EXTRACTA MATHEMATICAE Vol. **18**, Núm. 1, 91 – 106 (2003)

On Vector-Valued Hörmander-Beurling Spaces †

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(Presented by Oscar Blasco)

AMS *Subject Class.* (2000): 46F05, 46E40 *Received March* 8, 2002

1. INTRODUCTION

It is well-known the important role that the Hörmander spaces $\mathcal{B}_{p,k}$ play in the theory of linear partial differential operators (see [10], [11]). In [8] and [18] an interpolation theory for these spaces can be found, in [26] different (local) Hörmander spaces are represented by sequence spaces, and in [16] new results about $\mathcal{B}_{p,k}$ spaces are obtained and these spaces are extended to the vector-valued setting. On the other hand, in [3] Björck, by using Beurling's ultradistributions, studies questions of existence, approximation and interior regularity of solutions of linear partial differential equations with constant coefficients in more general spaces than those of Hörmander (we call these spaces "Hörmander-Beurling spaces"). In this paper we extend these Hörmander-Beurling spaces to the vector-valued setting studying some of their properties. In Section 2 we collect some basic facts about vector-valued ultradistributions. In Section 3 we calculate the dual of the space $\mathcal{B}_{p,k}(E)$ without supposing that E' posseses the Radon-Nikodým property (by using finitely additive E'-measures of bounded p'-variation). We also generalize a Favini's result about interpolation of Hörmander spaces, and we use the Goulaouic's procedure to interpolate some classes of countable projective limits of vector-valued Hörmander-Beurling spaces. In Section 4 we prove that the spaces $\mathcal{B}_{p,k}(E)$ and $\mathcal{B}^0_{\infty,k}(E)$ have the property of approximation by cutting.

NOTATION. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $\mathcal{L}_b(E,F)$ (resp. $\mathcal{L}_c(E,F)$) is the locally convex

†The results of this research paper belong to the doctoral thesis currently being carry out by the author under the supervision of Prof. Joaquín Motos.

space of all continuous linear operators equipped with the bounded (resp. precompact) convergence topology. If $E = F$ we write $\mathcal{L}_b(E)$ (resp. $\mathcal{L}_c(E)$). $E\widehat{\otimes}_\epsilon F$ (resp. $E\widehat{\otimes}_\pi F$) denotes the completion of the injective (resp. projective) tensor product of E and F. We write $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous. We replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F. The topological dual of E is denoted by E' and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $\mathcal{C}^m$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{D}'$ and $\mathcal{S}'$ have the usual meaning (see [20]). In the E-valued case we write $\mathcal{C}^m(E)$, $\mathcal{D}(E)$, $\mathcal{S}(E)$, $\mathcal{D}'(E)$ and $\mathcal{S}'(E)$ (see [21]). Let $1 \leq p \leq \infty$, $k : \mathbb{R}^n \longrightarrow]0, \infty[$ a Lebesgue measurable function, and E a Banach space. Then $L_p(E)$ is the set of all measurable Bochner functions $f : \mathbb{R}^n \longrightarrow E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p \, dx\right)^{1/p}$ is finite (if $p = \infty$ we assume $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|f\|_E < \infty$). $L_{p,k}(E)$ denotes the set of all measurable Bochner functions $f : \mathbb{R}^n \longrightarrow E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for $f \in L_{p,k}(E)$, $L_{p,k}(E)$ becomes a Banach space isometrically isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transformation of f, $\widehat{f}$ or $\mathcal{F}f$, is defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\xi x} dx$. If f is a function on $\mathbb{R}^n$ then $\widetilde{f}(x) = f(-x)$, $(\tau_h f)(x) = f(x - h)$, $x, h \in \mathbb{R}^n$.

2. Spaces of vector-valued ultradistributions

In this section we collect some basic facts about vector-valued ultradistributions. The results are "elementary" in the sense that the usual "scalar proofs" carry over to the vector-valued setting by using obvious modifications only. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [3], [4], [5], [14], and [15]. Our notations are based on [3] (cf. also [19, pp. 14-19]).

Let $\mathcal{M}$ be the set of all continuous real-valued functions $\omega(x)$ on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:

(i) $\sigma(0) = 0$,

(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2} dt < \infty$ (Beurling's condition),

(iii) there exists a real number a and a positive number b such that $\sigma(t) \geq a + b \log(1 + t)$ for $t \geq 0$.

The main assumption is (ii), which is essentially the Denjoy-Carleman non-quasi-analyticity condition (see [3]). The two most prominent examples of functions $\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1 + |x|)^d$ with $d > 0$, and $\omega(x) =$

$|x|^\beta$ with $0 < \beta < 1$. If $\omega \in \mathcal{M}$ and E is a Banach space, we denote by $\mathcal{D}_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that

$$\|f\|_\lambda^{(\omega)} = \int_{\mathbb{R}^n} \left\|\widehat{f}(x)\right\|_E e^{\lambda\omega(x)} dx < \infty$$

for all $\lambda > 0$. For each compact subset K of $\mathbb{R}^n$, $\mathcal{D}_\omega(K,E) = \{f \in \mathcal{D}_\omega(E) : \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of norms $\left\{\|\cdot\|_\lambda^{(\omega)} : \lambda > 0\right\}$, is a Fréchet space and $\mathcal{D}_\omega(E)$ with the inductive limit topology, $\mathcal{D}_\omega(E) = \varinjlim_{\overrightarrow{K}} \mathcal{D}_\omega(K,E)$, becomes a strict (LF)-space. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\widehat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with

$$\overrightarrow{p}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\|_E < \infty$$

and

$$\overrightarrow{\pi}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha (\mathcal{F}f)(x)\|_E < \infty$$

