REDIMAT
Journal of Research in Mathematics Education

Hipatia Press
www.hipatiapress.com

Instructions for authors, subscriptions and further details:

http://redimat.hipatiapress.com

Onto-semiotic complexity of the Definite Integral. Implications for teaching and learning Calculus

María Burgos[1], Seydel Bueno[2], Juan D. Godino[1], Olga Pérez[2]

1) Granada University (Spain)
2) Camaguey University (Cuba)

Date of publication: February 24th, 2021
Edition period: February 2021-June 2021

To cite this article: Burgos, M., Bueno, S., Godino, J.D., & Pérez, O. (2021). Onto-semiotic complexity of the Definite Integral. Implications for teaching and learning Calculus *REDIMAT – Journal of Research in Mathematics Education*, *10*(1), 4-40. doi: 10.17583/redimat.2021.6778

To link this article: http://dx.doi.org/10.17583/redimat.2021.6778

REDIMAT, Vol. 10 No. 1 2021 pp. 4-40

Onto-semiotic complexity of the Definite Integral. Implications for teaching and learning Calculus

María Burgos, *Granada University*
Seydel Bueno, *Camaguey University*
Juan D. Godino, *Granada University*
Olga Pérez, *Camaguey University*

Abstract

Teaching and learning Calculus concepts and procedures, particularly the definite integral concept, is a challenge for teachers and students in their academic careers. In this research, we supplement the analysis made by different authors, applying the theoretical and methodological tools of the Onto-Semiotic Approach to mathematical knowledge and instruction. The goal is to understand the diverse meanings of the concept of the definite integral and potentials semiotic conflicts based on the given data. We focus attention on a first intuitive meaning, which involves mainly arithmetic knowledge, and the definite integral formal meaning as Riemann's sums limit predominantly in the curricular guidelines. The recognition of the onto-semiotic complexity of mathematics objects is considered as a key factor in explaining the learning difficulties of concepts, procedures and its application for problem-solving, as well as to make grounded decisions on teaching. The methodology analysis of a mathematical text, which we exemplify in this work applying the tools of Onto-Semiotic Approach, provides a microscopic level of analysis that allows us to identify some semiotic-cognitive facts of didactic interest. This also allows for the identification of some *epistemic strata*, that is, institutional knowledge that should have been previously studied, which usually goes unnoticed in the teaching process.

Keywords: Definitive integral, mathematical practices, onto-semiotic complexity, teaching, learning.

2021 HipatiaPress
ISSN: 2014-3621
DOI: 10.17583/redimat.2021.6778

Hipatia Press
www.hipatiapress.com

REDIMAT, Vol. 10 No. 1 2021 pp. 4-40

Complejidad Onto-Semiótica de la Integral Definida. Implicaciones para la Enseñanza y el Aprendizaje del Cálculo

María Burgos
Granada University

Seydel Bueno
Camaguey University

Juan D. Godino
Granada University

Olga Pérez
Camaguey University

Resumen

La enseñanza y el aprendizaje de los conceptos y procedimientos del Cálculo, en particular del concepto de integral definida, es un reto para profesores y estudiantes en su trayectoria académica. En esta investigación, complementamos el análisis realizado por diferentes autores, aplicando las herramientas teóricas y metodológicas del Enfoque Onto-Semiótico al conocimiento y la instrucción matemática. El objetivo es comprender los diversos significados del concepto de integral definida y los potenciales conflictos semióticos a partir de los datos aportados. Centramos la atención en un primer significado intuitivo, que implica principalmente conocimientos aritméticos, y en el significado formal integral definida como límite de las sumas de Riemann predominantemente en las directrices curriculares. El reconocimiento de la complejidad onto-semiótica de los objetos matemáticos se considera un factor clave para explicar las dificultades de aprendizaje de los conceptos, los procedimientos y su aplicación para la resolución de problemas, así como para tomar decisiones fundamentadas sobre la enseñanza. El análisis metodológico de un texto matemático, que ejemplificamos en este trabajo aplicando las herramientas del Enfoque Onto-Semiótico, proporciona un nivel microscópico de análisis que permite identificar algunos hechos semiótico-cognitivos de interés didáctico. También permite identificar algunos estratos epistémicos, es decir, conocimientos institucionales que deberían haber sido estudiados previamente y que suelen pasar desapercibidos en el proceso de enseñanza.

Palabras clave: Integral definitiva, prácticas matemáticas, complejidad onto-semiótica, enseñanza, aprendizaje.

2021 HipatiaPress
ISSN: 2014-3621
DOI: 10.17583/redimat.2021.6778

The review prepared by Bressoud, Ghedams, Martinez-Luances & Törner (2016) describes the evolution of research and main trends on various topics involved in the teaching and learning of calculus, such as student difficulties, task design, class practices, technology usage, etc. These authors identify issues related to the epistemological, cognitive, institutional, and instructional aspects raised by the research in this field. They also highlight the concern about the relationship between student thoughts in Calculus concepts and learning expectations in the curricula. In this relationship, they identify at least three dialectical tensions: the potential infinity versus actual infinity, the dynamic versus the static, and the visualization versus formalization. These dialectics lead to some open questions: "What epistemological considerations should be taken into account to address such tensions? Which are the roles of teaching and the practices in the classroom?" (Bressoud, et al., 2016, p. 30).

Bressoud et al (2016) mention the main theoretical frameworks that have been applied in educational research on Calculus with an emphasis on cognitive development. They refer to the approaches "Concept Image and Concept Definition" (Tall & Vinner, 1981), the theory of Semiotic Representation Register (Duval, 1995), the theory Action-Process- Object-Schema (Dubinsky, 1991), and the Three Worlds of Mathematics theory (Tall, 2004). Some research focus on institutional, socio-cultural, and discursive aspects such as the Didactic Situations Theory (Brousseau, 1997), the Anthropological Theory of Didactics (Chevallard, 1992), and the Cognitive Framework (Sfard, 2008). Others have carried out experiments in teaching the definite integral based on different theoretical frameworks. Attorps, Björk, Radic &Tossavainen (2013) apply the Learning Study model grounded on the Variation Theory (Marton, Runesson & Tsui,2004) and Kouropatov & Dreyfus (2013; 2014) adopt a research methodology based on the Abstraction in Context theory (AiC) (Hershkowitz, Schwarz & Dreyfus, 2001).

Despite the great amount of research done about the teaching and learning of Calculus, we consider that it is important to problematize the meanings of key concepts in Calculus, like limit, derivative, and integral. The purpose is to identify a sequence of situations-problems whose resolution allows them to contextualize the knowledge and to develop progressively the student's mathematical comprehension and competence.

In this research, we focus the attention on the definite integral concept. In some countries its study begins in the later grades high school, presenting to the students an informal/intuitive first encounter with this mathematical object, preparing them to understand the most general and formal meanings required in university studies. While in other countries, the study of the integral is introduced abruptly, with all its generality, mainly in the case of students from experimental sciences, engineering, and other technical careers; this approach can make conceptual understanding and justification of procedures difficult.

