_2$ or $m_2 \otimes m_2$. Proposition 5.3.13 says tuple $(n, 1, 0)$ generates a chain. So our case is reduced to checking if $m_4 = V_{4n-2}$ appears in $m_1 \otimes m_3$ decomposition, which it is true; and studying Jacobi identity for

[5] In general, we have $\alpha_{113}(\cdot,\cdot)_{\frac{2k-n+2}{2}}$ and $\alpha_{114}(\cdot,\cdot)_{j+k-n+1}$. But, as we only have non-null α_{113} and α_{114} for some λ we can simplify to those which have them.

two elements in m_1 and another in m_2. But this last condition, seen in equation (5.19), is equivalent to $[f, g, h, n, 0, 0, 0]_3 = 0$ for $n \geq 1$, $f, g \in V_n$ and $h \in V_{2n-2}$.

Suppose now $(n_1, n_2, n_3, n_4) = (n, 1, 0, 2)$, and note that $\alpha_{113} = 0$ and $\alpha_{114} \neq 0$ (respectively $\alpha_{124} \neq 0$) only if $n = 2$ (respectively $n = 2, 4$). For the $\mathbb{N}$-graded condition $\alpha_{113} = \alpha_{114} = \alpha_{124} = 0$, Proposition 5.3.13 says that tuple $(n, 1, 0)$ generates a chain. So this case is reduced to checking if $m_4 = V_{4n-6}$ appears in $m_1 \otimes m_3$ decomposition, which it is true; and studying Jacobi identity for two elements in m_1 and another in m_2. But this last condition, seen in equation (5.19), is equivalent to

$$\begin{aligned}[h, f, g, n, 0, 2, 0]_1 + [f, h, g, n, 0, 2, 0]_2 - [h, g, f, n, 0, 2, 0]_1 \\ - [g, h, f, n, 0, 2, 0]_2 + \frac{14n - 18}{9n - 12}\big([f, g, h, n, 0, 2, 0]_3 \\ - [g, f, h, n, 0, 2, 0]_3\big) + \frac{(n-1)(2n-4)}{(3n-4)(3n-2)} G = 0\end{aligned}$$

for $n \geq 2$, $f, g \in V_n$ and $h \in V_{2n-2}$, where

$$G = \frac{2n-3}{n-1}[h, f, g, n, 0, 1, 1]_1 + [g, h, f, n, 0, 1, 1]_2 + [f, g, h, n, 0, 1, 1]_3,$$

which will appear again in our final case $(n_1, n_2, n_3, n_4) = (n, 1, 1, 1, 1)$. By reapplying Proposition 5.3.13, tuple $(n, 1, 1)$ generates a chain. So, assuming $\alpha_{113} = \alpha_{114} = \alpha_{124} = 0$, our case is reduced to checking if $m_4 = V_{4n-6}$ appears in $m_1 \otimes m_3$ decomposition, which it is true; and studying Jacobi identity for two elements in m_1 and another in m_2. But this last condition, seen in equation (5.19), is equivalent to

$$\frac{2n-3}{n-1}[h, f, g, n, 0, 1, 1]_1 + [g, h, f, n, 0, 1, 1]_2 + [f, g, h, n, 0, 1, 1]_3 = 0,$$

for $n \geq 2$, $f, g \in V_n$ and $h \in V_{2n-2}$. Here also appear the particular tuples $(2, 1, 1, 1)$ and $(4, 1, 1, 1)$ for which $(\alpha_{113}, \alpha_{114}, \alpha_{12,4}) \neq (0, 0, 0)$ and $\alpha_{113} = \alpha_{114} = 0$ but $(\alpha_{113}, \alpha_{114}, \alpha_{124}) \neq 0$. Exceptions $(2, 1, 0, 1)$, $(4, 1, 1, 1)$, $(2, 1, 1, 1)$ and $(4, 1, 1, 1)$ are covered by the particular sevenfold λ found at the end of the proposition. □

Overview Finally, to sum up, we will see the relation among all the previously described chains. As on any t-chain we can take the quotient by their last $\mathfrak{sl}_2$-modules to obtain smaller chains, there is a strong relation between chains. This is idea is expressed in the following lemma.

Lemma 5.3.18. *The tuple $(n_1, n_2, \ldots, n_t)$ could form a $\mathfrak{sl}_2$-chained Lie algebra if $(n_1, n_2, \ldots, n_k)$ forms a valid $\mathfrak{sl}_2$-chained Lie algebra for every $k \leq t$.*

Proof. If $(n_1, n_2, \ldots, n_t)$ forms a valid $\mathfrak{sl}_2$-chained Lie algebra $L = \mathfrak{sl}_2 \oplus N$ for $N = m_1 \oplus m_2 \oplus \cdots \oplus m_t$, then $L/N^{k+1} = \mathfrak{sl}_2 \oplus m_1 \oplus m_2 \oplus \cdots \oplus m_k$ would be a Lie algebra for every k. $\square$

This is the same as saying that, if $(n_1, n_2, \ldots, n_k)$ does not produce any valid chain, then $(n_1, n_2, \ldots, n_k, \ldots, n_t)$ would never produce a valid chain.

Example 5.3.12. From Remark 5.3.16, tuples $(n, 1, 2, *)_{n\geq 2}$ or $(2n+1, 2n+1, 0, *)_{n\geq 1}$ do not produce Lie algebras with t-chain ideal lattice for $t \geq 4$.

This result is interesting for creating a tree-dependency between these chains, which can be seen in Figure 5.11.

5.3.2.4 Quadratic Lie chains

Along the memoir we have built series of naturally graded Lie algebras of nilindex 4 and 5 whose derivation algebra contains a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. The question now arises is which chains are quadratic Lie algebras. As seen in Proposition 5.3.6, one necessary condition is that the ideal lattice is self-dual. This imposes strong restrictions over our algebras, as it will only work for symmetric chains, i.e. $\mathfrak{sl}_2 \bigoplus_{i=1}^{t} V_{k_t}$ where $V_{k_i} \cong V_{k_{t-i}}$ for $i = 1, \ldots, t-1$ and $k_t = 2$ so $V_{k_t} \cong \mathfrak{sl}_2$. This way, the chain will be something like

$$\mathfrak{sl}_2 \oplus V_n \oplus V_m \cdots \oplus V_m \oplus V_n \oplus V_2.$$

Among all previous algebras, the only symmetric ones of 5 or 6 ideals are:

- $\mathfrak{g}^{\rho,\lambda}_{(2,1,1)}$ from Proposition 5.3.13.
- $\mathfrak{g}^{\rho,\lambda}_{(4n+2,2n+1,4n+1)}$ from Proposition 5.3.14.
- $\mathfrak{g}^{\rho,\lambda}_{(1,1,0,0)}$ and $\mathfrak{g}^{\rho,\lambda}_{(2,1,1,1)}$ from Proposition 5.3.17.
- $\mathfrak{g}^{\rho,\lambda}_{(4,1,3,3)}$ and $\mathfrak{g}^{\rho,\lambda}_{(4,3,1,3)}$ from Table 5.4.

- V_n
  - $V_n \oplus V_{2n-2k}$
    - $V_n \oplus V_{2n-2} \oplus V_{3n-2}$
      - $V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-2}$
      - $V_n \oplus V_{2n-2} \oplus V_{3n-2} \oplus V_{4n-6}$
    - $V_n \oplus V_{2n-2} \oplus V_{3n-4}$
      - $V_n \oplus V_{2n-2} \oplus V_{3n-4} \oplus V_{4n-6}$
      - $V_3 \oplus V_4 \oplus V_5 \oplus V_2$
    - $V_n \oplus V_{2n-2} \oplus V_{3n-8}$
      - $V_3 \oplus V_4 \oplus V_1 \oplus V_2$
      - $V_4 \oplus V_6 \oplus V_4 \oplus V_2$
      - $V_5 \oplus V_8 \oplus V_7 \oplus V_2$
      - $V_6 \oplus V_{10} \oplus V_{10} \oplus V_6$
    - $V_n \oplus V_{2n-6} \oplus V_{3n-8}$
      - $V_4 \oplus V_2 \oplus V_4 \oplus V_2$
      - $V_5 \oplus V_4 \oplus V_7 \oplus V_2$
    - $V_{4n} \oplus V_{4n-2} \oplus V_4$
    - $V_{4n+1} \oplus V_{4n} \oplus V_1$
    - $V_{4n+2} \oplus V_{4n+2} \oplus V_2$
    - $V_{4n+3} \oplus V_{4n+4} \oplus V_1$
    - $V_{4n+4} \oplus V_{4n+6} \oplus V_4$

Figure 5.11: Relationships of $\mathfrak{sl}_2$-chained Lie algebras up to length 6.