for all multi-indices α and all positive numbers λ. $\mathcal{S}_\omega(E)$ with the topology induced by the family of seminorms $\{\overrightarrow{p}_{\alpha,\lambda}, \overrightarrow{\pi}_{\alpha,\lambda}\}$ is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}_\omega(E)$. If $E = \mathbb{C}$ then $\mathcal{D}_\omega(E)$ and $\mathcal{S}_\omega(E)$ coincide with the spaces $\mathcal{D}_\omega$ and $\mathcal{S}_\omega$. In this case we write, $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$ instead of $\overrightarrow{p}_{\alpha,\lambda}$ and $\overrightarrow{\pi}_{\alpha,\lambda}$. Let us recall that, by the Beurling's condition, the space $\mathcal{D}_\omega$ is non-trivial and the usual procedure of the resolution of unity can be established with $\mathcal{D}_\omega$-functions (cf.[2], [3, Th. 1.3.7]). Furthermore, $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{D}$ (cf. [3, Th. 1.3.18]) and $\mathcal{D}_\omega$ is nuclear (cf. [26, Cor. 7.5]). On the other hand, $\mathcal{D}_\omega = \mathcal{D} \cap \mathcal{S}_\omega$, $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{S}_\omega \overset{d}{\hookrightarrow} \mathcal{S}$ (cf. [3, Prop. 1.8.6, Th. 1.8.7]) and $\mathcal{S}_\omega$ is a Fréchet-Schwartz space with the approximation property (by [3, Th. 1.8.7], [19, Prop. 1.2.2/2] and [21, Prop. 3, p. 9]). Using the above results and [14, Th. 1.12] we can identify $\mathcal{D}_\omega(E)$ with $\mathcal{D}_\omega \widehat{\otimes}_\epsilon E$ and $\mathcal{S}_\omega(E)$ with $\mathcal{S}_\omega \widehat{\otimes}_\epsilon E$. A continuous linear operator from $\mathcal{D}_\omega$ into E is said to be a (Beurling) ultradistribution with values in E. We write $\mathcal{D}'_\omega(E)$ for the space of all E-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $\mathcal{D}'_\omega(E) = \mathcal{L}_b(\mathcal{D}_\omega, E)$ is isomorphic to $\mathcal{D}'_\omega \widehat{\otimes}_\epsilon E$. A continuous linear operator from $\mathcal{S}_\omega$ into E is said to be an E-valued tempered ultradistribution. $\mathcal{S}'_\omega(E)$ is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology. Also $\mathcal{S}'_\omega(E) = \mathcal{L}_b(\mathcal{S}_\omega, E)$ is isomorphic to $\mathcal{S}_\omega \widehat{\otimes}_\epsilon E$ ($\mathcal{S}'_\omega$ has the approximation

property) and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}'_\omega(E)$. Next we recall the definition of $\mathcal{K}_\omega$ given in [3, p. 383]. If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a constant $\lambda > 0$ such that

$$k(x+y) \leq e^{\lambda\omega(x)} k(y)$$

for all x and y in $\mathbb{R}^n$. If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1 k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [3,Th. 2.1.3]). If $u \in L_1^{loc}$ and $\int_{\mathbb{R}^n} \varphi(x)\, u(x)\, dx = 0$ for all $\varphi \in \mathcal{D}_\omega$, then $u = 0$ a.e (see [3]). This result, the Hahn-Banach theorem and [6, Cor. II.27] prove that if $k \in \mathcal{K}_\omega$ and $p \in [1, \infty]$ we can identify $f \in L_{p,k}(E)$ with the E-valued tempered ultradistribution

$$\varphi \to \langle \varphi, f \rangle = \int_{\mathbb{R}^n} \varphi(x)\, f(x)\, dx,$$

$\varphi \in \mathcal{S}_\omega$. Summarizing, we have the embeddings

$$\begin{array}{ccccccc}
\mathcal{D}_\omega(E) & \overset{d}{\hookrightarrow} & \mathcal{S}_\omega(E) & \overset{d}{\hookrightarrow} & \mathcal{S}'_\omega(E) & \overset{d}{\hookrightarrow} & \mathcal{D}'_\omega(E) \\
\downarrow d & & \downarrow d & & \uparrow d & & \uparrow d \\
\mathcal{D}(E) & \overset{d}{\hookrightarrow} & \mathcal{S}(E) & \overset{d}{\hookrightarrow} & \mathcal{S}'(E) & \overset{d}{\hookrightarrow} & \mathcal{D}'(E)
\end{array}$$

(commutative diagram) and

$$\mathcal{S}_\omega(E) \overset{d}{\hookrightarrow} L_{p,k}(E) \overset{d}{\hookrightarrow} \mathcal{S}'_\omega(E),$$

$p < \infty$.

Let G be a locally convex space such that $G \hookrightarrow \mathcal{D}'_\omega(E)$, then G is said to be an "E-valued space of ultradistributions". For $\varphi \in \mathcal{S}_\omega$, $T \in \mathcal{S}'_\omega(E)$ and $\psi \in \mathcal{S}_\omega$, we define $\langle \psi, \varphi T \rangle = \langle \psi\varphi, T \rangle$. The "point-wise multiplication" $\mathcal{S}_\omega \times \mathcal{S}'_\omega(E) \to \mathcal{S}'_\omega(E) : (\varphi, T) \mapsto \varphi T$ is a well-defined separately continuous bilinear map (and hypocontinuous when $\mathcal{S}_\omega$ is nuclear). If $\varphi \in \mathcal{S}_\omega$ and $T \in \mathcal{S}'_\omega(E)$, we define $\varphi * T(x) = \langle \tau_x \widetilde{\varphi}, T \rangle$, $x \in \mathbb{R}^n$. The function $\varphi * T : \mathbb{R}^n \to E$ is called the convolution of φ and T. $\varphi * T$ is continuous and there exist positive constants C and Λ such that $\|\varphi * T(x)\| \leq C e^{\Lambda\omega(x)}$ for all $x \in$

$\mathbb{R}^n$. Thus, we can identify $\varphi * T$ with the E-valued tempered ultradistribution $\psi \to \langle \psi, \varphi * T \rangle = \int_{\mathbb{R}^n} \psi(x) (\varphi * T)(x)\, dx$, $\psi \in \mathcal{S}_\omega$. The bilinear map $\mathcal{S}_\omega \times \mathcal{S}'_\omega(E) \to \mathcal{S}'_\omega(E) : (\varphi, T) \mapsto \varphi * T$ is separately continuous (and hypocontinuous if $\mathcal{S}_\omega$ is nuclear). One easily checks that

$$\langle \psi, \varphi * T \rangle = \langle \widetilde{\varphi} * \psi, T \rangle, \ (\varphi * T)^\wedge = \widehat{\varphi}\, \widehat{T}, \ \ (\varphi T)^\wedge = (2\pi)^{-n} \left(\widehat{\varphi} * \widehat{T} \right),$$

for all $\varphi, \psi \in \mathcal{S}_\omega$ and $T \in \mathcal{S}'_\omega(E)$.

3. The vector-valued Hörmander-Beurling spaces $\mathcal{B}_{p,k}(E)$

In this section we generalize the spaces $\mathcal{B}_{p,k}$ of [3] to the vector-valued setting. We shall use the notations of the previous section and we shall begin by extending the Definition 2.2.5 of [3] and the Definition 1 of [16].