Our research deals with the problem of clarifying some key meanings of definite integral, applying the theoretical and methodological tools of the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino & Batanero, 1998; Font, Godino & Gallardo, 2013; Godino, Batanero & Font, 2019). The notion of partial meaning (or sense) of a mathematical concept, understood in a pragmatic and anthropological way, provides a macroscopic view of the overall meaning of a mathematical object. Besides, the OSA provides tools to make a microscopic analysis of the mathematical activity, allowing to identify the configuration of objects and processes that intervene in the mathematical practices required in problems solving, which are the *raison d'être* of a concept. These analyses allow us to become aware of the concept complexity, which will lead to the reflection by the teacher's educator and the teacher about the possible difficulties that may arise in the organization of the teaching and learning processes.

The article is organized in the following sections. After this introduction, which we consider as section 1, in section 2, we describe the theoretical tools, the specific research problem and the methodology applied. In section 3 we included a synthesis of previous research on the definite integral components and structure and the characterization of its meanings. In sections 4 and 5 we applied some OSA tools to analyze the complexity of the mathematical practices involved in the Riemann's integral study. To underscore the dialectic tension between the intuitive and formal meanings, firstly, in section 4, we analyze the presentation proposed by Starbird (2006), that can be considered as a first informal encounter with the integral. Later on, in section 5, we analyze the general and formal definition that Stewart (2016) makes, which is preceded by a previous contextualization based on the study of problems related to the area and distances calculations. The microscopic analysis of the integral meanings made in these sections show how the ontological and semiotic-cognitive problem of mathematical education is approached using the OSA tools. The analysis in these sections lead us in section 6 to establish a discussion in relation to the articulation of our analysis with other research. The implications of our investigation and some open questions are included at the end in the section 7.

Theoretical Framework, Specific Problem, and Methodology

We will apply some tools of the Onto-Semiotic Approach (OSA) of mathematical knowledge and instruction to analyze the diversity and complexity of mathematical objects and its meanings. We consider this complexity as an explanatory factor of the students' learning difficulties and conflicts.

Theoretical Framework

The OSA approaches the epistemological, ontological, and semiotic-cognitive problems in mathematics teaching and learning process, by proposing a system of theoretical notions and methodological tools (Godino et al., 2019). Below we include a synthesis of the main notions used in this research work.

The epistemological problem

The OSA gives answers to the epistemological problem of how mathematical knowledge emerges and develops, by assuming anthropological (Wittgenstein, 1953) and pragmatist (Peirce, 1958) views. It considers people's activity in solving problems as a central element in mathematical knowledge construction. This epistemological view is made with the notion of mathematical practice, by assuming its institutional and personal relativity. In the OSA framework, a *mathematical practice* is (Godino & Batanero, 1998, p. 182): "any action or manifestation (linguistic or otherwise) carried out by someone to solve mathematical problems, to communicate the solution to others, so as to validate and generalize that solution to other contexts and problems."

The institutional genesis of the mathematical knowledge is investigated in the following methods: (1) The identification and categorization of the problems-situations that require an answer; (2) The description of the practice sequences in the resolution.

The ontological problem

Mathematics is not only human activity, but also a logically organized system of objects. In OSA, a *mathematical object* is any material or immaterial entity that intervenes in mathematical practices, by supporting and regulating its realization. This general idea of an object is useful when it is complemented with a typology of mathematical objects considering their different roles in mathematical activity. Symbols, external representations, and manipulatives intervene in the school and professional mathematical activity, in a public, material, and perceptible way. Therefore, they are considered *ostensive mathematical objects*. The *concepts* of numbers, fraction, derivative, integral, etc., are mathematical objects of different natures and roles than their ostensive representations. They are not ostensive objects, but mental objects (when they intervene in personal practices). They are also institutional objects (when they participate in sociocultural or shared practices). Together with the *propositions*, *procedures*, and the *arguments* that justify them, they are objects that regulate the mathematical activity, while its ostensive representations serve as support to facilitate the realization of this work. There is no mathematical activity without objects or mathematical objects without activity. As the practices, it could be seen from the social (institutional, shared) or personal (individual, idiosyncratic) perspectives, objects can also be conceived from the institutional-personal duality which originates the following principle:

In institutional or personal mathematical practices, different kinds of objects intervene that fulfill different roles: instrumental/representational, regulatory (setting rules on practices), explanatory, justifying.

Given the generality of the practice and object notions, as well as the diversity of sequences of practices (*processes*) that can be made, it is considered necessary and useful to propose a typology of objects and basic processes, which are reflected in figure 1, designed as *onto-semiotic configuration*. These configurations can be epistemic (institutional objects networks) or cognitive (personal objects networks). Other processes from figure 1 are considered in the psychological and educational literature, for example, problem solving and modeling processes, among others. Those processes can be described by using the proposed basic processes by the OSA, so they are treated as *mega processes*.

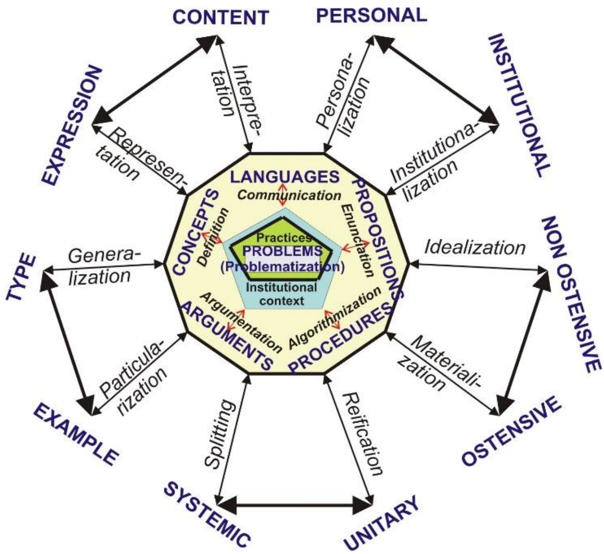


Figure 1. Onto-semiotic configuration of practice objects and processes. Source: Godino (2014, p. 23).

The concepts, propositions and procedures, are understood, in its *unitary* version (treated as a whole or unit), according to Wittgenstein's proposal, that is to say, as grammatical rules of the languages used in the practices carried out to describe our worlds and give answer to the problems-situations we face to. In the OSA, the concepts and other mathematical objects can also be considered from a *systemic* perspective, that is, as a system of components.

The semiotic-cognitive problem

The *knowledge* is assumed in the OSA as the integration of relations that the subject (person or institution) establishes between the objects and the practices, relations that are modeled by the *semiotic function* notion. A semiotic function is understood as the correspondence between an antecedent object (expression, signifier) and a consequent one (content, meaning) established by a subject (person or institution), according to a criteria or correspondence rule. An act of interpretation or semiosis relates an antecedent object with another consequent object according to a certain agreement or correspondence rule. The act of semiosis may consist in giving sense, use or purpose to an action, or sequence of actions, within the framework of the activity being carried out. Knowledge is about the content of one (or many) semiotic functions, so there is a variety of types of knowledge in correspondence with the diversity of semiotic functions that can be established between the various types of practices and objects. The next principle is based on that:

The institutional or personal 'meaning of the object' is interpreted as the correspondence between an object and the system of practices where such an object occurs.

