Nereis. Revista Iberoamericana Interdisciplinar de Métodos, Modelización y Simulación	14	175-182	Universidad Católica de Valencia San Vicente Mártir	Valencia (España)	ISSN 1888-8550

Efficient numerical evaluation of weak restricted compositions

Evaluación numérica eficiente de las composiciones restringidas débiles

Fecha de recepción y aceptación: 27 de enero de 2022 y 25 de julio de 2022

DOI: 10.46583/nereis_2022.14.1018

Juan Luis González-Santander

Universidad de Oviedo
Correspondencia: gonzalezmarjuan@uniovi.es

Universidad
Católica de Valencia
San Vicente Mártir

ABSTRACT

We propose an algorithm to calculate the number of weak compositions, wherein each part is restricted to a different range of integers. This algorithm performs different orders of approximation up to the exact solution by using the Inclusion-Exclusion Principle. The great advantage of it with respect to the classical generating function technique is that the calculation is exponentially faster as the size of the numbers involved increases.

KEYWORDS: *restricted compositions, numerical efficiency, combinations with repetition, Inclusion-Exclusion Principle.*

RESUMEN

Proponemos un algoritmo para calcular el número de composiciones débiles, en el que cada parte está restringida a un rango diferente de números enteros. Este algoritmo realiza diferentes órdenes de aproximación hasta la solución exacta utilizando el Principio de Inclusión-Exclusión. La gran ventaja que tiene con respecto a la técnica clásica de la función generadora es que el cálculo es exponencialmente más rápido a medida que aumenta el tamaño de los números involucrados.

PALABRAS CLAVE: *composiciones restringidas, eficiencia numérica, combinaciones con repetición, principio de inclusión-exclusión.*

INTRODUCTION

If we want to count the different ways of gathering n elements from m different types in such a way we have n_k elements of type $k = 1,...,m$, i.e.

$$n = n_1 + n_2 + \ldots + n_m, \tag{1}$$

we use the formula of combinations with repetition [5, Eqn. 5.2]

$$M = \binom{n+m-1}{m-1}. \tag{2}$$

However, (2) is based on the fact that there are available as many elements of any type k as we need. Therefore, a natural generalization of this problem is to consider that each n_k is bounded by lower and upper limits. An approach to solve this generalization is to count the number of weak compositions with some restrictions. According to (1), the number of weak compositions of n in m parts is precisely the binomial coefficient given in (2). If $A = \{a_1, a_2, \ldots, a_l\}$ is a finite subset of positive integers, an A-restricted composition is an ordered collection of one or more elements in A whore sum is n. The number M_A of A-restricted compositions of n with m parts is given by [6]:

$$M_A(n,m) = \left[x^n\right] \left(\sum_{a\in A} x^a \right)^m, \tag{3}$$

where the square brackets indicate the extraction of the coefficient of x^n in the polynomial that follows it. The coefficients given in (3) are called *extended binomial coefficients* [3]. Recently, there is a new interest in compositions research [7]. Applications of this research can be found in [4,8].

According to the generalization mentioned before, define $n_{\min,k}$ and $n_{\max,k}$ as the minimum and maximum number of items we can select of type k. In this case, the number of different compositions M is given by,

$$M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right) = \left[x^n\right] \prod_{k=1}^{m} \left(\sum_{j=n_{\min,k}}^{n_{\max,k}} x^j \right), \tag{4}$$

where $\vec{n}_{\min} = \left(n_{\min,1},\ldots,n_{\min,m}\right)$ and $\vec{n}_{\max} = \left(n_{\max,1},\ldots,n_{\max,m}\right)$. Notice that if we allow $\forall k$ that $n_{\min,k}$, $n_{\max,k} \in \mathbb{Z}$, then $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$ will provide the number of integral solutions of (1) with $n_{\min,k} \leq n_k \leq n_{\max,k}$. The generating function approach given in (4) has the drawback that its numerical evaluation is complex and time-consuming since requires symbolic computation facilities [8]. However, to compute $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$, we can consider the approach given in [1, Sect. 6.2], which uses the combination of repetition formula (2) as well as the Inclusion-Exclusion Principle [2, p.177]. In this paper, we have modified the latter approach in order to obtain an efficient algorithm to compute $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$.

This paper is organized as follows. The first Section is devoted to the different orders of approximation to $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$ until we arrive to an order *h* for which the approximation is exact. For this purpose, we transform previously $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$ in order to reduce the order *h*, making the algorithm more efficient. The second Section compares the timing performance of the generation formula (4) with respect to the algorithm proposed in the first Section. In the last Section, we collect our conclusions.