Definition 3.1. Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leq p \leq \infty$ and E a Banach space. We denote by $\mathcal{B}_{p,k}(E)$ the set of all E-valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\left\langle \varphi, \widehat{T} \right\rangle = \int_{\mathbb{R}^n} \varphi(x) f(x)\, dx$, $\varphi \in \mathcal{S}_\omega$. Obviously $\mathcal{B}_{p,k}(E)$ is a linear subspace of $\mathcal{S}'_\omega(E)$ (if we identify $L_{p,k}(E)$ with a subspace of $\mathcal{S}'_\omega(E)$ then $\mathcal{B}_{p,k}(E) = \mathcal{F}^{-1} L_{p,k}(E)$). $\mathcal{B}_{p,k}(E)$ with the norm

$$\|T\|_{p,k} = \begin{cases} \left((2\pi)^{-n} \int_{\mathbb{R}^n} \left\| k(x)\, \widehat{T}(x) \right\|_E^p dx \right)^{1/p} & \text{if } p < \infty \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} \left\| k(x)\, \widehat{T}(x) \right\|_E & \text{if } p = \infty \end{cases}$$

becomes a Banach space isometrically isomorphic to $L_{p,k}(E)$ and, therefore, to $L_p(E)$. (In the previous formulae we have written $\widehat{T}(x)$ instead of $f(x)$, we shall frequently commit this abuse of notation.)

Remark 3.2. (1) Of course, $\mathcal{B}_{p,k}(\mathbb{C})$ is the Hörmander-Beurling space $\mathcal{B}_{p,k}$ considered by Björck in [3], and our definition coincides with Definition 1 of [16] when $\omega(x) = \log(1 + |x|)$.

(2) The study of the spaces $\mathcal{B}_{p,k}(E)$ is not reduced to the study of the spaces $\mathcal{B}_{p,k} \widehat{\otimes}_\epsilon E$ since, as is well-known, $L_p(E)$ and $L_p \widehat{\otimes}_\epsilon E$ are not isomorphic in general. For sake of completeness we recall some examples: The space $L_1(\ell_p)$ is not isomorphic to $L_1 \widehat{\otimes}_\epsilon \ell_p$ if $2 \leq p < \infty$ (see [6, p. 117 & Cor.

p. 258]). If $\dim E = \infty$ and $1 < p < \infty$ then $L_p(E)$ is not isomorphic to $L_p \widehat{\otimes}_\epsilon E$ (see [6, p. 253]). Finally, we show that $L_\infty(\ell_2)$ is not isomorphic to $L_\infty \widehat{\otimes}_\epsilon \ell_2$: Since the space $K(L_1, \ell_2)$ of compact linear operators from L_1 into ℓ_2 is not complemented in $\mathcal{L}(L_1, \ell_2)$ (cf [23]), it follows from Proposition 1 of [13] that $K(L_1, \ell_2)$ is not isomorphic to a dual space. This concludes the proof since $L_\infty \widehat{\otimes}_\epsilon \ell_2 \simeq K(L_1, \ell_2)$ and $L_\infty(\ell_2) \simeq (L_1(\ell_2))'$.

The next result is the vector-valued counterpart of the Theorem 2.2.3 of [3].

PROPOSITION 3.3. *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leq p \leq \infty$ and E a Banach space. Then*

$$\mathcal{S}_\omega(E) \hookrightarrow \mathcal{B}_{p,k}(E) \overset{d}{\hookrightarrow} \mathcal{S}_\omega'(E)$$

and $\mathcal{D}_\omega \otimes E$ is dense in $\mathcal{B}_{p,k}(E)$ if $p < \infty$.

Proof. It is enough to take into account the two diagrams of the Section 2, that the Fourier transformation is an automorphism of $\mathcal{S}_\omega(E)$ and of $\mathcal{S}'_\omega(E)$ and an isomorphism from $\mathcal{B}_{p,k}(E)$ onto $L_{p,k}(E)$, and that $\mathcal{D}_\omega \otimes E$ is dense in $\mathcal{D}_\omega(E)$. ∎

In the next proposition we shall determine the dual space of $\mathcal{B}_{p,k}(E)$ when $p < \infty$ by extending Theorem 10.1.14 of [11] and Theorem 6 of [16]. In the second part of the proposition we shall use some well-known results about finitely additive vector measures and finitely additive vector measure spaces (see [1], [6], [7], [17] and [25]): Let (Ω, Σ, μ) be a σ-finite measure space and $\Sigma_0 \subset \Sigma$ the ring of sets of finite μ-measure. A Σ_0-partition π of Ω is any finite disjoint collection $\{A_n\} \subset \Sigma_0$ (the Σ_0-partitions of Ω are partially ordered by defining $\pi_1 \leq \pi_2$ whenever each element of π_1 is a union of elements of π_2). If E a Banach space and $p \in [1, \infty]$, we denote by $\mathcal{V}_p(\mu, E)$ the space

$$\Big\{ m : \Sigma_0 \to E,\ m \text{ finitely additive, } m \ \mu\text{-continuous,} \\ m(A) = 0 \text{ if } \mu(A) = 0 \text{ and } |m|_p < \infty \Big\}$$

(here $|m|_p = \sup\{(\sum_\pi \|m(A)\|_E^p / \mu(A)^{p-1})^{1/p} : \pi = \Sigma_0\text{-partition of } \Omega\}$ if $p < \infty$ and $|m|_\infty = \sup\{\|m(A)\|_E / \mu(A) : A \in \Sigma_0\}$) of all finitely additive E-valued measures of bounded p-variation. With the norm $\|m\|_{\mathcal{V}_p(\mu,E)} = |m|_p$, $\mathcal{V}_p(\mu, E)$ is a Banach space. The map $L_p(\mu, E) \to \mathcal{V}_p(\mu, E) : f \mapsto m_f$, where $m_f(A) = \int_A f d\mu$ when $A \in \Sigma_0$, is an isometric embedding which becomes an isometric isomorphism when E has the Radon-Nikodým property. On the

other hand, for $1 \leq p < \infty$, the map $\mathcal{V}_{p'}(\mu, E') \to (L_p(\mu, E))' : m \mapsto \int_\Omega \cdot dm$ always is an isometric isomorphism (here the integral is the integral of Bartle: $\int_\Omega f dm = \lim_\pi \Sigma_\pi \langle \frac{1}{\mu(A)} \int_A f d\mu, m(A) \rangle$, $|\int_\Omega f dm| \leq \|f\|_p |m|_{p'}$ and $\int_\Omega f dm_g = \int_\Omega \langle f, g \rangle d\mu$ if $g \in L_{p'}(\mu, E')$).