The problem-situations depend on the individuals and communities of practice (institutions). Hence, meanings, and therefore, knowledge, are relative. However, it is possible to reconstruct a global or holistic meaning of an object through the systematic exploration of the object contexts of use and the systems of practices required in solving problems. This holistic meaning is an epistemological and cognitive reference model for the object meanings, and constitutes an onto-semiotic-cognitive methodological tool:

The method to delimit the different meanings of mathematical objects, and, therefore, to reconstruct the epistemological and cognitive reference models is to analyze the practice systems (personal and institutional) and of the onto-semiotic configurations involving them.

The reconstruction of each partial meaning of a mathematical object, for example, the definite integral notion, implicates the description of the system of practices required to solve a type of problems in a context of use, as well as the objects and processes put at stake in these practices.

Specific Research Problem and Methodology

The educational-instructional issue of trying to understand the student's learning difficulties about the definite integral, and to intervene in a grounded way in the teaching processes, could be formulated using the OSA theoretical and methodological tools. We should begin by characterizing the different meanings of the definite integral and the diverse generalizations that have been developed to solve the application problems. To characterize the meanings, the problem-situations that are the *raison d'être* of each meaning and the systems of operative, discursive and normative practices involved in their resolution must be identified and categorized. This is the perspective adopted by authors like Ordoñez (2011) and Crisóstomo (2012), who identify and analyze the partial meanings of the definite integral from a historical-epistemological approximation. Other studies that have applied the OSA framework on different aspects of integral learning are Mateus (2016); Pino-Fan, Font, Gordillo, Larios, & Breda (2018); Borji & Font (2019).

The main contribution of this research work in teaching and learning Calculus is essentially theoretical and methodological. We focus our attention on the ontological and semiotic-cognitive problem based on two partial meanings of the definite integral, one informal / intuitive meaning, and the other formal meaning of the integral as the limit of Riemann Sums. With this aim, we apply a microscopic analysis oriented to the explicit recognition of objects (problems, languages, definitions, propositions, procedures, and arguments) that intervene and emerge in mathematical practices. We also identify the processes involved (interpretation / meaning, representation, argumentation, generalization, etc.).The onto-semiotic analysis is applied to two texts with a certain international diffusion: the intuitive presentation of the definite integral proposed by Starbird (2006), and the exposition of the integral as the Limit of Riemann Sums by Stewart (2016).

Background on the Definite Integral Characterization

Different authors have studied the meaning of definite integral, its components and structure, in order to design suitable teaching processes to enable students to apply this object in problem solving. They have also identified the ways in which students understand the definite integral. In this section we include a synthesis of these studies.

Components and Structure of the Definite Integral

Although a large part of the educational research on the definite integral has been carried out to characterize the types of students' learning, with a cognitive approach (Orton, 1983; Grundmeier, Hansen & Sousa, 2006; Rasslan and Tall, 2002; Serhan, 2015), we have found some authors who analyze the structure and components of the integral from an epistemic perspective. Such as Sealey (2014) who develops a framework to characterize students' understanding of Riemann Sums and the definite integral. This author indicates the conceptualization of the product $f(x)\Delta x$ as the most complex part of the problem-solving process involving the integral, despite the simplicity of the mathematical operations required in this step. Sealey presents a reference frame for understanding the structure of the Riemann integral that distinguishes four layers: *Product, Summation, Limitand Function*. The layers correspond to the mathematical operations involved in calculating the integral as$\sum_{i=1}^{n} F(x_i)\Delta x$ where x_i represents any value in the i-th interval associated with a partition of $[a, b]$ and $\Delta x = b - a/n$.

The Product layer is composed of the multiplication of two quantities, $f(x_i)$ and Δx, where $f(x_i)$ (for example, the speed of an object) is a rate and Δx a difference (elapsed time). The product of both quantities corresponds to the distance traveled in aninterval. The Summation layer, including the sum from i=1 to i=n, gives the Riemann Sums $\sum_{i=1}^{n} f(x_i)\Delta x$. Thus, if $f(x_i)$ is the speed of an object during a time interval Δx, the sum would be approximate of the distance in the interval from a to b, which are the extreme points of the interval

The third layer, Limit, corresponds to the limit when n approaches to infinity in the expression of the other two layers, which brings the Riemann integral. The fourth layer considers the definite integral as a function whose variable is the upper limit (i.e., right endpoint) and the value of the function is the numerical value of the definite integral.

As a result of her experimental research, Sealey (2014) introduced a preliminary layer, which she calls *Orienting*, to take into account the activities that students perform to visualize the situation, understand the problem variables, recognize the magnitudes, and quantities (pressure, force, etc.). The product of the quantities given by $f(x)$ and Δx represents new quantities in the context of the problem that needs to be visualized.

Thompson and Silverman (2008) present the definite Riemann integral in terms of the mathematical idea of the accumulation function $F(x) = \int_a^x f(t)dt$. They explain the students' understanding and use difficulties, based on constituent parts. The concept of accumulation is the core of understanding of Calculus concepts and applications, but the meaning it takes in this branch of mathematics is not simple. Students find it difficult to think of something that accumulates when they are unclear about the "bits" that accumulate. For example, to understand the idea of work done as something that accumulates incrementally means that the total amount of work at each moment must be thought of as the sum of previous increments, and each additional incremental bit of work is made up of an applied force over a distance.

Jones (2013, 2015) approaches the topic of the definite integral by distinguishing three conceptualizations: a) as the area under a curve; b) as the values of an anti-derivative; c) as the limit of Riemann Sums. Conceptualization a) uses the representation "perimeter and area" to indicate the integrating function, the area of a region of the Cartesian plane, and the region contour. Conceptualization b) uses the notation indicating the relationship among anti-derivative, derivative, and the original function, $F'(x) = f(x)$. Conceptualization c) uses the sum of infinitely small pieces.

Greefrath, Oldenburg, Siller, Ulm & Weigand (2016) propose a model for understanding the concept of definite integral that considers three axes or dimensions: 1) Aspects of the content that serve as the basis for a definition of the concept (antiderivative, product sum, measure); 2) Conceptions or mental models (Grundvorstellungen) about the concept (reconstruction, area, average value, accumulation); 3) Levels of understanding of the concept (intuitive, subject matter, integrated, critical). Combining the different aspects, conceptions, and understandings, they elaborate a three-dimensional model with 48 cells (8 of them empty) in each of which specific knowledge, skills, and abilities are put into play for the understanding and use of the integral concept.

Cognitive Schemas on the Integral

Rasslan and Tall (2002) describe the cognitive schemas that high school students have about the definite integral, applying the notions of *concept image* and *concept definition*. Although the definition of the concept is introduced in school, students use concept image, that is, “all the mental pictures, properties and processes associated with the concept in their mind” (Tall & Vinner, 1981). Rasslan and Tal (2002) consider that “the concept definition is essentially an incidental part of the process which is far more concerned in practice with developing experience and images of the concepts themselves” (volume 4, p. 96).

Serhan (2015) investigates students' conceptual and procedural knowledge of the definite integral, applying the cognitive framework of

Hiebert and Lefevre (1986) and the notions of concept image and concept definition of Tall and Vinner (1981). Meanwhile, Hiebert and Lefevre (1986) express “conceptual knowledge is characterized most clearly as knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network” (p. 3). Procedural knowledge, “is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks” (p. 6).