All previously mentioned symmetric $\mathfrak{sl}_2$-chains, when considered naturally graduated ($\alpha_{ijk} = 0$ for $k \neq i + j$) are quadratic, as they admit at least the following non-degenerate symmetric bilinear form. Let

$$\mathfrak{sl}_2 \bigoplus_{i=1}^{t} V_{k_i}$$

be the chain such that $k_t = 2$ and $k_i = k_{t-i}$ for $i = 1, \ldots, t-1$. We define a bilinear form such that

$$\mathfrak{sl}_2^{\perp} = \mathfrak{sl}_2 \bigoplus_{i=1}^{t-1} V_{k_i},$$
$$V_{k_j}^{\perp} = \mathfrak{sl}_2 \bigoplus_{i=1}^{t-j-1} V_{k_i} \bigoplus_{i=t-j+1}^{t} V_{k_i}.$$

Then, for $\{x_i^{k_i}, x_i^{k_i-1}y_i, \ldots, x_i y_i^{k_i-1}, y_i^{k_i}\}$ a basis of each V_{k_i}, we have

$$B(e, y_t^2) = B(h, x_t y_t) = -B(f, x_t^2) = \theta \neq 0.$$

For $i = 1, \ldots, t-1$ and $j = 0, \ldots, k_i > 0$

$$B(x_i^{k_i}, y_{t-i}^{k_i}) = (-1)^j \binom{k_i}{j} B(x_i^j y_i^{k_i-j}, x_{t-i}^{k_i-j} y_{t-i}^j) = \frac{\alpha_{i,t-i,t} B(e, y_t^2)}{k_i} = \frac{\alpha_{i,t-i,t}\theta}{k_i};$$

and, when $k_i = 0$, thus $k_{i-1} = k_1 \neq 0$ and $2i = t$, our bilinear form is defined as

$$B(1,1) = \frac{\alpha_{i-1,i,2i-1} B(x_1^{k_1}, y_{2i-1}^{k_1})}{\alpha_{1,i-1,i}} = \frac{\alpha_{i-1,i,t-1} B(x_1^{k_1}, y_{t-1}^{k_1})}{\alpha_{1,i-1,i}} = \frac{\alpha_{i-1,i,t-1} \cdot \alpha_{1,t-1,t} \cdot \theta}{\alpha_{1,i-1,i} \cdot k_1}.$$

So we end up with

$$\mathrm{span}\{x_i^{k_i-j} y_i^j\}^{\perp} = V_{k_i}^{\perp} \oplus \mathrm{span}\{x_i^{k_i}, \ldots, x_i^{k_i-j+1} y_i^{j-1}, x_i^{k_i-j-1} y_i^{j+1}, \ldots, y_i^{k_i}\}$$

and with

$$\mathrm{span}\{e\}^{\perp} = \mathfrak{sl}_2^{\perp} \oplus \mathrm{span}\{x_t^2, x_t y_t\},$$
$$\mathrm{span}\{f\}^{\perp} = \mathfrak{sl}_2^{\perp} \oplus \mathrm{span}\{x_t y_t, y_t^2\},$$
$$\mathrm{span}\{h\}^{\perp} = \mathfrak{sl}_2^{\perp} \oplus \mathrm{span}\{x_t^2, y_t^2\}.$$

Apart from the bilinear form described, some of algebras in their naturally graded form can be seen as quadratic using other arguments:

- $\mathfrak{g}_{(2,1,1)}^{\rho,\lambda}$ from Proposition 5.3.13 and $\mathfrak{g}_{(2,1,1,1)}^{\rho,\lambda}$ from Proposition 5.3.17 both form a chain $\mathfrak{sl}_2 \oplus V_2 \oplus \cdots \oplus V_2 \cong \mathfrak{sl}_2 \otimes \mathbb{F}[x]/\langle x^i \rangle$ for $i = 4$ and 5 respectively. These algebras are quadratic as seen in Section 2.2.2. This idea can also be applied over bigger chains. For instance, $\mathfrak{g}_{(2,1,\ldots,1)}^{\rho,\lambda}$, which produces $\mathfrak{sl}_2 \oplus V_2 \oplus \cdots \oplus V_2 \cong \mathfrak{sl}_2 \otimes \mathbb{F}[x]/\langle x^i \rangle$, is always a quadratic Lie algebra.

- $\mathfrak{g}^{\rho,\lambda}_{(2,1,1)}$ from Proposition 5.3.13 can be obtained as a double extension of $\mathfrak{n}_{3,2}$ by $\mathfrak{sl}_2$ and as a T^*-extension of $\mathfrak{sl}_2 \ltimes_{\rho_1} V_2$ as the square of the nilradical is a lagragian.

- $\mathfrak{g}^{\rho,\lambda}_{(2,1,1,1)}$ appears as the double extension of the Lie algebra 7, also named as $\mathfrak{n}^2_{3,3}$, in Section 6.2.2.1.

- $\mathfrak{g}^{\rho,\lambda}_{(1,1,0,0)}$ from Proposition 5.3.17 can be obtained as the double extension of $\mathfrak{n}_{2,3}$ by $\mathfrak{sl}_2$.

Moreover, we point out these algebras are mixed, double extensions and local quadratic Lie algebras. And, we can question ourselves what happens in a general symmetric chain. As we are not sure if our general bilinear form works for every symmetric chain, it is just a conjecture, we need other arguments. Let $\mathfrak{g} = \mathfrak{sl}_2 \oplus V_{i_1} \oplus V_{i_2} \oplus \cdots \oplus V_{i_t}$ be quadratic symmetric chain. Some ideas which can be applied are:

(a) The nilradical is $\mathfrak{n} = V_{i_1} \oplus V_{i_2} \oplus \cdots \oplus V_{i_t}$ and $\mathfrak{n}$ is t-step ($\mathfrak{n}^{t+1} = 0$).

(b) The ideals contained in $\mathfrak{n}$ satisfies $(\mathfrak{n}^k)^\perp = \mathfrak{n}^{t+1-k}$ for $k = 1, \ldots, t$. This can be seen in Figure 5.12.

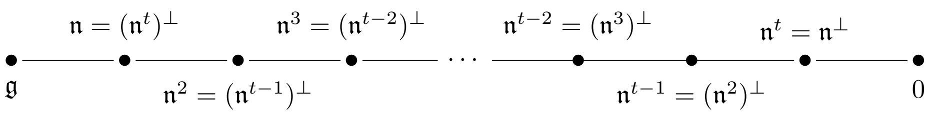

Figure 5.12: Ideals in a quadratic chain $\mathfrak{g}$.

(c) Applying Corollaries 3.1.11 and 3.1.12, $\mathfrak{n}/\mathfrak{n}^\perp$ must be quadratic and obtainable as successive double extensions.

(d) If $t = 2k - 1$ then $(\mathfrak{n}^k)^\perp = \mathfrak{n}^k$ is an isotropic ideal of dimension half of the algebra and can be obtained as a T^*-extension by [Bordemann, 1997, Theorem 3.2].

Remark 5.3.19. According to [Kath and Olbrich, 2006, Section 3], there are ascending and descending series of ideals, referred as higher socles and radicals respectively, related by orthogonality. In our chains they match. Apparently, this is one of the most natural ways of studying general quadratic Lie algebras, as stated in some unpublished notes titled “Structure des espaces symetriques pseudo-riemanniens” by Berard Bergery.

5.4 Summary

In contrast with previous chapters which restrict the problem of studying quadratic Lie algebras to smaller families, this chapter focuses on the inverse: extending those small algebras into bigger and more general ones.

The first section is devoted to the study of derivations, automorphisms and bilinear forms. Derivations are a key tool to obtain split and double extensions, while automorphisms can be used to tackle isomorphism problems. In the beginning of Section 5.1, derivations and automorphisms of any nilpotent Lie algebra are given in Theorems 5.1.5 and 5.1.6. The first of these results was a known result from [Satô, 1971], while the second one is a new contribution to the matter. Both results are supported on the UMP. Moreover, in combination to Proposition 5.1.1 and Corollary 5.1.2, derivations and automorphisms of free nilpotent Lie algebras can be deduced from how the map acts of their m.s.g. All these ideas, described in Theorems 5.1.5 and 5.1.6, can be computationally implemented (see Section 6.2.2) to generate many examples included in the chapter. The last part of the section is a remark on invariant bilinear forms on general Lie quotients. This was the tool used in Section 3.1.3 (originally in [Benito, 2017]) to obtain the different nilpotent quadratic Lie algebras of low dimension.

Section 5.2 is focused on the study of local algebras (one maximal ideal). These algebras appear when considering indecomposable quadratic Lie algebras of quadratic dimension two. The restriction to only one maximal ideal transfers interesting properties gathered in Proposition 5.2.3 (originally in [Bajo and Benayadi, 2007]). The most general algebras in this family can be obtained applying our previous Theorem 3.1.8 as double extensions. This is illustrated in several examples where, from free nilpotent quadratic algebras, we obtain local algebras (solvable and mixed). A key example of local algebras is the variety of generalized oscillator algebras. These algebras are defined along this thesis as double extensions of abelian metric spaces and decompose as split extensions of GHA (see Proposition 5.2.7). From the derivation subalgebra of GHA, we are able to describe derivations and skew-derivations of real oscillator algebras in Theorem 5.2.10. This provides new series of mixed quadratic Lie algebras.