Now we assume that E is a Banach space, $p \in [1, \infty]$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$. If $1 \leq p < \infty$, the map $L_{p,k}(E) \longrightarrow \mathcal{V}_p(k^p dx, E) : f \mapsto m_f(A) = \int_A f k^p dx$, $A \in \Sigma_0$, is an isometric embedding. Furthermore, the map $I : \mathcal{V}_p(k^p dx, E) \longrightarrow \mathcal{S}'_\omega(E)$ given by $\langle \varphi, I(m) \rangle = \int_{\mathbb{R}^n} \varphi k^{-p} dm$, $\varphi \in \mathcal{S}_\omega$, also is an embedding. In fact, for each $\varphi \in \mathcal{S}_\omega$ the function $\varphi k^{-p} \in L_{p'}(k^p dx)$, so the integral $\int_{\mathbb{R}^n} \varphi k^{-p} dm$ is well-defined and

$$\|\langle \varphi, I(m) \rangle\|_E \leq |m|_p \left\| \varphi k^{-p} \right\|_{L_{p'}(k^p dx)} \leq c\, |m|_p\, p_{0,\Lambda}(\varphi)$$

where Λ is a certain constant > 0 independent of φ. This yields that the map I is well-defined and is linear and continuous. Let us see that it is injective: If $I(m) = 0$, that is, if $\int_{\mathbb{R}^n} \varphi k^{-p} dm = 0$ for all $\varphi \in \mathcal{S}_\omega$, then $\int_{\mathbb{R}^n} \varphi k^{-p} d(e' \circ m) = 0$ for each $\varphi \in \mathcal{S}_\omega$ and $e' \in E'$. Since $e' \circ m \in \mathcal{V}_p(k^p\, dx)$, there is a function $f_0 \in L_p(k^p\, dx)$ such that $e' \circ m(A) = \int_A f_0\, k^p\, dx$ for each $A \in \Sigma_0$ and such that $\int_{\mathbb{R}^n} \varphi k^{-p} d(e' \circ m) = \int_{\mathbb{R}^n} \varphi k^{-p} f_0\, k^p dx = \int_{\mathbb{R}^n} \varphi\, f_0\, dx$ for all $\varphi \in \mathcal{S}_\omega$. Thus $f_0 = 0$ a.e. and, consequently, $e' \circ m = 0$. Since e' is arbitrary we get $m = 0$ and so I is injective. If $p = \infty$ the map $L_{\infty,k}(E)\big(\overset{1}{\simeq} L_\infty\left(\frac{1}{k} dx, E\right)\big) \longrightarrow \mathcal{V}_\infty\left(\frac{1}{k} dx, E\right) : f \mapsto m_f(A) = \int_A f\, dx$, $A \in \Sigma_0$, is an isometric embedding and the map $I : \mathcal{V}_\infty\left(\frac{1}{k} dx, E\right) \longrightarrow \mathcal{S}'_\omega(E)$ given by $\langle \varphi, I(m) \rangle = \int_{\mathbb{R}^n} \varphi\, dm$, $\varphi \in \mathcal{S}_\omega$, also is an embedding. The above analysis allows to give the following definition which generalizes the definition of $\mathcal{B}_{p,k}(E)$: $\mathcal{BV}_{p,k}(E)$ is the set of all E-valued tempered ultradistributions T for which there exists $m \in \mathcal{V}_p(k^p\, dx, E)$ ($m \in \mathcal{V}_\infty\left(\frac{1}{k} dx, E\right)$ if $p = \infty$) such that $\langle \varphi, \widehat{T} \rangle = \int_{\mathbb{R}^n} \varphi k^{-p} dm$ ($\langle \varphi, \widehat{T} \rangle = \int_{\mathbb{R}^n} \varphi\, dm$ if $p = \infty$), $\varphi \in \mathcal{S}_\omega$. Putting

$$\|T\|_{\mathcal{BV}_{p,k}(E)} = \begin{cases} \left\| (2\pi)^{-n/p}\, \widehat{T} \right\|_{\mathcal{V}_p(k^p dx, E)} & \text{if } p < \infty, \\ \left\| \widehat{T} \right\|_{\mathcal{V}_\infty(\frac{1}{k} dx, E)} & \text{if } p = \infty. \end{cases}$$

$\mathcal{BV}_{p,k}(E)$ becomes a Banach space isometric to the space $\mathcal{V}_p(k^p dx, E)$ (to the space $\mathcal{V}_\infty\left(\frac{1}{k} dx, E\right)$ when $p = \infty$). By using this notation we have

PROPOSITION 3.4. *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leq p < \infty$ and E a Banach space.*

(1) *If the dual E' has the Radon-Nikodým property then the map*

$$Z : \mathcal{B}_{p',\frac{1}{k}}(E') \longrightarrow (\mathcal{B}_{p,k}(E))'$$

defined by

$$\langle T, Z(S)\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \langle \widehat{T}(x), \widehat{S}(x)\rangle dx,$$

$S \in \mathcal{B}_{p',\frac{1}{k}}(E')$, $T \in \mathcal{B}_{p,k}(E)$, is an isometric isomorphism.

(2) *The map $Z : \mathcal{BV}_{p',\frac{1}{k}}(E') \longrightarrow (\mathcal{B}_{p,k}(E))'$ defined by*

$$\langle T, Z(S)\rangle = \begin{cases} (2\pi)^{-n} \int_{\mathbb{R}^n} k^{p'}(x)\ \widehat{T}(x) d\widehat{S}(x) & \text{if } 1 < p < \infty, \\ (2\pi)^{-n} \int_{\mathbb{R}^n}\ \widehat{T}(x) d\widehat{S}(x) & \text{if } p = 1, \end{cases}$$

$S \in \mathcal{BV}_{p',\frac{1}{k}}(E')$, $T \in \mathcal{B}_{p,k}(E)$, is an isometric isomorphism.