ANALYTICAL SOLUTION

We want to know how many different subsets of *n* elements can be formed choosing elements from *m* different types, wherein the minimum $n_{\min,k}$ and maximum $n_{\max,k}$ number of selectable items of each type $k = 1,\dots,m$ is fixed. First, consider the case $\vec{n}_{\min} = \vec{0}$ denoting for simplicity $M\left(n,\vec{n}_{\max}\right) = M\left(n,\vec{0},\vec{n}_{\max}\right)$. Also, define the sum of the coordinates of a vector as

$$\left\langle \vec{v} \right\rangle = \sum_{k=1}^{m} v_k .$$

The number of different subsets $M\left(n,\vec{n}_{\max}\right)$ can be counted choosing *n* elements, or discarding $\left\langle \vec{n}_{\max} \right\rangle - n$ elements.

$$\begin{aligned} M\left(n,\vec{n}_{\max}\right) &= M\left(\left\langle \vec{n}_{\max} \right\rangle - n,\vec{n}_{\max}\right) \\ &= M\left(\min\left(n,\left\langle \vec{n}_{\max} \right\rangle - n\right)\right) \end{aligned} \tag{5}$$

In the general case $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$, consider that we must have at least $n_{\min,k}$ items of type *k*, thus we calculate $M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right)$ extracting in advance the minimum number of items of each type, i.e. $\left\langle \vec{n}_{\min} \right\rangle$, and then taking compositions with the remaining elements. Therefore, the maximum number of selectable items of each type is now given by $\vec{n}_{\max} - \vec{n}_{\min}$. According to the latter and taking into account (5), we have

$$\begin{aligned} M\left(n,\vec{n}_{\min},\vec{n}_{\max}\right) &= M\left(n - \left\langle \vec{n}_{\min} \right\rangle, \vec{n}_{\max} - \vec{n}_{\min}\right) \\ &= M\left(n^{*},\vec{n}\right). \end{aligned}$$

where we have defined

$$n^* = \min\left(n - \langle \vec{n}_{\min} \rangle, \langle \vec{n}_{\max} \rangle - \vec{n}\right),$$
$$\vec{n} = \vec{n}_{\max} - \vec{n}_{\min}.$$

Since $M\left(n, \vec{n}_{\min}, \vec{n}_{\max}\right)$ is reduced to the calculation of $M\left(n^*, \vec{n}\right)$, next we calculate the latter case.

Zeroth order approximation

The lowest approximation to the problem is to consider the combinatorial formula (2), i.e.

$$M^{(0)}\left(n^*, \vec{n}\right) = \binom{n^* + m - 1}{m-1},$$

Where $m = \dim\left(\vec{n}\right)$ Notice that this approximation is exact when we have available enough items of each type. Thereby, if $n^* \leq \min\left(\vec{n}\right)$, then

$$M\left(n^*, \vec{n}\right) = M^{(0)}\left(n^*, \vec{n}\right). \tag{6}$$

First order approximation

If $\exists k \in \{1,\ldots,m\}$, such that $n_k < n^*$, then the above approximation (6) is not exact. In this case, we have to subtract all the impossible compositions from (6), i.e. all the compositions containing at least $n_k + 1$ elements of type k. Notice that we have $n^* - n_k - 1$ undetermined elements in this kind of impossible compositions, thus the number of them is

$$\binom{n^* - n_k + m - 2}{m-1}.$$

Therefore, defining the function

$$f_1\left(n^*, n_k\right) = \begin{cases} \binom{n^* - n_k + m - 2}{m-1}, & n^* > n_k \\ 0, & n^* \leq n_k \end{cases}$$

the first order approximation reads as

$$M^{(1)}\left(n^{*},\vec{n}\right)=\binom{n^{*}+m-1}{m-1}-\sum_{k=1}^{m}f_{1}\left(n,n_{k}\right). \tag{7}$$

Second order approximation

However, if $\exists j,\, k\in\{1,\dots,m\}, j\neq k$, such that $n_{j}+n_{k}+1<n^{*}$, then we would have discounted more than once the same composition in the first order formula (7). We can avoid the latter defining the function

$$f_{2}\left(n^{*},n_{j},n_{k}\right)=\begin{cases}\binom{n^{*}-n_{j}-n_{k}+m-3}{m-1}, & n^{*}>n_{j}+n_{k}+1\\ 0, & n^{*}\leq n_{j}+n_{k}+1\end{cases}$$

Thereby, the second order approximation reads as

$$M^{(2)}\left(n^{*},\vec{n}\right)=\binom{n^{*}+m-1}{m-1}-\sum_{k=1}^{m}f_{1}\left(n,n_{k}\right)+\sum_{j=1}^{m}\sum_{k=j+1}^{m}f_{2}\left(n,n_{j},n_{k}\right). \tag{8}$$