The final section is related to the study of the lattice of ideals of quadratic Lie algebras. It follows the results on ideal arrangement given in Section 3.1. First, we note lattices of Lie algebras must be complete, bounded, modular, and, if finite, also distributive (check Theorem 5.3.4). This restricts the possible lattices up to 8 ideals as seen in Figure 5.6. When working of quadratic Lie algebras, lattices must be self-dual (symmetric) as stated in Proposition 5.3.6. Main examples of self-dual lattices are boolean lattices and n-chains. There is a wide variety of quadratic Lie algebras with this type of lattice.

At the end of the chapter, we start the study of n-chain lattices from a general perspective using free nilpotent algebras in combination with simple ones via representations, or using current algebras. Both generate mixed Lie algebras, but only the last one provides quadratic Lie algebras. In order to obtain other construction methods, supporting ourselves in results from [Benito, 1992b] and [Šnobl, 2010], we develop Theorem 5.3.9. This much more general approach, in combination with $\mathfrak{sl}_2$ using its irreducible representations V_n and some suitable $\mathfrak{sl}_2(\mathbb{F})$-invariant bilinear products $V_n \otimes V_m \to V_{m+n-2k}$ that appeared in [Dixmier, 1984], allows us to explicitly describe algorithms to obtain Lie algebras whose lattice is a chain. These algorithms, inspired by [Dixmier, 1984] and [Bremner and Hentzel, 2004], will produce parametric families of $\mathbb{N}$-graded (also named quasi-cyclic or Carnot) Lie algebras of arbitrary dimension and nilpotent index up to 5 (although bigger versions can be achieved by extensions of these algorithms using Theorem 5.3.9). The algebras are easily described by taking basis and defining their respective structure constants. Up to length six, all possible chains are shown in Figure 5.11. Despite their self-dual structure, only three specific algebra and two countable series are quadratic thanks to the explicit invariant bilinear form given.

Algorithms

6

CHAPTER

During the development of this thesis, several algorithms have been developed in order to assist ourselves in obtaining examples and getting conjectures of what is happening for developing new results after. In this chapter, we present these algorithms. Although they are implemented for *Wolfram Mathematica*, the idea behind them can be translated to different languages that support symbolic expressions such as *SageMath*.

All the information in this Chapter is referred to package `LieFunctions` version 1.0.0 and is subject to improvements. In order to facilitate its use, each method uses a descriptive name and preserves Mathematica naming convections, for instance an ending Q for tests. Apart from the explanations we will give along this chapter, each method includes an inline description shown in the autocomplete popup or prepending a question mark to its name.

6.1 Availability and installation

The package and the all its source code is available at the GitHub repository `https://github.com/joroldan/MathematicaLieFunctions`. There, anyone can download or inspect the code of all the package and functions.

The license of the source code is GNU General Public License v3.0 so anyone is free to use it, even for commercial or private uses, modify it, distribute it... The only restrictions applied is that the derived works must preserved the same openness (open source and license), and cite the original work.

After downloading the package, in order to use it, you need to install and, after that, import it in every Mathematica Notebook in which you make use of it. The installation can be done through the menu of the application: File ▶ Install. Once installed, before using it, you need to import it writing `<<LieFunctions‘` or whatever new name you have chosen.

If you do not have access to an installation of Mathematica, you can use it online at the webpage `https://www.wolframcloud.com`. Here, with a free account, you can still import the package and call every function. However, there is a limit about the computation time available to used in that free mode. This is why the most demanding functions can present problems. Anyway, to install the package, as there is no File menu, the alternative is executing the command:

```
PacletInstall["https://github.com/joroldan/\
    MathematicaLieFunctions/releases/download/\
    Releases/LieFunctions-1.0.0.paclet"]
```

changing the final part `1.0.0` for whatever version we are interested in. This same command also works in the offline version. The rest works exactly the same.

6.2 Functions included

The package we have developed covers different topics of the dissertation, ranging from Hall bases to chains of ideals, including derivations, automorphisms, bilinear forms, and more.

The common pattern used in the package to make all of it work together is the list of the adjoint matrices in a basis, that is their structure constants. Most of the methods accept or produce that list of adjoint matrices. This is why the first methods we need to know are the following:

- `AdjointListSymmetricQ` given a list of matrices representing those adjoints in same basis checks if $ad_x(y) = -ad_y(x)$ for every x, y in the basis.
- `AdjointListJacobiQ` checks the Jacobi identity.
- `AdjointListQ` combines the two previous methods in just one. The reason why there exists a separate version is in order to better detect misspellings in the adjoint list and known exactly what is wrong with it.

Once you have a valid list of adjoint matrices, you can make use of the method `ProductCoordinates[v1,v2,adjointList]` to calculate the Lie bracket of a vector with coordinates `v1` against other with coordinates `v2`, obtaining the coordinates of the result. Also, you can get the product table using the function `ProductTableList[basis,adjointList]`. This method returns a list of the products given by the adjoints using to denote the elements the names given in `basis`. This same function can be call as `NonNullProductTableList` to remove null products or changing `List` into `Print` for a human readable notation. See Figure 6.1.

```
In[1]:= ProductTableList[{e, f, h}, SL2AdjointList]
Out[1]= {{e, f, h}, {e, h, -2 e}, {f, h, 2 f}}

In[2]:= ProductTablePrint[{e, f, h}, SL2AdjointList]
        [e, f] = h
Out[2]= [e, h] = -2 e
        [f, h] = 2 f
```

Figure 6.1: Example of a product table given the adjoint matrices.

Remark 6.2.1. Along all models, we consider right side products for matrices.

6.2.1 Hall Basis

From the generator set $\mathfrak{m} = \{x_1, \ldots, x_d\}$, we easily get the standard monomials $[x_{i_1}, \ldots, x_{i_r}]$ that linearly generate Lie algebra $\mathfrak{n}_{d,t}$. However, the anticommutativity law ($[x_i, x_j] + [x_j, x_i] = 0$) and the Jacobi identity

$$J(x_i, x_j, x_k) = \sum_{\text{cyclic}} [[x_i, x_j], x_k] = 0,$$

both set linear dependency relations. This makes it difficult to find a basis formed by monomials. The problem was solved by M. Hall in 1950. Focusing on the behavior of algorithms, the most natural basis to work on free nilpotent Lie algebras, is the *Hall basis* (see [Hall, 1950] for definition, and [Serre, 1992, Chapter IV, Section 5] for a detailed construction).

Starting with the total order $x_d < x_{d-1} < \ldots < x_1$, the definition of Hall basis states recursively if a given standard monomial depends on the previous ones. The recursive algorithm is covered by the pseudocode given in Table 6.1 and provides a Hall basis that we will denote as $\mathcal{H}_{d,t}(U_<)$ or $\mathcal{H}_{d,t}$ if the total order in U is clear. This algorithm checks if an element v belongs to the Hall basis once we have defined a monomial order. For some small d and t values, the output of Hall basis algorithm is given in Table 6.2.

---

isCanonical(v):
 if $\deg v == 1$ **then true**;
 else if (**not** *isCanonical*(v_1) **or not** *isCanonical*(v_2) **or** $v_2 > v_1$) **then false**;
 else if $\deg v_1 > 1$ **then** (*isCanonical*$(v_{1,1})$ **or** *isCanonical*$(v_{1,2})$ **or** $v_2 \geq v_{1,2}$);
 else true;

Table 6.1: Hall basis algorithm pseudocode. Note that this is a recursive algorithm. Here $v = [v_1, v_2]$. In order to generate Hall Basis elements of degree n we can combine v_1 and v_2 in level $n-k$ and k respectively, where $k = 1, \ldots, n/2$.

This algorithm is built into our Mathematica package under divided into several functions:

- `HallBasisLevel[d,t]` returns the degree t monomials in the Hall basis with d generators.
- `HallBasisUntilLevel[d,t]` returns the full $\mathcal{H}_{d,t}$.