Proof. (1) Let us consider the following diagram

$$\begin{array}{ccccc} \mathcal{B}_{p',\frac{1}{k}}(E') & & \xrightarrow{\qquad Z \qquad} & & (\mathcal{B}_{p,k}(E))' \\ \Big\downarrow {\scriptstyle c_1\mathcal{F}} & & & & \Big\uparrow {\scriptstyle {}^t(c_2\mathcal{F})} \\ L_{p',\frac{1}{k}}(E') & \xrightarrow{A} & L_{p'}(k^p dx, E') & \xrightarrow{B} & (L_{p,k}(E))' \end{array}$$

where $\mathcal{F}$ denotes the Fourier transformation both in $\mathcal{S}'_\omega(E')$ and in $\mathcal{S}'_\omega(E)$, $c_1 = (2\pi)^{-n/p'}$, $c_2 = (2\pi)^{-n/p}$, A is the multiplication operator $A(h) = \frac{h}{k^p}$, B is the operator defined by

$$\langle f, B(g)\rangle = \int_{\mathbb{R}^n} \langle f(x), g(x)\rangle k^p(x)\, dx$$

and Z is the corresponding composed operator. Since $c_1\mathcal{F}$, ${}^t(c_2\mathcal{F})$, A and B are isometric isomorphisms (although in [6, p. 98] it is shown that if (Ω, Σ, μ) is a finite measure space and $1 \leq p < \infty$ then $(L_p(\mu, E))' \overset{1}{\simeq} L_{p'}(\mu, E')$

if and only if E' has the Radon-Nikodým property, it is well-known that this result holds if the measure μ is σ-finite because in this case writing $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, with the sets Ω_n pairwise disjoints, $\mu(\Omega_n) < \infty$ and $\mu|_{\Sigma \bigcap \Omega_n} = \mu_n$, we get the natural isometric isomorphisms $(L_p(\mu, E))' \simeq (\ell_p(L_p(\mu_n, E)))' \simeq \ell_{p'}\left((L_p(\mu_n, E))'\right) \simeq \ell_{p'}\left(L_{p'}(\mu_n, E')\right) \simeq L_{p'}(\mu, E')$) also Z is an isometric isomorphism given by

$$\begin{aligned}\left\langle T, Z(S)\right\rangle &= \left\langle T, {}^t(c_2\mathcal{F})BA(c_1\mathcal{F})(S)\right\rangle = c_1c_2\left\langle T, {}^t\mathcal{F}\left(BA\widehat{S}\right)\right\rangle \\ &= (2\pi)^{-n}\left\langle \widehat{T}, BA\widehat{S}\right\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n}\left\langle \widehat{T}(x), \widehat{S}(x)\right\rangle dx\end{aligned}$$

for $T \in \mathcal{B}_{p,k}(E)$ and $S \in \mathcal{B}_{p',\frac{1}{k}}(E')$.

(2) If $1 < p < \infty$ the appropriate diagram is

$$\begin{array}{ccccc}
\mathcal{BV}_{p',\frac{1}{k}}(E') & & \xrightarrow{\qquad Z \qquad} & & \left(\mathcal{B}_{p,k}(E)\right)' \\
\Big\downarrow{\scriptstyle c_1\mathcal{F}} & & & & \Big\uparrow{\scriptstyle {}^t(c_2\mathcal{F})} \\
\mathcal{V}_{p'}\left(k^{-p'}dx, E'\right) & \xrightarrow{A} & \left(L_p(k^{-p'}dx, E)\right)' & \xrightarrow{B} & \left(L_{p,k}(E)\right)'
\end{array}$$

where c_1, c_2 and $\mathcal{F}$ are as in part 1, B is the adjoint of the multiplication operator $L_{p,k}(E) \longrightarrow L_p\left(k^{-p'}dx, E\right) : g \mapsto gk^{p'}$, A is the operator defined by $\left\langle f, A(m)\right\rangle = \int_{\mathbb{R}^n} f dm$, and Z is the composed operator. As in the preceding case, Z becomes an isometric isomorphism and it is given by

$$\begin{aligned}\left\langle T, Z(S)\right\rangle &= c_1c_2\left\langle T, {}^t\mathcal{F}\left(BA\widehat{S}\right)\right\rangle = c_1c_2\left\langle \widehat{T}k^{p'}, A\widehat{S}\right\rangle \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} k^{p'}(x)\widehat{T}(x)\, d\widehat{S}(x)\end{aligned}$$

for $T \in \mathcal{B}_{p,k}(E)$ and $S \in \mathcal{BV}_{p',\frac{1}{k}}(E')$. If $p = 1$ we consider the following diagram

$$\begin{array}{ccc} \mathcal{BV}_{\infty,\frac{1}{k}}(E') & \xrightarrow{\quad Z \quad} & \big(\mathcal{B}_{1,k}(E)\big)' \\ \Big\downarrow \mathcal{F} & & \Big\uparrow {}^t(c_2\mathcal{F}) \\ \mathcal{V}_\infty(k\,dx, E') & \xrightarrow{\quad A \quad} & \big(L_{1,k}(E)\big)' \end{array}$$

where c_2 and $\mathcal{F}$ are as in part 1, A is the operator defined by $\langle f,\, A(m)\rangle = \int_{\mathbb{R}^n} f dm$, and Z is the composed operator. Z become an isometric isomorphism and it is given by

$$\big\langle T, Z(S)\big\rangle = c_2\big\langle T,\, {}^t\mathcal{F}\big(A\widehat{S}\big)\big\rangle = (2\pi)^{-n}\big\langle \widehat{T}, A\widehat{S}\big\rangle = (2\pi)^{-n}\int_{\mathbb{R}^n} \widehat{T}(x)\, d\widehat{S}(x)$$

for $T \in \mathcal{B}_{1,k}(E)$ and $S \in \mathcal{BV}_{\infty,\frac{1}{k}}(E')$. (It is easy to check that these formulae coincide with the one obtained in first part of the proposition provided E' has the Radon-Nikodým property.) ∎

In [8] Favini interpolates Hörmander spaces extending partially some results of Schechter (see [18]). In the following proposition we extend (for $p_0, p_1 < \infty$) the Corollary of Theorem 14 of [8] to vector-valued Hörmander-Beurling spaces by using direct methods.

PROPOSITION 3.5. *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p_i < \infty$, $i = 0, 1$, $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and E a Banach space. Then*

$$\big(\mathcal{B}_{p_0,k_0}(E), \mathcal{B}_{p_1,k_1}(E)\big)_\theta = \big(\mathcal{B}_{p_0,k_0}(E), \mathcal{B}_{p_1,k_1}(E)\big)_{\theta,p} = \mathcal{B}_{p,k}(E),$$

where $k(x) = k_0^{1-\theta}(x)\, k_1^\theta(x)$ and the corresponding norms are equivalent.