Specifically, Serhan (2015, p. 85) asks the following questions: 1) Which is the most dominant knowledge of the definite integral for students? Is it the procedural knowledge or conceptual knowledge? 2) Are students capable of dealing with negative areas and explaining their answers? 3) Which concept images do Calculus students associate with the definite integral concept? The results coincide with those of Orton (1983) in that students can use procedural knowledge and solve integration problems but have a limited understanding of the integration of basic concepts.

Meanings of the Definite Integral

The integral concept has been generated and evolved throughout the history of Mathematics, starting from its applications to solving problems, fundamentally those related to Physics and Geometry. After a period where the emphasis was on the calculation of primitives, approximate integration emerged (through numerical, graphical, and mechanical methods) where processes were sought to find an approximate value for the definite integral of functions whose primitive could not be determined. Next, the mathematical development of the definite integral was interested in its foundation with the elaboration of more precise definitions, independent of geometry and based on the limit calculation. The final formalization of the integral would be supported by the measurement theory.

The universe of the definite integral meanings can be structured according to different criteria, considering the fields of problems that are solved with the integral, the resolution techniques, and the degree of generality and formalization with which they are treated. Contreras and Ordóñez (2006), Contreras, Ordóñez and Wilhelmi (2010), the doctoral thesis of Ordóñez (2011), and Crisostomo (2012), study the meanings of the definite integral applying the OSA assumptions and tools. Ordóñez (2011) performs a historical-epistemological study of the definite integral meanings; she uses it as a reference frame in a study of the institutional and personal meanings of this mathematical object in High School. Also, Crisóstomo (2012) has a similar study to investigate the institutional meanings of integral in university studies for secondary school mathematics teachers training.

From the point of view of the processes of mathematical instruction at High School and University levels, we consider useful to distinguish four types of meanings, the integral as:

1) Quantity of magnitude bounded between two sequences of convergent quantities. The magnitude can be geometric, physical, length, area, volume, distance, work, density, etc.

2) Limit of Riemann Sums, $\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x$.

3) Cumulative function, $G(x) = \int_a^x f(t)dt$.

4) Incremental difference of the cumulative function, $\int_a^b f(x)dx = G(b) - G(a)$, if $G'(x) = f(x)$.

Each of these integral partial meanings provide relevant aspects of the global meaning of this mathematical object, understood as the articulated network of partial meanings. The meaning1) constitutes a first encounter with the integral, which lends itself to an intuitive treatment connected with the applications, which constitutes its original *raison d'être*. This conception alone is not enough for robust understanding of integrals. The integral has additional layers of meaning above and beyond the limit (Jones, 2013; Thompson & Silverman, 2008). As suggested by Sealey (2014), the interpretation of "area under a curve," could limit the integral applicability to other areas.

The partial meaning 2) as the limit of Riemann Sums introduces precision to the strategy of calculating the integral as "an infinite summation of infinitely small pieces" associated with meaning 1). "Adding up pieces is not equivalent to the Riemann integral, since the students tend to think of the summation happening with an infinite number of infinitesimally small pieces. By contrast, the Riemann integral constructs a sequence of finite summations and then considers the limit of this sequence" (Jones, 2015, p. 11).

However, the calculation of the definite integral applying the meaning 2), that is, by calculating the limit of Riemann Sums, is complex in most cases, if it is compared to the application of the meanings 3) and 4), that is, finding the antiderivative function first and then the incremental difference.

The curricular planning of the integral study should consider this macroscopic perspective on the various partial meanings and their articulation as a strategy to promote the students' understanding and competence. But the management of the learning and teaching processes requires the teacher to be aware of the ontological and semiotic complexity of each one of the meanings, even the meanings that we consider as informal or intuitive. In sections 4 and 5 we include the analysis of two integral meanings by applying the onto-semiotic analysis methodology.

Intuitive Meaning of the Riemann Integral

In this section we analyze Starbird's intuitive presentation of the integral, where he finally gets to justify the Riemann Sums for the case of calculating the covered distance by a car with a variable velocity (Starbird, 2006, p. 18-21). The presentation is supported by the following problem:

Imagine you are kidnapped and tied up in the back of a car and driven off on a straight road. You cannot see out of the car, but fortunately, you can see the speedometer, and you have a video camera to take time-stamped pictures of the speedometer. (There is no odometer in sight.) After an hour, you are dumped on the side of the road. How far have you gone?

Resolution Process: Sequence of Operative and Discursive Practices

We include below the abbreviated process of solving the problem. Starbird comes to state the Riemann Sums and the concept of definite integral. For the analysis, we break the process down into units (epistemic configurations), and in each one we identify the elementary mathematical practices that compose them.

Configuration 1 (EC1): Constant velocity

P1.1: Let's take a simple case: The car was moving at a constant velocity.

P1.2: If we go 1 mile per minute for 60 minutes, we will have gone 60 miles during that hour. If we go 2 miles per minute for 20 minutes, we will have travelled 2 × 20, or 40 miles; in one hour the distance travelled will be 3 × 40, that is, 120 miles.
P1.3: On a graph, this constant velocity would appear as a horizontal line.

Configuration 2 (EC2): Piecewise constant velocity

P2.1: Suppose the velocity is steady for some time, then abruptly change to another velocity and so on.
P2.2: Suppose we travel at 1 mile per minute between times 0 and 10 minutes, 2 miles per minute for the time between 10 and 20 minutes, 3 miles per minute for the time between 20 and 30 minutes, 4 miles per minute for the time between 30 and 40 minutes, 5 miles per minute for the time between 40 and 50 minutes, and 6 miles per minute for the time between 50 and 60 minutes.
P2.3: Our total distance travelled will be: (1 × 10) + (2 × 10) + (3 × 10) + (4 × 10) + (5 × 10) + (6 × 10); that is, 210 miles.

Configuration 3 (EC3): Variable velocity

P3.1: Let's consider a car that is moving at each time t at velocity 2t miles per minute.
P3.2: That is, at 1 minute, it is traveling at a speed of 2 miles per minute; at 2 minutes, it is traveling at 4 miles per minute; and so forth.
P3.3: The strategy we are going to follow is to underestimate the distance travelled, then overestimate it and affirm that the distance will be between these two estimates.
P3.4: Every minute the car accelerates smoothly, so the velocity at the beginning of the minute is less than at the end.
P3.5: Since we already know how to calculate the covered distance by a car jumping in each interval, we calculate the distance travelled of two "jerky" cars, one with the velocity of our car at the beginning of the interval and another with the velocity at the end of the interval.
P3.6: In the first 3 minutes, if we divide each minute into 0.5-minute intervals, the covered distance by the delayed "jerky" car is 7.5 miles and the advanced "jerky" car 10.5 miles.

P3.7: Let's try to break down the time into intervals 1/10th of a minute long. Once again, we can get an underestimate distance (8.7 miles) and an overestimate distance (9.3 miles).
P3.8: The correct answer would be somewhere between those two estimates. The smaller the intervals, the more accurate will be the approximation to distance travelled.
P3.9: The exact distance can't be found with any single division of the time interval. It is obtained by looking at the infinitely many increasingly improved approximations.
P3.10: The finer approximations get closer and closer to a single value—the limit of the approximations.
P3.11: This infinite process is the second fundamental idea of Calculus—the integral. If we know the velocity of a car at every moment in the interval of time, then the integral tells us how far the car travelled during that interval.