Both methods have a sibling function to obtain the dimension of each of them just adding the word `Dimension` at the end of their names. These algorithms can be used to obtain Table 6.2. However, they do not used `x` as their variable like in our example. In order to do so, we must set it manually using the

(d,t)	$\mathcal{H}_{d,t}$
$(2,6)$	$x_2, x_1, [x_1,x_2], [[x_1,x_2],x_2], [[x_1,x_2],x_1], [[[x_1,x_2],x_2],x_2], [[[x_1,x_2],x_2],x_1],$ $[[[x_1,x_2],x_1],x_1], [[[[x_1,x_2],x_2],x_2],x_2], [[[[x_1,x_2],x_2],x_2],x_1],$ $[[[x_1,x_2],x_2],[x_1,x_2]], [[[[x_1,x_2],x_2],x_1],x_1], [[[x_1,x_2],x_1],[x_1,x_2]],$ $[[[[x_1,x_2],x_1],x_1],x_1], [[[[[x_1,x_2],x_2],x_2],x_2],x_2], [[[[[x_1,x_2],x_2],x_2],x_2],x_1],$ $[[[[x_1,x_2],x_2],x_2],[x_1,x_2]], [[[[[x_1,x_2],x_2],x_2],x_1],x_1],$ $[[[[x_1,x_2],x_2],x_1],[x_1,x_2]], [[[[[x_1,x_2],x_2],x_1],x_1],x_1],$ $[[[x_1,x_2],x_1],[[x_1,x_2],x_2]], [[[[x_1,x_2],x_1],x_1],[x_1,x_2]],$ $[[[[[x_1,x_2],x_1],x_1],x_1],x_1]$
$(4,3)$	$x_4, x_3, x_2, x_1, [x_3,x_4], [x_2,x_4], [x_2,x_3], [x_1,x_4], [x_1,x_3], [x_1,x_2], [[x_3,x_4],x_4],$ $[[x_3,x_4],x_3], [[x_3,x_4],x_2], [[x_3,x_4],x_1], [[x_2,x_4],x_4], [[x_2,x_4],x_3], [[x_2,x_4],x_2],$ $[[x_2,x_4],x_1], [[x_2,x_3],x_3], [[x_2,x_3],x_2], [[x_2,x_3],x_1], [[x_1,x_4],x_4], [[x_1,x_4],x_3],$ $[[x_1,x_4],x_2], [[x_1,x_4],x_1], [[x_1,x_3],x_3], [[x_1,x_3],x_2], [[x_1,x_3],x_1], [[x_1,x_2],x_2],$ $[[x_1,x_2],x_1]$
$(6,2)$	$x_6, x_5, x_4, x_3, x_2, x_1, [x_5,x_6], [x_4,x_6], [x_4,x_5], [x_3,x_6], [x_3,x_5], [x_3,x_4], [x_2,x_6],$ $[x_2,x_5], [x_2,x_4], [x_2,x_3], [x_1,x_6], [x_1,x_5], [x_1,x_4], [x_1,x_3], [x_1,x_2]$

Table 6.2: Hall basis of $\mathfrak{n}_{d,t}$. Note from expanded basis $\mathcal{H}_{4,3}$ and $\mathcal{H}_{2,6}$ we can recover Hall basis of $\mathfrak{n}_{4,2}$ and $\mathfrak{n}_{2,t}$ for $t = 2,3,4,5$

command `$HallVar=x`. In addition, due to Mathematica limitations the Lie bracket are denote using curly brackets or braces as if it were a list. An example is shown in Figure 6.2.

```
In[1]:= $HallVar = a;
        HallBasisLevel[2, 3]
Out[2]= {{{a[1], a[2]}, a[2]}, {{a[1], a[2]}, a[1]}}

        (*This last result represents {[[a1,a2],a2], {[a1,a2],a1]}*)

In[3]:= HallBasisUntilLevelDimension[4, 3]
Out[3]= 30
```

Figure 6.2: Example of Hall basis commands.

Apart from the Hall basis, we can also obtain the Lie bracket of the free nilpotent Lie algebras via its adjoint matrices in the Hall basis. The command to get that is `HallBasisAdjointList[d,t]`.

Finally, as free nilpotent algebras have a natural gradation observed directly from the Hall basis monomials degree (see Example 2.1.7), there is an auxiliar method which helps us showing the matrices (for example their adjoints or derivations) with vertical lines separating them in blocks. This is achieved using `HallGrid[d,t]` before or after a matrix.

6.2.2 Derivations and automorphisms

Given any Lie algebra via their adjoint list you can obtain derivations and automorphisms using methods:

- `DerivationQ` and `AutomorphismQ` given a matrix and a list of adjoints checks whether that matrix could represent a derivation or automorphism.
- `GetDerivation` and `GetAutomorphism` given a variable and the structure constants returns the desired matrix.
- Specific to derivations we have two additional methods:
  - `GetInnerDerivations[var,adjointList]`
  - `GetSkewDerivation[var,adjointList,B]` which finds the derivations skew-symmetric respect the bilinear form `B`.

6.2.2.1 Free nilpotent algebras quotients

In Section 5.1 we saw, among other things, how to find derivations of $\mathfrak{n}_{d,t}/I$. This procedure can be automated using Mathematica. The idea behind can be explained in the next steps:

1. Find generic derivations δ and δ' in $\mathfrak{n}_{d,t}$.
2. Change basis in that derivation to a basis formed by the union of a basis of a complement of I and a basis of I. This splits the derivation into 4 submatrices considering projections.
3. Impose $\delta(I) \subseteq I$ and $\delta'(\mathfrak{n}_{d,t}) \subseteq I$, which refers to the upper right and upper submatrices respectively.
4. After seeing both derivations we apply the quotient (equivalence class).

An example of this idea is found is Figure 6.3.

```
In[1]:= n = HallBasisUntilLevelDimension[2, 5];
        ideal = {EC[5, n], EC[7, n], EC[8, n],
           EC[10, n] + EC[11, n], EC[12, n], EC[13, n], EC[14, n]};
        complement = Table[EC[i, n], {i, {1, 2, 3, 4, 6, 9, 10}}];
        m = Length[complement];
        cB = Transpose[Join[complement, ideal]];
        aL = HallBasisAdjointList[2, 5];
        d = GetDerivation[x, aL];
        dN = Inverse[cB].d.cB;
        cond1 = dN⟦ ;; m, m + 1 ;;⟧ == NullMatrix[m, n - m];
        cond2 = dN⟦ ;; m,  ;;⟧ == NullMatrix[m, n];
        cond3 = SkewSymmetricRespectToQ[B, dN⟦ ;; m,  ;; m⟧];
        B = Reverse[DiagonalMatrix[{-1, 1, -1, 1, -1, 1, -1}]];
        Grid[dN /. Solve[And[cond1, cond3]]⟦1⟧,
         Dividers → {{m + 1 → Black}, {m + 1 → Black}}, Frame → True]
        Grid[dN /. Solve[And[cond2, cond3]]⟦1⟧,
         Dividers → {{m + 1 → Black}, {m + 1 → Black}}, Frame → True]
```

Figure 6.3: Finding $\mathrm{Der}_\phi(\mathfrak{n}_{2,5}/I)$ for the algebra which appears later named as Algebra 3.

For automorphisms the strategy is really simular. Although, from an algebraically point of view, this procedure is really good, when implementing it, we can find a more general and easy way to obtain those derivations. This new approach consists on finding the adjoint matrices in $\mathfrak{n}_{d,t}/I$ and finding their derivations and automorphisms as normal on that new adjoints. In order to do so when can we the following methods:

- `AdjointListQuotient[cB,iB,adjointList]`. Let `cB` a basis of C with elements $\{c_1, \ldots, c_n\}$ and `iB` a basis of an ideal I, both expressed by their coordinates, such that $L = C \oplus I$. This method returns the list of adjoints of L/I in basis $\{c_1 + I, \ldots, c_n + I\}$.

- `ChangeAdjointListBasis[cM,adjointList]` is a more generic method which serves as support for the previous one. Here `cM` represents the matrix which changes coordinates from the new basis to the old one.

This new technique simplifies the code seen in Figure 6.3 to the one in Figure 6.4. Both strategies allow us to find skew-derivations for the nilpotent algebras found in the low-dimensional classification from [Benito et al., 2017].

```
In[1]:= n = HallBasisUntilLevelDimension[2, 5];
ideal = {EC[5, n], EC[7, n], EC[8, n],
   EC[10, n] + EC[11, n], EC[12, n], EC[13, n], EC[14, n]};
complement = Table[EC[i, n], {i, {1, 2, 3, 4, 6, 9, 10}}];
adj25Q = AdjointListQuotient[complement, ideal,
   HallBasisAdjointList[2, 5]];
B = Reverse[DiagonalMatrix[{-1, 1, -1, 1, -1, 1, -1}]];
GetSkewDerivation[x, adj25Q, B] // MatrixForm
```

Figure 6.4: Finding $\mathrm{Der}_\phi(\mathfrak{n}_{2,5}/I)$ for the algebra which appears later named as Algebra 3.

First, note that in all these algebras

$$\dim \operatorname{Inner} L = \dim L - \dim Z(L) = \dim L - (\dim L - \dim L^2) = \dim L^2,$$

and that if $\delta \in \operatorname{Inner} L$, then δ is nilpotent and ϕ-skew-symmetric. For each algebra we will give two matrices, on the left $\mathrm{Der}_\phi(\mathfrak{n}_{d,t}/I)$ and on its right $\operatorname{Inner}(\mathfrak{n}_{d,t}/I)$.

Algebra 1 The 1-dimensional abelian Lie algebra $(\mathfrak{n}_{1,1}, \phi)$ with basis $\{a_1\}$ and $\phi(a_1, a_1) = 1$. Here, all ϕ-skew derivations are null.