Proof. It is clear from the definition that the Fourier transformation $\mathcal{F}$ in $\mathcal{S}'_\omega(E)$ is an isomorphism from $\mathcal{B}_{p_i,k_i}(E)$ onto $L_{p_i,k_i}(E)$, $i = 0, 1$, which implies that $\mathcal{F}$ is also an isomorphism from the space $\big(\mathcal{B}_{p_0,k_0}(E), \mathcal{B}_{p_1,k_1}(E)\big)_\theta$ (resp. $\big(\mathcal{B}_{p_0,k_0}(E), \mathcal{B}_{p_1,k_1}(E)\big)_{\theta,p}$) onto the space $\big(L_{p_0,k_0}(E), L_{p_1,k_1}(E)\big)_\theta$ (resp. $\big(L_{p_0,k_0}(E), L_{p_1,k_1}(E)\big)_{\theta,p}$). To finish, it suffices to take into account (see, f.i, [21, p. 130]) that

$$\big(L_{p_0,k_0}(E), L_{p_1,k_1}(E)\big)_\theta = \big(L_{p_0,k_0}(E), L_{p_1,k_1}(E)\big)_{\theta,p} = L_{p,k}(E)$$

(with equivalent norms) and that $\mathcal{F}$ is also an isomorphism from $\mathcal{B}_{p,k}(E)$ onto $L_{p,k}(E)$. ∎

Some functional spaces can be represented by countable projective limits of Hörmander-Beurling spaces. For example, if $\omega(x) = \log(1+|x|)$ and $k_j(x) = \left(1+|x|^2\right)^{j/2}$, $j = 1, 2, \ldots$, the Plancherel theorem shows that $\mathcal{B}_{2,k_j}$ coincides with the Sobolev space W_2^j, thus, by the Sobolev embedding theorem, we get

$$\mathcal{B}_{2,\overline{k}} = \bigcap_{j=1}^{\infty} \mathcal{B}_{2,k_j} = \bigcap_{j=1}^{\infty} W_2^j = \left\{u \in L_2 : \partial^\alpha u \in L_2,\ \forall \alpha \in \mathbb{N}_0^n\right\} = D_{L_2}$$

where D_{L_2} is the Schwartz space of all functions $f \in \mathcal{C}^\infty$ such that $\partial^\alpha f \in L_2$ for all $\alpha \in \mathbb{N}_0^n$ (see [20, p. 199]). By using the preceding proposition it is easy to interpolate $\mathcal{B}_{p,\overline{k}}(E)$ spaces using the Goulaouic [9] procedure. Let us briefly review this procedure: Let $(A_i)_{i=1}^{\infty}$ be a decreasing sequence of Banach spaces such that the natural injections $A_{i+1} \longrightarrow A_i$ are continuous. If $A = \bigcap_{i=1}^{\infty} A_i$ is dense in A_i, $i \geq 1$, and A is equipped with the projective limit topology, then A becomes a Fréchet space denoted by $\varprojlim_i A_i$. Let us now consider two Fréchet spaces B_0, B_1 of the form $B_0 = \varprojlim_i B_{0,i}$, $B_1 = \varprojlim_i B_{1,i}$ and such that all spaces $B_{0,i}, B_{1,i}$ are continuously embedded in a common Hausdorff topological vector space $\mathfrak{B}$. We also suppose that, for each (i,j), $B_0 \cap B_1$ is dense in $B_{0,i} \cap B_{1,j}$ (with the norm $\max\{\|\cdot\|_{B_{0,i}}, \|\cdot\|_{B_{0,j}}\}$). Then we write $(B_0, B_1) = \varprojlim_i (B_{0,i}, B_{1,i})$. For $0 < \theta < 1$ and $1 \leq p < \infty$, $(B_0, B_1)_{\theta,p}$ denotes the space $\{b \in B_0 + B_1 : r_i(b) = \left[\int_0^\infty \frac{1}{t}\left(t^{-\theta} k_i(t,b)\right)^p dt\right]^{1/p} < \infty,\ i = 1, 2, \ldots\}$ with the topology defined by the sequence of norms $\{r_i : i \geq 1\}$ (k_i is the Peetre k-functional associated with the couple $(B_{0,i}, B_{1,i})$). By [9, Ch.3], we have $(B_0, B_1)_{\theta,p} = \varprojlim_i (B_{0,i}, B_{1,i})_{\theta,p}$.

We can now establish the following interpolation result:

PROPOSITION 3.6. *Let $\omega \in \mathcal{M}$, $\overline{k}_0 = (k_{0,i})_{i=1}^{\infty}$, $\overline{k}_1 = (k_{1,i})_{i=1}^{\infty}$ increasing sequences in $\mathcal{K}_\omega$, $1 \leq p_0, p_1 < \infty$ and E a Banach space. Then we have*

$$\left(\mathcal{B}_{p_0,\overline{k}_0}(E), \mathcal{B}_{p_1,\overline{k}_1}(E)\right)_{\theta,p} = \mathcal{B}_{p,\overline{k}}(E)$$

where

$$\overline{k} = \left(k_{0,i}^{1-\theta} k_{1,i}^{\theta}\right)_{i=1}^{\infty}, \quad \frac{1}{p} = \frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}, \quad \mathcal{B}_{p_0,\overline{k}_0}(E) = \varprojlim_i \mathcal{B}_{p_0,k_{0,i}}(E),$$

$$\mathcal{B}_{p_1,\overline{k}_1}(E) = \varprojlim_i \mathcal{B}_{p_1,k_{1,i}}(E), \quad \text{and} \quad \mathcal{B}_{p,\overline{k}}(E) = \varprojlim_i \mathcal{B}_{p,k_i}(E).$$

Proof. All these vector-valued Hörmander-Beurling spaces are continuously embedded in $\mathcal{S}'_\omega(E)$ and, by Proposition 3.3, $\mathcal{D}_\omega \otimes E$ is dense in each of them. $\mathcal{D}_\omega \otimes E$ also is dense in the Banach space $\mathcal{B}_{p_0,k_{0,i}}(E) \bigcap \mathcal{B}_{p_1,k_{1,i}}(E)$ for each (i,j). Consequently,

$$\left(\mathcal{B}_{p_0,\overline{k}_0}(E), \mathcal{B}_{p_1,\overline{k}_1}(E)\right) = \varprojlim_i \left(\mathcal{B}_{p_0,k_{0,i}}(E), \mathcal{B}_{p_1,k_{1,i}}(E)\right).$$