Configuration 4 (EC4): Space as a function of time and variable velocity

P4.1: In the example above where the speedometer always reads 2t, where are we after 1 minute, 2 minutes, 2.5 minutes, 3 minutes?
P4.2: In each case, we will use this infinite procedure to see how far we travelled from 0 to these times.
P4.3: Let's look at the table and see if there is a pattern.

Instantaneous velocity: $v(t) = 2t$

Time (min)	Distance
0.5	0.25
1	1
1.5	2.25
2	4
3	9

Position: $p(t) = t^2$

P4.4: We see that at any time t the odometer will mark the distance t2: the distance travelled is, thus, the square of the time interval taken, or p(t) = t2.

Configuration 5 (EC5): The integral as a limit of sums

P5.1: The integral process involves dividing the interval of time into small increments, seeing how far the car would have travelled if it had gone at a steady velocity during each small interval of time, and then adding up those distances to approximate the total distance.

P5.2: Therefore, the formula to determine the distance travelled between time a and b is: $(v(a) \times \Delta t) + (v(a + \Delta t) \times \Delta t) + (v(a + 2\Delta t) \times \Delta t) + \cdots + (v(b - \Delta t) \times \Delta t)$, as$\Delta$t becomes increasingly smaller.
P5.3: By taking smaller subdivisions and taking a limit, we arrive at the actual value of the integral.

Analysis of Practices: Intentionality, Objects, and Processes

Below we include the detailed analysis of the network of practices, objects, and processes identified in each of the above configurations where Starbird introduces the intuitive meaning of Riemann integral. In the below tables from 1 to 5, the first column indicates the groups of practices (epistemic configurations) mentioned above, the second column identifies the usage or intentionality of each practice, columns 3 and 4 describe the types of objects (languages, concepts, propositions, procedures, and arguments) and the involved processes in the practices or groups of practices. In the OSA, every sequence of practices is considered as a mathematical process, and 16 basic processes are proposed (Figure 1). We consider it important to become aware of the processes of interpretation, particularization/generalization, algorithmization, argumentation, among others. The mega-process of problem-solving is made up of the sequence of mathematical practices described above; this mega-process can be broken down into more basic processes, focusing on the sequences of elementary practices that are being carried out.

Table 1
EC1. Onto-semiotic configuration: Constant velocity

Practice	Use/ intentionality	Objects	Processes
P1.1 P1.2 P1.3	Pose a simple case of uniform motion.	Language: natural, numerical and graphic. Concepts: uniform movement; magnitudes distance, time, speed; units of measure, miles, hours, minutes; function *v(t)* = constant.	*Interpretation:* - The general statement of the problem cannot be solved unless a hypothesis about the car velocity is introduced. - Meaning is assigned to the terms used to describe uniform motion: constant velocity; measurement

Table 1 (Continued)
EC1. Onto-semiotic configuration: Constant velocity

	State and solve a case of uniform movement.	Cartesian graph; constant function. Proposition: In one hour the distance covered will be 120. Procedure: arithmetic calculation of the distance covered. Proposition: Constant velocity will appear as a horizontal line. Arguments: in uniform movement $p(t) = vt$.	units of the magnitudes velocity, time and space. - Graphic is related to the Cartesian representation of functions; the graphical representation of the constant function is a line parallel to the abscissa axis. *Particularization:* - The movement is supposed to be uniform, the speed is constant. - Give specific values to velocity. *Algorithmization/Calculation:* Sequence of arithmetic calculations to find the miles covered in the given conditions. *Argumentation* (implicit): In uniform motion, the space covered is obtained by multiplying the constant velocity by the elapsed time.

The practices in EC1 seek to solve an elementary case of calculating the distance in a uniform motion. This is particularized to some velocity values, and the miles covered are calculated in simple and specific situations. The argumentation appears implicit, supported by the knowledge of the physical conditions that characterize the uniform motion.

In CE2, the procedures used in the EC1 configuration are "locally" applied by assuming that velocity is now piecewise constant. EC1 has gained entity as a new mathematical object that can be involved as an argument or technique in a new configuration.

Table 2

EC2. Onto-semiotic configuration: Piecewise constant velocity

Practice	Use/ intentionality	Objects	Processes
P2.1 P2.2 P2.3	State and solve a more general case; uniform piecewise motion. First encounter with Riemann-type sums.	Language: natural and numerical. Concepts: piecewise constant function. Procedure: arithmetic calculation of the distance covered. Proposition: Total distance covered is 210 miles. Arguments: In each section the motion is uniform.	*Generalization*: from constant velocity to piecewise constant velocity. *Algorithmization/calculation*: sequence of arithmetic calculations. *Argumentation*: In uniform motion $e=vt$; distance is an additive magnitude.

Table 3

EC3. Onto-semiotic configuration: Variable velocity

Practice	Use/ intentionality	Objects	Processes
P3.1	State the case of variable velocity when $v(t) = 2t$.	Concepts: continuous function, linear function $v=2t$; Estimation by defect and excess of the distance; acceleration.	*Generalization*: From piecewise constant velocity it is passed to linear velocity (continuous).
P3.2	Explain the function $v(t) = 2t$.	Language: natural, symbolic.	*Particularization*: From the linear function to specific velocity values for some time values.
P3.3	Describe the approximate distance calculation procedure fixing upper and lower limits.	Proposition: the relationship between time, t, and velocity $v=v(t)$, is $v(t)=2t$.	
P3.4	Set the car acceleration.	Proposition: The velocity at the beginning of the .	*Argumentation*: explanation of the problem and the procedure to follow.

Table 3. (Continued)
EC3. Onto-semiotic configuration: Variable velocity

Practice	Use/ intentionality	Objects	Processes
		minute is less than at the end. Argument: it is assumed that the car accelerates smoothly, the function $v(t)=2t$ is increasing.	
P3.5	Description of the approximation procedure.	Concept: initial speed, final speed, discontinuous speed change; time interval; interval width. Procedures: arithmetic calculation of the distance covered.	*Algorithmization*: Calculation of estimates for excess and defect of the distance covered. *Particularization*: Obtaining the distance for two subdivisions of the intervals.
P3.6	Present and solve a case of approximation as an illustrative example.		
P3.7	Idem, with finer approximation.		
P3.8	Set the approximation for the solution, exact distance covered.	Propositions: p1: the actual distance is between two estimates; p2: the estimation is more precise when the intervals are smaller.	*Enunciation*: lower and upper bound of the exact distance covered, precision of the bound.
P3.9 P3.10	Identify the relationship between the exact distance and the limit of the approximations.	Concept: limit value of approximations. Proposition: p3: the sequence of estimates approaches a single limit value Arguments: Practices P3.5, P3.6, and P3.7 plausibly justify.	*Enunciation*: distance as the limit of approximations. *Argumentation*: deductive based on previous practices.