Algebra 2 The 5-dimensional free nilpotent Lie algebra $(\mathfrak{n}_{2,3}, \phi)$ with basis $\{a_i\}_{i=1}^5$ and $\phi(a_i, a_j) = (-1)^{i-1}$ for $i \leq j$ and $i + j = 6$ and $\phi(a_i, a_j) = 0$ otherwise. The product table is:

$$\begin{aligned}
a_1 &= e_1 = x_2, \\
a_2 &= e_2 = x_1, \\
a_3 &= e_3 = [a_2, a_1] = [x_1, x_2], \\
a_4 &= e_4 = [a_3, a_1] = [[x_1, x_2], x_2], \\
a_5 &= e_5 = [a_3, a_2] = [[x_1, x_2], x_1].
\end{aligned}$$

Here, all ϕ-skew derivations are of the form:

$$\left(\begin{array}{cc|c|c|c} m_1 & m_2 & 0 & 0 & 0 \\ m_3 & -m_1 & 0 & 0 & 0 \\ \hline m_4 & m_5 & 0 & 0 & 0 \\ \hline m_6 & 0 & m_5 & m_1 & m_2 \\ 0 & m_6 & -m_4 & m_3 & -m_1 \end{array}\right) \supseteq \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline v_2 & v_1 & 0 & 0 & 0 \\ \hline v_3 & 0 & v_1 & 0 & 0 \\ 0 & v_3 & -v_2 & 0 & 0 \end{array}\right).$$

Algebra 3 The 7-dimensional Lie algebra $(\mathfrak{n}^1_{2,5}, \phi)$ with basis $\{a_i\}_{i=1}^7$ and nonzero products:

$$\begin{array}{lll} [a_2, a_1] = a_3, & [a_3, a_1] = a_4, & [a_4, a_1] = a_5, \\ [a_5, a_1] = a_6, & [a_5, a_2] = a_7, & [a_3, a_4] = a_7, \end{array}$$

where $\phi(a_i, a_j) = (-1)^i$ for $i \leq j$ and $i + j = 8$ and $\phi(a_i, a_j) = 0$ otherwise. The product table is:

$$\begin{aligned} a_1 = e_1 &= x_2, \\ a_2 = e_2 &= x_1, \\ a_3 = e_3 &= [a_2, a_1] = [x_1, x_2], \\ a_4 = e_4 &= [a_3, a_1] = [[x_1, x_2], x_2], \\ a_5 = e_6 &= [a_4, a_1] = [[[x_1, x_2], x_2], x_2], \\ a_6 = e_9 &= [a_5, a_1] = [[[[x_1, x_2], x_2], x_2], x_2], \\ a_7 = e_{10} &= [a_5, a_2] = [[[[x_1, x_2], x_2], x_2], x_1], \end{aligned}$$

As $a_7 = [a_3, a_4] = [[x_1, x_2], [[x_1, x_2], x_2]] = -e_{11}$, then $e_7 + e_{11} \in I$, so the ideal is $I = \operatorname{span}\langle e_5, e_7, e_8, e_{10} + e_{11}, e_{12}, e_{13}, e_{14}\rangle$. Here, all ϕ-skew derivations are of the form:

$$\left(\begin{array}{cc|c|c|c|c|c} m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_2 & -2m_1 & 0 & 0 & 0 & 0 & 0 \\ \hline m_3 & m_4 & -m_1 & 0 & 0 & 0 & 0 \\ \hline m_5 & 0 & m_4 & 0 & 0 & 0 & 0 \\ \hline m_6 & m_7 & 0 & m_4 & m_1 & 0 & 0 \\ \hline m_8 & 0 & m_7 & 0 & m_4 & 2m_1 & 0 \\ 0 & m_8 & -m_6 & m_5 & -m_3 & m_2 & -m_1 \end{array}\right) \supseteq \left(\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & \\ \hline v_1 & v_2 & \\ \hline v_3 & 0 & \cdots \\ \hline v_4 & 0 & \\ \hline v_5 & 0 & \\ 0 & v_5 & \end{array}\right).$$

Algebra 4 The 8-dimensional Lie algebra $(\mathfrak{n}^2_{2,5}, \phi)$ with basis $\{a_i\}_{i=1}^8$ and nonzero products:

$$\begin{aligned}
&[a_2, a_1] = a_3, \qquad &&[a_3, a_1] = a_4, \qquad &&[a_3, a_2] = a_5,\\
&[a_4, a_1] = a_6, &&[a_6, a_1] = a_7, &&[a_6, a_2] = a_8,\\
&[a_5, a_2] = a_6, &&[a_3, a_4] = a_8, &&[a_5, a_3] = a_7,
\end{aligned}$$

where $\phi(a_i, a_j) = (-1)^i$ for $0 \le i \le 3$ and $i + j = 9$, $\phi(a_4, a_4) = \phi(a_5, a_5) = 1$ and $\phi(a_i, a_j) = 0$ otherwise. The product table is:

$$\begin{aligned}
a_1 &= e_1 &&= x_2,\\
a_2 &= e_2 &&= x_1,\\
a_3 &= e_3 &&= [a_2, a_1] = [x_1, x_2],\\
a_4 &= e_4 &&= [a_3, a_1] = [[x_1, x_2], x_2],\\
a_5 &= e_5 &&= [a_3, a_2] = [[x_1, x_2], x_1],\\
a_6 &= e_6 &&= [a_4, a_1] = [[[x_1, x_2], x_2], x_2],\\
a_7 &= e_9 &&= [a_6, a_1] = [[[[x_1, x_2], x_2], x_2], x_2],\\
a_8 &= e_{10} &&= [a_6, a_2] = [[[[x_1, x_2], x_2], x_2], x_1].
\end{aligned}$$

In this algebra, we also have

$$\begin{aligned}
a_6 &= [a_5, a_2] = [[[x_1, x_2], x_1], x_1] = e_8 && \Rightarrow \quad e_8 - e_6 \in I,\\
a_7 &= [a_5, a_3] = [[[x_1, x_2], x_1], [x_1, x_2]] = e_{13} && \Rightarrow \quad e_{13} - e_9 \in I,\\
a_8 &= [a_3, a_4] = [[x_1, x_2], [[x_1, x_2], x_2]] = -e_{11} && \Rightarrow \quad e_{10} + e_{11} \in I,\\
a_8 &= [a_6, a_2] = [[[[x_1, x_2], x_1], x_1], x_1] = e_{14} && \Rightarrow \quad e_{14} - e_{10} \in I.
\end{aligned}$$

So $I = \operatorname{span}\langle e_7, e_8 - e_6, e_{10} + e_{11}, e_{12}, e_{13} - e_9, e_{14} - e_{10}\rangle$. Here, all ϕ-skew derivations are of the form:

$$\left(\begin{array}{cc|c|cc|c|cc}
0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0\\
-m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
m_2 & m_3 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
m_4 & m_5 & m_3 & 0 & m_1 & 0 & 0 & 0\\
m_5 & m_6 & -m_2 & -m_1 & 0 & 0 & 0 & 0\\
\hline
m_7 & m_8 & 0 & m_3 & -m_2 & 0 & 0 & 0\\
\hline
m_9 & 0 & m_8 & -m_5 & -m_6 & m_3 & 0 & m_1\\
0 & m_9 & -m_7 & m_4 & m_5 & -m_2 & -m_1 & 0
\end{array}\right) \supseteq \left(\begin{array}{cc|c}
0 & 0 & \\
0 & 0 & \\
\hline
v_1 & v_2 & \\
\hline
v_3 & 0 & \cdots\\
0 & v_3 & \cdots\\
\hline
v_4 & v_5 & \\
\hline
v_6 & 0 & \\
0 & v_6 &
\end{array}\right).$$

Algebra 5 The 6-dimensional free nilpotent Lie algebra $(\mathfrak{n}_{3,2}, \phi)$ with basis $\{a_i\}_{i=1}^{6}$ and nonzero products:

$$[a_2, a_1] = a_4, \qquad [a_3, a_1] = a_5, \qquad [a_3, a_2] = a_6,$$

where $\phi(a_i, a_j) = (-1)^{i-1}$ for $i \leq j$ and $i + j = 7$ and $\phi(a_i, a_j) = 0$ otherwise. The basis of the article is:

$$\begin{aligned}
a_1 &= e_1 = x_3, \\
a_2 &= e_2 = x_2, \\
a_3 &= e_3 = x_1, \\
a_4 &= e_4 = [a_2, a_1] = [x_2, x_3], \\
a_5 &= e_5 = [a_3, a_1] = [x_1, x_3], \\
a_6 &= e_6 = [a_3, a_2] = [x_1, x_2].
\end{aligned}$$