Then, by the previous Goulaouic's result and Proposition 3.5, we get

$$\begin{aligned}\left(\mathcal{B}_{p_0,\overline{k}_0}(E), \mathcal{B}_{p_1,\overline{k}_1}(E)\right)_{\theta,p} &= \varprojlim_i \left(\mathcal{B}_{p_0,k_{0,i}}(E), \mathcal{B}_{p_1,k_{1,i}}(E)\right)_{\theta,p} \\ &= \varprojlim_i \mathcal{B}_{p,k_i}(E) = \mathcal{B}_{p,\overline{k}}(E)\end{aligned}$$

and the proof is complete. ∎

4. An approximation theorem

By extending the definition given by Schwartz in [21, p.7] to the setting of the E-valued ultradistributions, we shall say that a linear subspace H of $\mathcal{D}'_\omega(E)$, equipped with a locally convex topology finer than the one induced by $\mathcal{D}'_\omega(E)$, has the property of approximation by cutting (CAP), if, for each $\alpha \in \mathcal{D}_\omega$, the operator $[\alpha] : H \longrightarrow H$ given by $[\alpha](T) = \alpha T$ is well-defined and it is continuous, and if, when $\varphi \in \mathcal{D}_\omega$ with $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $|x| \leq 1$, $[\varphi_\epsilon] \to I$ in $\mathcal{L}_c(H)$ ($\varphi_\epsilon(x) = \varphi(\epsilon x)$ and I is the identity) when $\epsilon \to 0+$. In this section we shall prove that $\mathcal{B}_{p,k}(E)$ is an $\mathcal{S}_\omega$-topological module and that $\mathcal{B}_{p,k}(E)$ has the CAP when $p < \infty$ (extending [3, Th. 2.2.7] and [16, Th. 3(3)]).

Proposition 4.1. *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leq p \leq \infty$, and E a Banach space. Let us put $\mathcal{B}^0_{\infty,k}(E) = \overline{\mathcal{D}_\omega(E)}^{\mathcal{B}_{\infty,k}(E)}$.*

(1) *If $\varphi \in \mathcal{S}_\omega$ and $T \in \mathcal{B}_{p,k}(E)$, then $\varphi T \in \mathcal{B}_{p,k}(E)$ and $\|\varphi T\|_{p,k} \leq \|\varphi\|_{1,M_k} \|T\|_{p,k}$. Consequently, $\mathcal{B}_{p,k}(E)$ and $\mathcal{B}^0_{\infty,k}(E)$ are $\mathcal{S}_\omega$-topological modules.*

(2) *$\mathcal{B}_{p,k}(E)$ ($p < \infty$) and $\mathcal{B}^0_{\infty,k}(E)$ have the CAP.*

Proof. (1) Let $\varphi \in \mathcal{S}_\omega$ and $T \in \mathcal{B}_{p,k}(E)$. By Section 2, $\varphi T \in \mathcal{S}'_\omega(E)$ and $(\varphi T)^\wedge = (2\pi)^{-n}\left(\widehat{\varphi} * \widehat{T}\right)$. Let us see that $\widehat{\varphi} * \widehat{T} \in L_{p,k}(E)$. Since $\widehat{T} \in L_{p,k}(E)$

we have

$$\widehat{\varphi} * \widehat{T}(x) = \langle \tau_x \tilde{\widehat{\phi}}, \widehat{T} \rangle = \int_{\mathbb{R}^n} \widehat{\varphi}(x-y)\ \widehat{T}(y)\, dy, \quad x \in \mathbb{R}^n,$$

and taking into account that $k(x) \leq M_k(x-y)\, k(y)$ we get

$$\left\| k(x) \left(\widehat{\varphi} * \widehat{T}\right)(x) \right\|_E \leq \left(|M_k \widehat{\varphi}| * \left\| k\widehat{T} \right\|_E \right)(x), \quad x \in \mathbb{R}^n.$$

Since $M_k \widehat{\varphi} \in L_1$ and $\left\| k\widehat{T} \right\|_E \in L_p$ we can apply the Young's inequality and so

$$\begin{aligned} \left\| |M_k \widehat{\varphi}| * \left\| k\widehat{T} \right\|_E \right\|_{L_p} &\leq \|M_k \widehat{\varphi}\|_{L_1} \left\| \left\| k\widehat{T} \right\|_E \right\|_{L_p} \\ &= (2\pi)^{-n\left(1+\frac{1}{p}\right)} \|\varphi\|_{1,M_k} \|T\|_{p,k}. \end{aligned}$$

Thus $\widehat{\varphi} * \widehat{T} \in L_{p,k}(E)$, that is, $\varphi T \in \mathcal{B}_{p,k}(E)$, and

$$\|\varphi T\|_{p,k} \leq \|\varphi\|_{1,M_k} \|T\|_{p,k} \leq c\pi_{0,\Lambda}(\varphi) \|T\|_{p,k}$$

being c and Λ certain positive constants. These inequalities imply that the bilinear map $\mathcal{S}_\omega \times \mathcal{B}_{p,k}(E) \longrightarrow \mathcal{B}_{p,k}(E)$: $(\varphi, T) \mapsto \varphi T$ is continuous. To finish, it is enough to notice that if $T \in \mathcal{B}^0_{\infty,k}(E)$ and $(f_j)_{j=1}^\infty$ is a sequence in $\mathcal{D}_\omega(E)$ such that $f_j \to T$ in $\mathcal{B}_{\infty,k}$ then, for each $\varphi \in \mathcal{S}_\omega$, $(\varphi f_j)_{j=1}^\infty$ is a sequence in $\mathcal{D}_\omega(E)$ such that $\varphi f_j \to \varphi T$ in $\mathcal{B}_{\infty,k}(E)$.

(2) Let $\varphi \in \mathcal{D}_\omega$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in $|x| \leq 1$. Let us put $\varphi_\epsilon(x) = \varphi(\epsilon x)$ for $\epsilon > 0$. Then, for any $\epsilon \in]0,1[$, we have

$$\begin{aligned} \|\varphi_\epsilon\|_{1,M_k} &= (2\pi)^{-n} \int_{\mathbb{R}^n} M_k(x) \left|\widehat{\varphi_\epsilon}(x)\right| dx = (2\pi)^{-n} \int_{\mathbb{R}^n} M_k(\epsilon x) \left|\widehat{\varphi}(x)\right| dx \\ &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} e^{\lambda\omega(\epsilon x)} \left|\widehat{\varphi}(x)\right| dx \leq (2\pi)^{-n} \|\varphi\|^{(\omega)}_\lambda \end{aligned}$$

where λ is a certain number > 0 independent of ϵ. Thus, $\sup\{\|\varphi_\epsilon\|_{1,M_k} : 0 < \epsilon < 1\} = c < \infty$. Let us now assume $p < \infty$. From part 1 and the above estimation it follows that $\{[\varphi_\epsilon] : 0 < \epsilon < 1\}$ is an equicontinuous subset of $\mathcal{L}(\mathcal{B}_{p,k}(E))$. Consequently, the topologies of simple and precompact convergence coincide on $\{[\varphi_\epsilon] : 0 < \epsilon < 1\}$ (see, f.i., [12, p.156]). Therefore, to