Table 3. (Continued)
EC3. Onto-semiotic configuration: Variable velocity

Practice	Use/ intentionality	Objects	Processes
		propositions p1, p2, and p3.	
P3.11	Introduce the integral.	Concept: integral as the limit value of an infinite process.	*Definition*: The distance covered in a interval of time as an integral of the velocity in said interval.

In EC3 the velocity is directly proportional to time, $v(t) = 2t$. It changes from a constant piecewise function (with jump discontinuities) to a continuous function. The function growth allows us to calculate estimates by default (considering the speed at the beginning of each interval) and by excess (considering the final speed in each interval). In the practice P3.5, it is said that "since we already know how to calculate the distance covered by a car jumping rapidly in each interval", that is, EC2 is involved in this new configuration as a process that is applied in a particular way in each one of these estimates. P3.9 states the "need to look at infinite approximations." The improvement of these approximations supposes smaller and smaller divisions of the intervals, so that they "progressively" get closer until the estimates taken by default and by excess are indistinguishable. The integral appears as a "fundamental idea of Calculus after this infinite process."

In EC4 the generalization must allow obtaining the distance $p(t) = t^2$ as a primitive of the velocity as a function of time. Again, the "infinite procedure" established in EC3 is used to obtain the values that allow "deducing a pattern". In this configuration we observe two processes of "abusive generalization" that can be a source of potential semiotic conflicts. One of them refers to obtain the distance covered for the five values of time, applying the limit calculation processes previously mentioned; the other, the inference of the general formula of the distance for any value of t, using only 5 pairs of values.

Table 4
EC4. Onto-semiotic configuration: Space as a function of time and variable velocity

Practise	Use/ intentionality	Objects	Processes
P4.1	Pose the problem of calculating the distance for a collection of time values.	Problem: calculation of *p(v, t)* for 4 values of *t*. Languages: tabular and algebraic-functional.	*Problematization*: Find the distance as a function of time. *Algorithmizing:* Calculate 5 values of the space covered when $v=2t$ applying the limits procedure.
P4.2	Relate the solution of P4.1 to the procedure developed in CE3.	Concepts: instantaneous velocity; linear function. Procedure: calculating the distance for 4 times values.	*Representation*: Tabular arrangement of the 5 pairs of function values.
P4.3	Pose the problem of identifying a pattern in the table.	Concepts: quadratic function.	*Problematization*: Analyze the existence of a pattern. *Generalization*: The continuous case is inferred from the discrete case of five pairs of values.
P4.4	State the function that relates space to time	Proposition: $p(t)=t^2$ Argument: Inductive reasoning from the 5 values in the table.	*Enunciation*: the distance covered is the square of the time interval taken. *Argumentation*: Inductive, the expression of the continuous function is stated in the case of 5 pairs of values.

Table 5
EC5. Onto-semiotic configuration: The integral as a limit of sums

Practice	Use/ intentionality	Objects	Processes
P5.1	Describe the whole process in a discursive way.	Concepts: value of the integral; limit, formula.	*Algorithmization*: Regulate the steps to calculate the value of the integral (integral process).
P5.2	Symbolically describe the summation formula to find the distance covered in any finite time interval.	Procedure: product of instantaneous velocities for small time intervals; sum of products; limit calculation.	*Representation*: Symbolic expression of the sequence sums of space covered quantities.
P5.3	Define the value of the integral as a limit of the sums.	Argument: sequence of configurations CE1 - CE4.	*Definition*: The value of the integral is the limit of the sums of the sequences when the subdivisions become progressively smaller.

In the first elementary practice of EC5 it is established what is the essence of the "integration process": 1) split the time interval into small increments; 2) see how far the car would go if it had gone at constant speed during each small time interval; 3) add these distances to approximate the total distance covered. Taking limits when the subdivisions are getting smaller, we get the integral value, which appears linked to the distance covered during the time interval.

Formal-Algebraic Meaning

In this section, we carry out an onto-semiotic analysis of the definite integral meaning usually studied in university Calculus courses, which is characterized by its progressive generalization and formalization achieved with the use of algebraic tools. We choose Stewart's (2016) book widely used. Lesson 5 of this textbook begins with problems about areas and distances which are solved using the Riemann Sum limit calculation, strongly supported by graphical representations. This section serves as a context and foundation for the subsequent introduction of the general definite integral definition (section 5.2, p. 378).

In the following sections, we present a synthesis of the epistemic trajectory developed by Stewart, with the posing and solution of problems and exercises on calculating areas and distances, highlighting the main epistemic configurations developed. Since the analysis of the practices, objects, and processes of these configurations requires excessive space for this article, we have chosen to make this analysis only for the integral general definition, which we include in section 5.2.

Preparing the Definite Integral General Definition: Geometric and Kinematic Meaning

To understand the validity and usefulness of the generality with which the integral definition is presented, it is necessary to start from specific problems whose resolution brings into play the mathematical practices condensed in the definition. It is necessary to understand the *raison d'être*, the origin and motivation of the normative practice that constitutes the integral Definition as a whole. For this reason, Stewart's (2016) book includes section 5.1, which addresses two types of problems: calculation of areas, and distance traveled by a moving object.

The area calculation problem

The calculation of areas of plane surfaces limited by curves is proposed, using as an example the function $f(x) = x^2$ in the interval [0, 1] (Figure 2a).

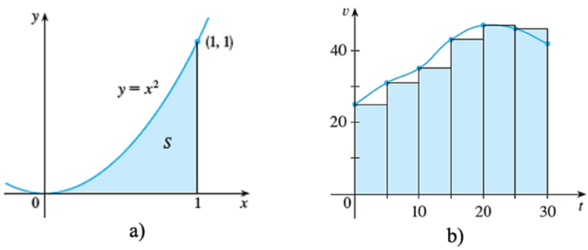
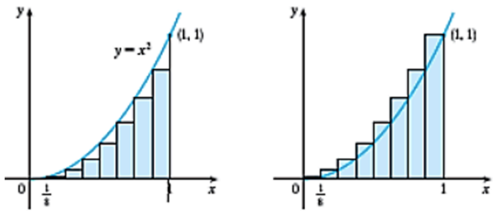


Figure 2. a) Area under a curve; b) Distance traveled

In Figure 3 is observed that as the number n of rectangles increases, both the sums of the areas of the approximation rectangles L_n (whose heights are taken at the left endpoints), and the sums of the areas of the approximation rectangles R_n (whose heights are taken at the right endpoints) are increasingly better approximations of the area of the region S under the graph. This generalization process is supported by graphical representations and tables with numerical estimates of areas by default and excess.



a) default b) excess

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

c) approximation sequences

Figure 3. Approximating rectangles

This allows to formulate the definition of the area A of the region S that lies under the graph of a continuous function f as the limit of the described sums (Figure 4).

2 Definition The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} [f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x]$$

Figure 4. Definition of the area of a region (Stewart, 2016, p. 371).

The existence of the limit "can be demonstrated" from the continuity of the function f; in the same way that it "can be demonstrated" that the same value is obtained if R_n is replaced by L_n.

The distance problem

The aim is to find the distance travelled by a moving object in a time period if the velocity of the object is known as a function of time (Figure 2b). In this case, after several examples, solved with arithmetic procedures and supported by their Cartesian representation (Figure 2b), the author proposes to observe the similarity of the sums to calculate the distance and the sums to calculate the areas. Now the area of each rectangle should be interpreted as a distance because the height represents velocity, and the width represents the time.