In this case, the ϕ-skew derivations are of the form:

$$\left(\begin{array}{ccc|ccc}
m_1 & m_2 & m_3 & 0 & 0 & 0 \\
m_4 & m_5 & m_6 & 0 & 0 & 0 \\
m_7 & m_8 & -m_1 - m_5 & 0 & 0 & 0 \\
\hline
m_9 & m_{10} & 0 & m_1 + m_5 & m_6 & -m_3 \\
m_{11} & 0 & m_{10} & m_8 & -m_5 & m_2 \\
0 & m_{11} & -m_9 & -m_7 & m_4 & -m_1
\end{array}\right) \supseteq \left(\begin{array}{ccc|c}
0 & 0 & 0 & \\
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \\
\hline
v_1 & v_2 & 0 & \\
v_3 & 0 & v_2 & \cdots \\
0 & v_3 & -v_1 &
\end{array}\right).$$

Algebra 6 The 8-dimensional Lie algebra $(\mathfrak{n}^1_{3,3}, \phi)$ with basis $\{a_i\}_{i=1}^{8}$ and nonzero products

$$\begin{aligned}
&[a_2, a_1] = a_4, \qquad [a_3, a_1] = a_5, \qquad [a_4, a_1] = a_6, \\
&[a_4, a_2] = a_7, \qquad [a_5, a_1] = a_8, \qquad [a_5, a_3] = a_7,
\end{aligned}$$

where $\phi(a_4, a_4) = \phi(a_5, a_5) = \phi(a_1, a_7) = -\phi(a_2, a_6) = -\phi(a_3, a_8) = 1$, and $\phi(a_i, a_j) = 0$ otherwise. The basis of the article is:

$$
\begin{aligned}
a_1 &= e_1 = x_3,\\
a_2 &= e_2 = x_2,\\
a_3 &= e_3 = x_1,\\
a_4 &= e_4 = [a_2, a_1] = [x_2, x_3],\\
a_5 &= e_5 = [a_3, a_1] = [x_1, x_3],\\
a_6 &= e_7 = [a_4, a_1] = [[x_2, x_3], x_3],\\
a_7 &= e_8 = [a_4, a_2] = [[x_2, x_3], x_2],\\
a_8 &= e_{10} = [a_5, a_1] = [[x_1, x_3], x_3],
\end{aligned}
$$

Moreover, $a_7 = [a_5, a_3] = [[x_1, x_3], x_1] = e_{12}$, so $e_8 - e_{12} \in I$. Therefore,

$$I = \operatorname{span}\langle e_6, e_9, e_{11}, e_{13}, e_{14}, e_8 - e_{12}\rangle.$$

The ϕ-skew derivations are of the form:

$$
\left(\begin{array}{ccc|cc|ccc}
m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
m_2 & -m_1 & m_3 & 0 & 0 & 0 & 0 & 0\\
m_4 & -m_3 & -m_1 & 0 & 0 & 0 & 0 & 0\\
\hline
m_5 & m_6 & m_7 & 0 & m_3 & 0 & 0 & 0\\
m_8 & m_7 & m_9 & -m_3 & 0 & 0 & 0 & 0\\
\hline
m_{10} & 0 & m_{11} & m_6 & m_7 & m_1 & 0 & m_3\\
0 & m_{10} & m_{12} & -m_5 & -m_8 & m_2 & -m_1 & m_4\\
m_{12} & -m_{11} & 0 & m_7 & m_9 & -m_3 & 0 & m_1
\end{array}\right)
\supseteq
\left(\begin{array}{ccc|c}
0 & 0 & 0 & \\
0 & 0 & 0 & \cdots\\
0 & 0 & 0 & \\
\hline
v_1 & v_2 & 0 & \\
v_3 & 0 & v_2 & \\
\hline
v_4 & 0 & 0 & \\
0 & v_4 & v_5 & \cdots\\
v_5 & 0 & 0 &
\end{array}\right).
$$

Algebra 7 The 9-dimensional Lie algebra $(\mathfrak{n}^2_{3,3}, \phi)$ with basis $\{a_i\}_{i=1}^9$ and nonzero products:

$$
\begin{array}{lll}
[a_2, a_1] = a_4, & [a_3, a_1] = a_5, & [a_3, a_2] = a_6,\\
[a_4, a_1] = a_7, & [a_4, a_2] = a_8, & [a_5, a_1] = a_9,\\
[a_5, a_3] = a_8, & [a_3, a_6] = a_7, & [a_6, a_2] = a_9,
\end{array}
$$

$\phi(a_4,a_4) = \phi(a_5,a_5) = \phi(a_6,a_6) = \phi(a_1,a_8) = -\phi(a_2,a_7) = -\phi(a_3,a_9) = 1$, and $\phi(a_i,a_j) = 0$ otherwise. The basis of the article is:

$$\begin{aligned}
a_1 &= e_1 = x_3,\\
a_2 &= e_2 = x_2,\\
a_3 &= e_3 = x_1,\\
a_4 &= e_4 = [a_2,a_1] = [x_2,x_3],\\
a_5 &= e_5 = [a_3,a_1] = [x_1,x_3],\\
a_6 &= e_6 = [a_3,a_2] = [x_1,x_2],\\
a_7 &= e_7 = [a_4,a_1] = [[x_2,x_3],x_3],\\
a_8 &= e_8 = [a_4,a_2] = [[x_2,x_3],x_2],\\
a_9 &= e_{10} = [a_5,a_1] = [[x_1,x_3],x_3],
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
a_7 &= [a_3,a_6] = [x_1,[x_1,x_2]] = -e_{14} &\Rightarrow&\quad e_7 + e_{14} \in I,\\
a_8 &= [a_5,a_3] = [[x_1,x_3],x_1] = e_{12} &\Rightarrow&\quad e_8 - e_{12} \in I,\\
a_9 &= [a_6,a_2] = [[x_1,x_2],x_2] = e_{13} &\Rightarrow&\quad e_{10} - e_{13} \in I,
\end{aligned}$$

so $I = \operatorname{span}\langle e_9, e_{11}, e_8 - e_{12}, e_{10} - e_{13}, e_7 + e_{14}\rangle$. The ϕ-skew derivations is of the form:

$$\left(\begin{array}{ccc|ccc|ccc}
0 & m_1 & m_2 & 0 & 0 & 0 & 0 & 0 & 0\\
-m_1 & 0 & m_3 & 0 & 0 & 0 & 0 & 0 & 0\\
-m_2 & -m_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
m_4 & m_5 & m_6 & 0 & m_3 & -m_2 & 0 & 0 & 0\\
m_7 & m_8 & m_9 & -m_3 & 0 & m_1 & 0 & 0 & 0\\
m_8 - m_6 & m_{10} & m_{11} & m_2 & -m_1 & 0 & 0 & 0 & 0\\
\hline
m_{12} & 0 & -m_{14} & m_5 & m_8 & m_{10} & 0 & m_1 & m_3\\
0 & m_{12} & m_{13} & -m_4 & -m_7 & m_6 - m_8 & -m_1 & 0 & -m_2\\
m_{13} & m_{14} & 0 & m_6 & m_9 & m_{11} & -m_3 & m_2 & 0
\end{array}\right) \supseteq \left(\begin{array}{ccc|c}
0 & 0 & 0 & \\
0 & 0 & 0 & \cdots\\
0 & 0 & 0 & \\
\hline
v_1 & v_2 & 0 & \\
v_3 & 0 & v_2 & \cdots\\
0 & v_3 & -v_1 & \\
\hline
v_4 & 0 & -v_6 & \\
0 & v_4 & v_5 & \cdots\\
v_5 & v_6 & 0 &
\end{array}\right).$$

6.2.3 Quadratic Lie algebras

This package can also generate symmetric invariant bilinear forms for a given Lie algebra. This is achieved using the methods:

- `InvariantBilinearFormQ[matrix,adjointList]` which gives the conditions `matrix` must satisfies in order to be invariant.

- `GetSymmetricInvariantBilinearForm[var,adjointList]` generates a matrix to define the bilinear form of a quadratic Lie algebra.

Internally, both methods use `SkewSymmetricRespectToQ[m1,m2]` to check if some matrix `m2` is skew-symmetric respect to `m1`.

Remark 6.2.2. In combination with the method `NumberOfVariables`, which will be introduced later, we can easily obtain the quadratic dimension.

6.2.4 Chains

In Section 5.3.2, we explained several algorithms to build Lie algebras whose lattice of ideals is a chain. These algebras were developed using $\mathfrak{sl}_2$ algebra and modules. The first structures integrated into the package are the different parts involved:

- `SL2AdjointList` (without parameters) is the list of adjoints of simple Lie algebra $\mathfrak{sl}_2$ in the standard basis.
- `VM[n]` is the basis of the of the $\mathfrak{sl}_2$-module V_n.