see that $\mathcal{B}_{p,k}(E)$ has the CAP it is enough to show that, for any $T \in \mathcal{B}_{p,k}(E)$, $\varphi_\epsilon T \mapsto T$ in $\mathcal{B}_{p,k}(E)$ when $\epsilon \to 0+$: Fix $T \in \mathcal{B}_{p,k}(E)$. For $f \in \mathcal{S}_\omega(E)$ we have

$$\begin{aligned}\|T - \varphi_\epsilon T\|_{p,k} &\leq \|T - f\|_{p,k} + \|f - \varphi_\epsilon f\|_{p,k} + \|\varphi_\epsilon f - \varphi_\epsilon T\|_{p,k} \\ &\leq \left(1 + \|\varphi_\epsilon\|_{1,M_k}\right) \|T - f\|_{p,k} + \|f - \varphi_\epsilon f\|_{p,k} \\ &\leq (1+c) \|T - f\|_{p,k} + \|f - \varphi_\epsilon f\|_{p,k},\end{aligned}$$

but this shows that $\varphi_\epsilon T \mapsto T$ in $\mathcal{B}_{p,k}(E)$ since $\mathcal{S}_\omega(E)$ is dense in $\mathcal{B}_{p,k}(E)$ (Proposition 3.3) and $\varphi_\epsilon f \mapsto f$ in $\mathcal{B}_{p,k}(E)$ (to see that $\varphi_\epsilon f \mapsto f$ in $\mathcal{S}_\omega(E)$ one can follow the scalar case, [3, Th. 1.8.7], step by step). The argument for $\mathcal{B}^0_{\infty,k}(E)$ is analogous. We leave the details to the reader. ∎

Remark 4.2. Let $\omega \in \mathcal{M}$ be such that $\mathcal{S}_\omega$ is nuclear. Let $k \in \mathcal{K}_\omega$, $1 \leq p \leq \infty$. Let E, F be Banach spaces. Then, by virtue of Proposition 4.1 and [22, Prop. 3, p.37], there exists an unique continuous bilinear map

$$\begin{aligned}\mathcal{S}_\omega(E) \times \left(\mathcal{B}_{p,k}\widehat{\otimes}_\epsilon F\right) &\longrightarrow \mathcal{B}_{p,k}\widehat{\otimes}_\epsilon \left(E\widehat{\otimes}_\pi F\right) \\ (\phi, T) &\mapsto \phi \cdot_\pi T\end{aligned}$$

such that $(\varphi \otimes e) \cdot_\pi (u \otimes f) = (\varphi u) \otimes (e \otimes f)$ for all $\varphi \in \mathcal{S}_\omega$, $e \in E$, $u \in \mathcal{B}_{p,k}$, $f \in F$.

We now consider the map $Z : (\mathcal{S}_\omega \otimes_\pi E) \times \mathcal{B}_{p,k}(F) \longrightarrow \mathcal{B}_{p,k}\left(E\widehat{\otimes}_\pi F\right)$ defined by

$$\langle \varphi, Z(\phi, T)\rangle = \left\{\phi = \sum_{i=1}^m \varphi_i \otimes e_i\right\} = \sum_{i=1}^m e_i \otimes \langle \phi\varphi_i, T\rangle, \quad \varphi \in \mathcal{S}_\omega.$$

It is clear that this definition is independent of the representation of ϕ. It is also immediate that $Z(\phi, T) \in \mathcal{S}'_\omega\left(E\widehat{\otimes}_\pi F\right)$, that $Z(\phi, T)^\wedge = \sum_{i=1}^m e_i \otimes \widehat{\varphi_i T}$ and that $Z(\varphi \otimes e, u \otimes f) = (\varphi u) \otimes (e \otimes f)$ for all $\varphi \in \mathcal{S}_\omega$, $e \in E$, $u \in \mathcal{B}_{p,k}$, $f \in F$. Since, by virtue of Proposition 4.1, $\widehat{\varphi_i T} \in L_{p,k}(F)$, it follows that $\sum_{i=1}^m e_i \otimes \widehat{\varphi_i T} \in L_{p,k}\left(E\widehat{\otimes}_\pi F\right)$. Therefore, Z is well-defined and is bilinear. Let us see that it is continuous: Let $p < \infty$. If $\phi = \sum_1^m \varphi_i \otimes e_i \in \mathcal{S}_\omega \otimes E$ and

$T \in \mathcal{B}_{p,k}(F)$, we have

$$\begin{aligned}
\left\| k\left(\sum_1^m e_i \otimes \widehat{\varphi_i T} \right) \right\|_p &= \left(\int_{\mathbb{R}^n} \left\| \sum_1^m e_i \otimes \widehat{\varphi_i T}(x) \right\|^p_{E \widehat{\otimes}_\pi F} k^p(x) dx \right)^{1/p} \\
&\leq \sum_1^m \|e_i\|_E \left(\int_{\mathbb{R}^n} \left\| \widehat{\varphi_i T}(x) \right\|^p_F k^p(x) dx \right)^{1/p} \\
&= (2\pi)^{n/p} \sum_1^m \|e_i\|_E \, \|\varphi_i T\|_{p,k} \\
&\leq C \left(\sum_1^m \|e_i\|_E \, \|\varphi_i\|_{1,M_k} \right) \|T\|_{p,k} \\
&\leq C \left(\sum_1^m \pi_{0,\lambda}(\varphi_i) \, \|e_i\|_E \right) \|T\|_{p,k} .
\end{aligned}$$

Thus

$$\|Z(\phi, T)\|_{p,k} \leq C \left(\pi_{0,\lambda} \otimes \|\cdot\|_E \right)(\phi) \, \|T\|_{p,k}$$

($C, \lambda > 0$ are independent of ϕ and T) and Z is continuous. (The case $p = \infty$ is analogous.) Since $\mathcal{S}_\omega$ is nuclear, $\mathcal{S}_\omega \widehat{\otimes}_\pi E = \mathcal{S}_\omega(E)$. This fact and the theorem of extension of bilinear mappings prove that there exists a uniquely determined continuous extension of Z to $\mathcal{S}_\omega(E) \times \mathcal{B}_{p,k}(F)$.

ACKNOWLEDGEMENTS

I would like to thank the referee for valuable comments and suggestions.

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