After the discussion of some examples, explained in arithmetic language, given that finite sets of particular numbers and arithmetic operations are used, Stewart proceeds to the general case *v=f(t)*, where $a \leq t \leq b$, and *f(t)*≥0, making it plausible that the exact distance travelled d is the limit of the expression, $d = \sum_{i=1}^{n} f(t_{i-1})\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t$.

Comparing the expressions of the area and distance calculations, it follows that the distance travelled is equal to the area under the graph of the velocity function. Later, Stewart considers the case of other relationships between magnitudes of interest in the natural and social sciences fields, which can be interpreted as the area under a curve.

Due to space limitations, it is not possible to make a detailed analysis of the mathematical practices described in this section like the one carried out in section 4. We apply that microscopic analysis of mathematical activity to the general definition of the integral as a Limit of Riemann Sums, enunciated by Stewart as the final milestone of some concepts and procedures introduced in section 5.1. In this case, the onto-semiotic analysis is applied to a text that does not correspond to the process of solving a problem, but to the process of the general definition of a mathematical concept.

Onto-Semiotic Analysis of the Integral Definition as a Riemann Sum Limit

The mathematical practice of the definition in Figure 5 can be broken down into more basic practices. In these practices, different types of (ostensive and non-ostensive) objects intervene, interrelated by means of semiotic functions, each of which constitutes specific knowledge. The recognition of the network of semiotic functions reveals the complexity of the required interpretation and performance processes, as well as the logic and intentionality of each action.

2 Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \ldots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Figure 5. Definite integral definition (Stewart, 2016, p. 378)

Next, we break down the definition (Figure 5) into elementary practices:

P0: Definition of a definite integral

P1: If f is a function defined for $a \leq x \leq$ b,

P2: We divide the interval [a,b] into n subintervals of equal width Δx =(b-a)/n

P3: We let $x_0(=a)$, x_1, x_2, . . . , x_n(=b) be the endpoints of these subintervals

P4: and we let x^*_1, x^*_2, . . . , x^*_n be any sample points in these subintervals, so x^*_i lies in the ith subinterval [x_{i-1}, x_i].

P5: Then the definite integral of f from a to b is $\int_a^b f(x)dx = \sum_{i=1}^{n} f(x_i^*)\Delta x$

P6: provided that this limit exists and gives the same value for all possible choices of sample points.

P7: If it does exist, we say that f is integrable on [a, b].

The intention of P0 is to name the sequence of practices that follow and classify it as a definition / rule. We consider practices P1, P2, P3 and P4 as an epistemic configuration of the general definition (EC1-DG), which have a preparatory role for the definition itself, done in practices P5, P6 and P7 (EC2-DG). Tables 6 and 7 show these elementary practices (column 1) related to their use or intentionality (column 2), the objects (column 3) and processes put at stake (column 4).

Table 6

EC1-DG Onto-semiotic configuration: Function, interval, partition, and sample points

Practice	Intentionality	Objects	Processes
P1	Establish a condition for the type of functions to be considered and represent it with *f*.	Languages: natural, symbolic. Concepts: function; real number interval; domain; variable.	*Interpretation*: -The term function refers to the concept of real function of real variable. - The domain of the function is the interval between two fixed but arbitrary real numbers, *a* and *b*. *Representation*: *f* designates any function of real variable; *x* the independent variable; $a \leq x \leq b$ represents the domain of the function.
P2	Set the interval width of the partition.	Concepts: interval extreme points, interval width. Procedure: Determination of the subinterval's amplitude.	*Interpretation*: A generic range of real numbers is assigned as content (meaning) to the expression [a, b]; The letters *a* (origin) and *b* (end) name the interval endpoints. Δx designates the amplitude of any subinterval; it intervenes as a variable. Splitting (the interval) into *n* subintervals means breaking it down into *n* disjoint parts. The letter *n* refers to the finite number, but any, in which the interval is broken down. *Definition*: $\Delta x =(b-a)/n$ *Representation*: Assignment of symbols to the interval endpoints, number of intervals and interval width.
P3	Assign a symbol to the endpoints of the subintervals of the partition.	Language: symbolic representation of points. Concepts: endpoints of subintervals; finite sequence of points. Procedure: choice of partition.	*Interpretation*: Endpoints of the subintervals. An interval is given by two points, lower (left) and upper (right). Because the subintervals are contiguous, the upper one matches the lower one, except for the first subinterval: $x_0(=a)$ and the last subinterval: $x_n(=b)$. *Representation*: Assignment of symbols $x_0, x_1, x_2, \ldots, x_n$ to the sequence of points that split the interval.
P4	Assign a symbol to each sample point of the intervals; Remember what sample point	Language: symbolic representation of a generic sample point. Concept: sample point.	*Definition*: Sample points in each interval refer to specific interior points in each interval. *Interpretation*:

Table 6. (Continued)

EC1-DG Onto-semiotic configuration: Function, interval, partition, and sample points

P4	means.	Procedure: choice of sample points in each interval.	- The sample points will represent the set of points of each subinterval. - Each sample point x_i^* belongs to the interval $[x_{i-1}, x_i]$. *Representation*: The symbols $x_1^*, x_2^*, \ldots, x_n^*$ refer to particular points of the *n* subintervals.

The subject who reads and understands the definition of definite integral (Figure 5) must mobilize a system of prior knowledge, for which he/she needs to relate the different objects involved in the practices and recognizing the role that each one plays in the definition process. In the network of processes recognized in Tables 6 and 7, those of interpretation/meaning are particularly noteworthy. These processes involve knowledge that an epistemic subject (expert or ideal) implicitly puts at stake when read, interpret, and understand the definition.

Table 7

EC2-DG Onto-semiotic configuration: Definition of definite integral and integrable function

Practice	Intentionality	Objects	Processes
P5	Define the definite integral of any function *f*, as the limit of a sum of products.	Language: symbolic. Concepts: definite integral; value of a function at one point; subinterval width; product; sum and limit.	*Interpretation*: -The expression $f(x_i^*)$ refers to the value of the function *f* at the sample point x_i^* of the interval *i*. -The expression Δx refers to the constant width of each subinterval given in P3. - The expression $f(x_i^*)\Delta x$ refers to the product of the value of the function at the sample point *i* bythe interval width.

Table 7 (Continued)

EC2-DG *Onto-semiotic configuration: Definition of definite integral and integrable function*

Practice	Intentionality	Objects	Processes
			-The expression $\sum_{i=1}^{n}$ refers to the sum of the *n* products. - The expression refers to the limit calculation of the summation sequence in *n*. $\int_a^b f(x)dx$ designates the value of the integral of the function *f* in the interval [*a,b*]. The resulting value of the limit is assigned to the definite integral, a symbol that intervenes as a variable used as a receiver of values. *Representation*: Assignment of symbols to the definite integral, limit and summation. *Algorithmization*: Products; sum of product; calculation of the limit of the sums sequence. *Definition*: The integral is defined as the number resulting from the calculation of the limit.
P6	Establish conditions on the definition validity.	Language: natural Concepts: limit; convergence sequence; sample point.	*Interpretation*: - Existence of the limit, refers to the fact that $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is a convergent sequence of natural numbers. - It gives the same value for all the possible choices of sample points; it refers to the fact that the value of the limit does not depend on the points at which the function is evaluated.
P7	Assign a name to the functions that meet a condition.	Concept: integrable function; interval; existence of the limit.	*Definition*: Function integrable in an interval.