As explained in their section, the product is given via actions over the modules and transvections. Those can also be computed directly:

- `SL2Module[coord,elto]` finds the action of the $\mathfrak{sl}_2$ element with coordinates `coord` onto the module element `elto`.
- `Transvection[f,g,k,n,m]` refers to $(f,g)_k\colon V_n \times V_m \to V_{n+m-2k}$ from equation (5.3.2.1). This method can also be called omitting `n` and `m`, as both can be obtain from `f` and `g` degrees.
- `SL2ChainAdjointList[n1,n2,...]` gives the adjoint list of the resulting chain for `n1`, `n2`... the same as in Algorithms 5.3.2 and 5.3.3. This function takes any $\alpha_{ijk} = 0$ whenever possible and works up to 4 modules (6 ideals).

Although it is not important, just for completeness, internally, products are calculated using a special, more visual, notation for elements. This can be observed in the basis from `SL2ChainBasis[n1,n2,...]`. This naming convection is used in `SL2ChainProduct`. But, for practical reasons, there is a method

SL2GetElementCoordinates which turns this notation into the classical coordinates.

Remark 6.2.3. All these methods are not foolproof and when they do not receive valid descriptions for modules the result is indeterminate. This is why Tables 5.1, 5.2, 5.3 and 5.4 are quite useful.

Finally, as a visual aid, we can get information about modules, dimensions or even the lattices of ideals of those chains:

- SL2ModuleSizes[n1,n2,...] receives up to 4 descriptors and returns the degree of modules m_1, m_2,... (see equation (5.15)).
- SL2ModuleDimensions[n1,n2,...], similar to the previous one but in this case it return $\dim m_1$, $\dim m_2$, ...
- SL2ChainDescription[n1,n2,...] describes in a natural way the decomposition of the algebra.
- SL2ChainDimensionList[n1,n2,...] gives the total algebra dimension and the dimension of each factor in its decomposition (including $\mathfrak{sl}_2$).
- SL2ChainGraph[n1,n2,...] shows the lattices of ideals.

An example of use of these last methods is given in Figure 6.5.

6.2.5 Other auxiliar methods

The package also includes some more generic functions to compute wider problems which are not specific to our matter. All these methods are used along previous sections in an auxiliar way. Their implementations is fairly simply and involve the use of the instruction Table to do the loops.

6.2.5.1 Matrices

Here we can find methods to generate matrices:

- Null matrices of any dimension build. Depending on whether the matrix is square or not, we use NullMatrix[rows,columns] or, when it is an square matrix, we use NullMatrix[order].

```
In[1]:= SL2ModuleSizes[2, 1, 0]
Out[1]= {2, 2, 4}

In[2]:= SL2ModuleDimensions[2, 1, 0]
Out[2]= {3, 3, 5}

In[3]:= SL2ChainDescription[2, 1, 0]
```

Out[3]= $\mathfrak{sl}_2 \oplus V(2) \oplus V(2) \oplus V(4)$

```
In[4]:= SL2ChainDimensionList[2, 1, 0]
Out[4]= {14, {3, 3, 3, 5}}

In[5]:= SL2ChainGraph[2, 1, 0]
```

Out[5]=

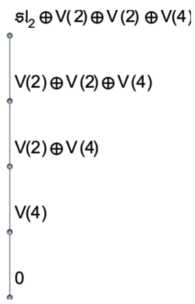

Figure 6.5: Example of chained structures, dimensions and lattices.

- Generic matrices (variables). We can obtain them calling the function `GenericMatrix[rows, columns, var]` or `GenericMatrix[order, var]` when `rows = columns`.

- Generic symmetric or skew-symmetric matrix full of variables. The syntax is the same as the previous one with name `GenericSymmetricMatrix` or `GenericSkewSymmetricMatrix` but this time only the square matrix case is valid.

We also have functions to check some conditions over matrices. These methods receive a matrix and return a boolean depending on whether the condition is fulfiled or not.

- Symmetry, checked with method `SymmetricQ`.

- Skew-symmetry, tested with `SkewSymmetricQ`.

- Maximum rank, or non-degenerate when it represents a bilinear form, is verified via `NonDegenerateQ`.
- Function `CubicMatrixQ` checks if the received list is a list of n matrices of order $n \times n$. It fails, when called with less matrices or when this are not square matrices for example. This is used for instances when checking if some list of matrices represent the list of adjoint matrices in some basis.

When the result is not clear, they return the condition needed. This is for example the difference between our method `SkewSymmetricQ` and the native version `SkewSymmetricMatrixQ`. All methods in this section up to this point can be observed in action in Figure 6.6.

```
In[1]:= NullMatrix[2, 3] // MatrixForm
Out[1]//MatrixForm=
        ( 0 0 0 )
        ( 0 0 0 )

In[2]:= MatrixForm[GenericSymmetricMatrix[2, x]]
Out[2]//MatrixForm=
        ( x[1] x[2] )
        ( x[2] x[3] )

In[3]:= SymmetricMatrixQ[{{a, b}, {c, d}}]
Out[3]= False

In[4]:= SymmetricQ[{{a, b}, {c, d}}]
Out[4]= b == c
```

Figure 6.6: Example of matrix generation and checking its properties.

In relation with basis we have a method to obtain the canonical vectors of any dimension. This can be obtained calling `EC[i,dim]` where `i` denotes the position and `dim` the length. For example `EC[2,4]` refers to $(0, 1, 0, 0)$.

Moreover, to help us compute products of matrices involving blocks instead of numbers we have two additional methods:

- `BTranspose` that given a blocked matrix computes is transpose.
- `BDot` that recibes two or more blocked matrices and returns its product.

The difference with the common transpose and product is that it takes into account the product of each block (submatrices) is not commutative. In order to do so, it uses product `**`, which is the native non-commutative product. Blocked matrices can contain inside variables, `1` to denote the identity matrix or `0` to denote the null matrix. In the following code we can see these methods being used in Figure 6.7.

```
In[1]:= bA = {{A, 1}, {0, B}};
        BDot[BTranspose[bA], bA] // MatrixForm
Out[2]//MatrixForm=
```

$$\begin{pmatrix} \text{Transpose[A]} ** \text{A} & \text{Transpose[A]} \\ \text{A} & 1 + \text{Transpose[B]} ** \text{B} \end{pmatrix}$$

Figure 6.7: Example of a blocked matrix operation $(bA)^t \cdot bA$.

6.2.5.2 Variables

The symbolic power of Mathematica is one of the key features that made us decide using it. But symbolic operations are full of variables and we need some help when working with them:

- `SymbolQ[expression]` checks if `expression` is a variable or not. This method comes with his brother `UndefinedSymbolQ`, which is similar but it also checks if there are values assigned on it or not. This way, if `a[1]=1` the value of only `a` remains unknown, thus `SymbolQ[a]` is true, but `UndefinedSymbolQ[a]` is false.

- `NumberOfVariables[expression]` is self-explanatory. One posible use of this method appears when counting variables in a generic bilinear form to obtain the quadratic dimension.

- Sometimes after applying several restrictions and assignments you end up with several non-consecutive variables. To sort that mess we can use `ReindexVariable[var,expression]`.

- In relation to those some indices given a variable and a expression we can also use `MaxIndexOfVariable`, `MinIndexOfVariable` or to obtain the list of indices `IndicesOfVariable`.

Some examples of use of these methods are found in Figure 6.8. Apart from this list we can also find `PositiveIntegerQ[n]` which was used to check valid indices which start at 1 in Wolfram language. And we also have the method `PolynomialDegree[polynomial]` which gives the sum of the degrees of the variables involved in each monomial when they are all the same degree.

```
In[1]:= v = {a[1], a[2], a[5] + a[2], b[2]};
        NumberOfVariables[v]
Out[2]= 4

In[3]:= IndicesOfVariable[a, v]
Out[3]= {1, 2, 5}

In[4]:= ReindexVariable[a, v]
Out[4]= {a[1], a[2], a[2] + a[3], b[2]}
```

Figure 6.8: Working with symbols and variables in Mathematica.

6.3 Support and future development

This Mathematica package is a continuos work, and it is open to the inclusion of new capabilities or refinements over the existing methods. This improvements will be made available at the repository of the package. In that same webpage, anyone can report bugs (code mistakes), make suggestions or contribute with new code. Even more, they can clone the repository and adapt everything to their needs.

Conclusions, results and open questions

In this dissertation we have been able to obtain our goals reviewing, sorting and expanding previous results on quadratic Lie algebras and developing new ones. These have been explained in detailed in the summary after each chapter. Some of the main results include:

- Ideal structure of quadratic Lie algebras (main ideals inclusion relationships, full lattices of ideals, algebras whose ideals form a chain…)
- Explicit and general formula for obtaining quadratic reduced Lie algebras as double extensions. This was achieved after locating some important ideals to make quotients.
- New method for constructing 2-step nilpotent Lie algebras based on multilinear algebra techniques. The name of the method is "n-quadratic family of matrices" and gives us a computational approach for getting this type of algebras.
- Equivalence theorem which relates n-quadratic family of matrices and double and T^*-extensions methods when applied to obtain 2-step nilpotent Lie algebras. It includes our telescopic expansion technique of successive 1-dimensional double extensions.
- Classification of quadratic reduced 2-step nilpotent Lie algebras up dimension 17 thanks to its relation with trivectors whose classification was already known.