Arbitrary order approximation

We can generalize (8) for an arbitrary order approximation considering k-tuples ($1\leq k\leq m$) of vector $\vec{n}$, i.e. $\vec{n}^{(k)}=\left(n_{i_1},n_{i_2},\dots,n_{i_k}\right)$. Thereby, in general, we have the function

$$f_{k}\left(n^{*},\vec{n}^{(k)}\right)=\begin{cases}\binom{n^{*}-t_{k}+m-k+1}{m-1}, & n^{*}>t_{k}+k-1\\ 0, & n^{*}\leq t_{k}+k-1\end{cases} \tag{9}$$

where we have set

$$t_k = \sum_{l=1}^{k} n_{i_l}$$

The number of different compositions of order h is then

$$M^{(h)}\left(n^*, \vec{n}\right) = \sum_{k=0}^{h} (-1)^k \, S_k\left(n^*, \vec{n}^{(k)}\right), \tag{10}$$

where we have defined

$$S_k\left(n^*, \vec{n}^{(k)}\right) = \sum_{i_1=1}^{m} \sum_{i_2=i_1+1}^{m} \cdots \sum_{i_k=i_1+\cdots+i_{k-1}+1}^{m} f_k\left(n^*, \vec{n}^{(k)}\right). \tag{11}$$

Sorting the coordinates of vector $\vec{n}$ to obtain a vector $\vec{n}\,' = (n'_1, n'_2, \ldots, n'_m)$ with $n'_1 \leq n'_2 \leq \ldots \leq n'_m$, the minimal order h for which $M\,(n^*, \vec{n}\,) = M^{(h)}\,(n^*, \vec{n}\,)$ can be calculated as follows:

$$h = \max_s \left\{ s \in \left\{0, 1, \quad , m-1\right\} : n^* > \sum_{k=1}^{s+1} n'_k + s - 1 \right\}, \tag{12}$$

which can be easily found testing the different values $s = 0, 1, \ldots, m - 1$. According to (9), the above procedure to calculate h assures that $f_h\left(n^*, \vec{n}^{(h)}\right) \neq 0$, but $f_{h+1}\left(n^*, \vec{n}^{(h+1)}\right) = 0$.

NUMERICAL ANALYSIS

The computation of $M\left(n, \vec{n}_{\min}, \vec{n}_{\max}\right)$ using the generating function approach (4) and the Inclusion-Exclusion Principle (10) has been coded in MATHEMATICA, as well as using brute force computation. These codes are available at https://bit.ly/30sQknY.

The great advantage of the code based on the Inclusion-Exclusion Principle is that it is much faster than the one based on the generating function. This is so because the Inclusion-Exclusion Principle code does not involve any symbolic processing. Fig. 1 shows the time ratio performance between both codes as a function of the size of the numbers involved in the calculations. This size is a scale factor, i.e. size 8 means that on average the numbers are double than size 4. Size 1 means that we consider 1 digit numbers. It is apparent that the time ratio performance increases exponentially with size, being the Inclusion-Exclusion Principle code much more efficient. To compute the calculations of Fig. 1, we have set $m = 8$, but similar patterns can be found for other values of m.

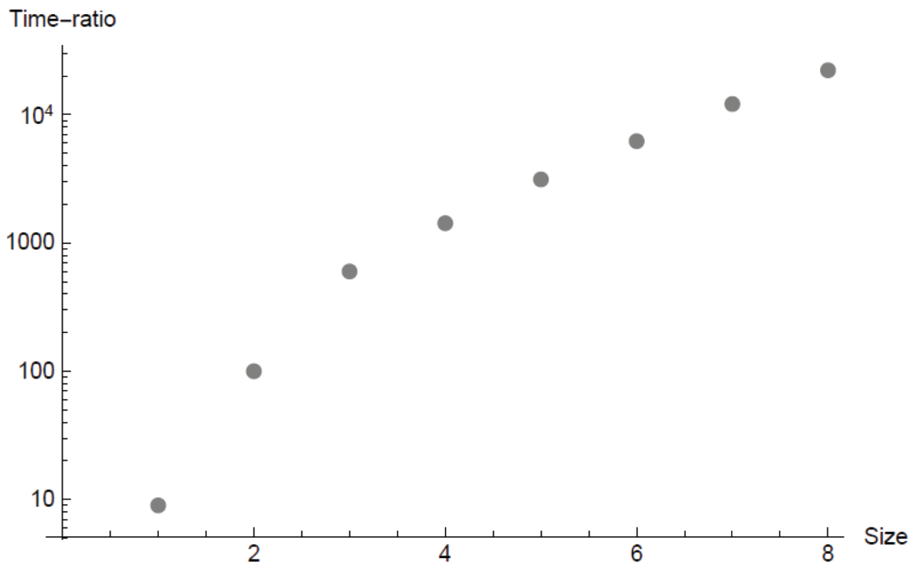

Figure 1. Time ratio between the Inclusion-Exclusion Principle coder and the generating code.

CONCLUSIONS

We have derived an efficient algorithm to calculate $M\left(n, \vec{n}_{\min}, \vec{n}_{\max}\right)$, i.e. the number of weak compositions of n with m parts, wherein each part $n_k \in Z$ is restricted within a defined range of integers, i.e. $n_{\min,k} \leq n_k \leq n_{\max,k}$. The great advantage of this algorithm is that computes $M\left(n, \vec{n}_{\min}, \vec{n}_{\max}\right)$ much faster than the classical formula using generating functions. Moreover, the time ratio between both calculations increases exponentially with the size of n, $\vec{n}_{\min}$, and $\vec{n}_{\max}$.

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