- Automorphisms of free nilpotent Lie algebras quotients. This result extends a similar property for derivations.
- Algorithms to obtain Lie algebras whose lattice of ideals is a chain using $\mathfrak{sl}_2$ representation theory.

All these results, among many others, come with a wide variety of examples to illustrate them.

Moreover, we have developed a Wolfram Mathematica package implementing many of the tools used in the thesis. This open source software is available online allowing people to use or modify it in order to adapt and build new algebras, check identities. . .

However, despite all the work included in the memoir, there are still some open questions which can be studied in the near future. These include:

- How double extensions affect the quadratic dimension.
- Identify, via invariants, for any quadratic 2-step Lie algebra the corresponding one in our classification list of small dimensional algebras.
- Partial classification of the variety of 3-step nilpotent quadratic Lie algebras, as the full classification seems unreachable.
- Quadratic extensions of 2 and 3.step quadratic Lie algebras.
- Quadratic structures on current Lie algebras, including our conjecture in $\mathfrak{sl}_2$-chains.
- Deep study on structure of generalized oscillator Lie algebras and their quadratic extensions.
- Different applications of the real oscillator algebras.
- Better and more general study of quadratic Lie algebras whose ideal is a chain.
- Relations of quadratic Lie algebras and other non-associative structures as: Lie triple systems and symmetric spaces, Manin pairs and triples, among others.
- Expanding our Mathematica package to include more functions. For example double extensions.

Conclusiones, resultados y problemas abiertos

En esta tesis hemos sido capaces de obtener nuestros objetivos revisando, ordenando y expandiendo resultados previos sobre álgebras de Lie cuadráticas, así como desarrollando nuevos. Estos han sido explicados en detalle en el resumen final de cada capítulo. Los principales logros son:

- Estructura de los ideales de las álgebras de Lie cuadráticas (relaciones de inclusión entre los ideales más importantes, retículo de ideales completo, álgebras cuyos ideales están en cadena…)

- Fórmula general y explícita para obtener álgebras cuadráticas reducidas a través de dobles extensiones. Esto se ha logrado tras localizar ciertos ideales para hacer cocientes.

- Nuevo método para construir álgebras 2-step nilpotentes basado en técnicas de álgebra multilineal. El nombre del método es "familia de matrices n-cuadráticas" y nos aporta una aproximación computacional para obtener estas álgebras.

- Teorema de equivalencia que relaciona las familias de matrices n-cuadráticas y los métodos de doble y T^*-extensión cuando obtenemos álgebras de Lie de índice de nilpotencia 2. Incluye una técnica de expansión telescópica a través de sucesivas dobles extensiones 1-dimensionales.

- Clasificación de las álgebras de Lie cuadráticas reducidas 2-step hasta dimensión 17 usando la conocida clasificación de trivectores.
- Automorfismos de cocientes álgebras de Lie nilpotentes libres. Este resultado extiende una propiedad similar en derivaciones.
- Algoritmos para obtener álgebras de Lie cuyo retículo de ideales es una cadena usando teoría de representación de $\mathfrak{sl}_2$.

Todos estos resultados, entre muchos otros, vienen acompañados de una gran variedad de ejemplos ilustrativos.

Además, hemos desarrollado un paquete en Wolfram Mathematica que implementa muchas de las herramientas utilizadas en la tesis. Este programa libre está disponible en internet permitiendo a cualquiera usarlo o modificarlo para adaptarlo y construir nuevas álgebras, comprobar identidades…

Sin embargo, a pesar de todo el trabajo realizado, todavía quedan preguntas abiertas a estudiadar en un futuro próximo. Aquí encontramos:

- Cómo la doble extensión afecta a la dimensión cuadrática.
- Identificar, vía invariantes, a qué algebra de Lie de nuestra clasificación en baja dimensión se corresponde un álgebra cuadrática de índice de nilpotencia dos.
- Clasificación parcial de la variedad de álgebras cuadráticas con índice de nilpotencia 3, ya que la clasificación completa parece inalcanzable.
- Extensiones cuadráticas de álgebras con índice de nilpotencia 2 o 3.
- Estructuras cuadráticas en álgebras de Lie *current*, incluyendo nuestra conjetura en $\mathfrak{sl}_2$-cadenas.
- Estudio profundo de las estructuras de la osciladora generalizada y sus extensiones cuadráticas.
- Diferentes aplicaciones de las álgebras osciladoras reales.
- Detallar el estudio de las álgebras cuadráticas con retículo en cadena.
- Relaciones de las álgebras de Lie cuadráticas y otras estructuras no asociativas: sistemas triples de Lie y espacios simétricos, pares y triples de Manin, entre otros.

- Continuar el desarrollo de nuestro paquete de Mathematica incluyendo más funciones. Por ejemplo la doble extensión.

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Index

Acronyms

2SP 2-step property 112

ACS ascending central series 20

DCS descending central series 20

DP diamond property 164

f.d. finite dimensional 56

GHA Generalized Heisenberg Algebras 42

i.e. latin locution 'id est' meaning 'that is' or 'in other words' 18

m.s.g. minimal set of generators 21, 22

NNP non-null property 112

poset partially ordered set 160

UMP Universal Mapping Property 22

w.l.o.g. without lost of generality 155

Symbols

A^-	twist product over algebra A 10
$J(x, y, z)$	Jacobi identity 10
L'	derived algebra of L 14
L^*	dual space of B 16
$L^{(k)}$	k^{th} derived algebra of L 19
L^k	k^{th} lower central term 20
V_n	$(n + 1)$-dim. irreducible module of $\mathfrak{sl}_2$ 176
$Z(L)$	centre of L 14
$\mathrm{Bi}_{\mathrm{inv}}(L)$	invariant bilinear forms of L 43
$\mathrm{Bi}_{\mathrm{inv}}^{\mathrm{as}}(L)$	invariant skew-symmetric bilinear forms of L 43
$\mathrm{Bi}_{\mathrm{inv}}^{\mathrm{s}}(L)$	invariant symmetric bilinear forms of L 43
End	endomorphism 11
$\mathrm{Hom}(L_1, L_2)$	Lie algebra homomorphisms from L_1 to L_2 15
$\mathrm{Hom}_L(V, W)$	L-module homomorphisms from V to W 18
Hom	homomorphism 136
Int	group of internal automorphisms 31
Orb	orbit 77
Skew	skew-symmetric Lie subalgebra 11
Sub	set of subspaces 165
Sym	symmetric Jordan subalgebra 11
Tr	trace 29
ad	adjoint representation 15
Alt	alternating 123

$\mathrm{asoc}(L)$	abelian socle of an algebra L 62
$\mathrm{Aut}(L)$	automorphism group of L 15
$\mathrm{NilpQuad}_{d,t}$	category of t nilpotent Lie algebras of type d 76
$\mathrm{Q}_{d,t}$	functor that associates $\mathrm{Sym}_0(d,t)$ to $\mathrm{NilpQuad}_{d,t}$ 76
$\mathrm{Sym}_0(d,t)$	category of certain symmetric invariant bilinear forms of $\mathfrak{n}_{d,t}$ 76
codim	codimension 21
ad*	coadjoint representation 16
$\mathrm{Der}(L)$	derivation algebra of L 12
$\mathrm{Der}_\varphi(L)$	φ-skew derivations of L 34
Id	identity 136
$\mathrm{Im}\,\varphi$	image of φ 15
$\mathrm{Inner}(L)$	inner derivations of L 12
κ	Killing form 29
$\ker\varphi$	kernel of φ 15
$\mathcal{F}(M_1,\dots)$	matrix representing the quadratic family of matrices 82
$\mathcal{H}_{d,t}$	Hall basis of $\mathfrak{n}_{d,t}$ 208
$\mathcal{J}(L)$	Jacobson radical of L 29
$\mathcal{M}_d(\mathbb{F})$	matrices $d\times d$ in $\mathbb{F}$ 10
$\mathbb{F}[x_1,\dots,x_n]$	polynomial ring with n variables 10
$\mathbb{F}^\times$	multiplicative group, the group under multiplication of the invertible elements of a field/ring 77
$\mathbb{F}x$	linear span of element x 13
$\mathbb{F}$	generic field 7
$\mathfrak{C}(\{m_i\},\{a_{ijk}\})$	$\mathfrak{sl}_2$-chained Lie algebras 183
$\mathfrak{FL}(d)$	free Lie algebra on d generators 21
$\mathfrak{d}_{2m+2}$	Generalized Oscillator algebras 152
$\mathfrak{gl}(V)$	general linear Lie algebra of V 11
$\mathfrak{gl}(n,\mathbb{F}),\mathfrak{gl}_n$	general linear Lie algebra in $\mathbb{F}$ using matrices of order $n\times n$ 11
$\mathfrak{h}_{2n+1}$	Heisenberg Lie algebras 42
$\mathfrak{n}_{d,t}$	free t-step nilpotent Lie algebra on d generators 21