Throughout the *definition process*, the alphabetic symbols *a* and *b* intervene as parameters: the real number to be designated as an integral of the function *f* depends in each case on the value assigned to the interval endpoints. The terms interval, subinterval and amplitude are related to their respective concepts/rules.

We find a *generalization process* related to the set of proposed and solved examples in section 5.1 of the book (Stewart, 2016, pp. 366-378). These refer to situations in which magnitudes, quantities, measurements and correspondences between magnitudes intervene. However, the general definition of integral refers to any function *f* of real variable; hence, the magnitudes disappear, and the correspondence is established between numbers.

In practice P6 a semiotic conflict may arise in the meaning and representation of the definite integral value of a function *f* in the interval [*a, b*]: the symbol *dx* has no previous referent. It could be expected to represent (for instance) this number as *S (f, a, b)*; it is a value that depends on the function *f* and the parameters *a* and *b*, extremes of integration.

In this system of practices, the integral is defined as a number, obtained as result of the application of the rule whose justification refers to the system of discursive, operative and normative practices previously included in section 5.1.

Discussion and Connections

In this section we synthesize the contributions made with the onto-semiotic analysis of the integral meanings carried out in sections 4 and 5, relating them to the insights proposed by other authors applying various theoretical tools. The analysis of the intuitive meaning (section 4) reveals that a first encounter with the integral is possible by selecting an introductory problem-situation, solvable with a sequence of practices involving arithmetic objects and procedures. It is inevitable, however, to look at the infinite process of obtaining the limit of the sum of products sequence of infinitesimal quantities.

However, the progression in the efficiency of mathematical work by future university professionals requires mastering the algebraic tool to express the generality of mathematical concepts and procedures and operate syntactically with such generality. The onto-semiotic analysis of the integral formal-algebraic meaning, established in the definition as limit of Riemann Sums (section 5), is also revealed as useful to become aware of the knowledge, skills and understanding acts involved.

The methodology of analysis of a mathematical text that we have exemplified in sections 4 and 5 applying the OSA theoretical tools provides a microscopic level of analysis that allows us to identify some epistemic and semiotic-cognitive facts of didactic interest, as well as some *epistemic strata*, that is, institutional knowledge that should have been previously studied, which usually goes unnoticed in the teaching processes. In both cases, we have been able to identify potential semiotic conflicts related particularly to the generalization processes that are carried out, often implicitly, in the process of defining the integral.

To recognize that a first encounter with the definite integral can be implemented by relying on non-algebraic means of expression is an epistemic event of educational interest. However, our analysis reveals that even in this intuitive approach, discursive practices that involve semiotic questioning on the part of the reader could be required. For example, in practice P5.2 (EC5, Section 4), Starbird abruptly introduces a general algebraic expression of the sum of increments of the distance covered between any time interval [*a, b*], that contrasts with the previous presentation made with natural language and particular numerical values. The reader unfamiliar with this symbolism to express the generality of the resolution process will have to question the meaning of each term of the expression.

In the case of the study of the integral proposed by Stewart (2016), Section 5, we also find a similar semiotic problem. For example, the solution to the problem of calculating the area under a curve is a number expressed as

$$A = R_n = [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x] =$$
$$L_n = [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x]$$

and the solution to the calculation of the distance traveled is also a number represented as

$$d = \sum_{i=1}^{n} f(t_{i-1})\Delta t = \sum_{i=1}^{n} f(t_i)\Delta t.$$

Furthermore, the reader should accept the existence and equality of such limits.

In the general definition of the integral (Figure 5) this number is expressed with a complex notation in which the reader have to understand that the symbol *dx* has no meaning by itself and that the expression, $\int_a^b f(x)dx$ must be seen as a single symbol to represent "a number that does not depend on *x*".

Students will read about Riemann Sums, limits, derivatives, area, and many other concepts while they are learning definite integral. In order to have a good understanding of the definite integral, students should be able to make connections between all these concepts. The research on definite integrals found that student's knowledge was limited to procedural knowledge, but they had difficulty making connections between different representations of the definite integral (Serhan, 2015).

Our analysis helps to reveal the complex network of objects involved in operative, discursive and normative mathematical practices that are inherent in mathematical activity. In these practices, not only conceptual and procedural objects intervene, but also problems that contextualize and give meaning to all mathematical activity; representations of various kinds (symbolic, graphic, diagrammatic, natural language, etc.); propositions that relate the concepts and synthesize the results that respond to problems; arguments of a descriptive, explanatory nature or those that justify procedures and propositions. Special attention is also required to mathematical processes, interpreted in terms of sequences of practices aimed at achieving the objectives of the mathematical activity. Thus, in the general process of problem solving, more specific processes are distinguished, such as representation, interpretation, definition, algorithmization, enunciation, argumentation, generalization, particularization, etc. In the practices analyzed in sections 4 and 5, we observe that not all these processes have the same presence and that their complexity could be also an explanatory factor for the various potential students' learning difficulties.

The onto-semiotic analysis exemplified here can complement the study carried out by other authors using different theoretical tools. The definite integral structure model elaborated by Sealey (2014) is undoubtedly useful and relevant for understanding the students' understanding difficulties. We consider that the analysis accomplished in sections 4 and 5 complements the Sealey model in two directions. On the one hand, in addition to the Orienting layer, related to the understanding of the magnitudes involved and the relationships with the problem contexts, there is a network of previous mathematical knowledge - linguistic, conceptual, procedural and argumentative - that should be considered in the instructional process. Furthermore, understanding each layer that Sealey describes also brings into play processes and objects that the onto-semiotic analysis reveals, which can be the source of potential semiotic conflicts.

We also consider that the onto-semiotic tools that we have presented are compatible with the notions of concept definition and concept image (Rasslan & Tall, 2002; Grundmeier, Hansen, & Sousa, 2006; Serhan, 2015), although they could broaden and deepen the analysis of the concepts, limitations and learning biases of the integral concept. The image concept is interpreted in the OSA as the personal meaning of the subject about a concept, which implies adopting a systemic and pragmatic perspective on mathematical knowledge. This meaning is described by the cognitive (or personal) onto-semiotic configuration, which involves a type of problem, the system of practices for its solution and the network of objects and processes involved in the practices. The concept definition is one of the objects of the configuration, such definition is a norm that regulates the operative and discursive practices that are carried out to solve the corresponding type of problems. The activity and mathematical knowledge analysis is done both from a personal or cognitive perspective, as well as from an institutional or epistemic perspective; consequently, one should also speak of the institutional "concept image", that is, the institutional meaning of the integral, also understood as an onto-semiotic configuration. Moreover, for the integral and any mathematical concept it is possible to find, not a single definition, but there are different definitions, each one linked to certain contexts or types of problems whose resolution brings into play different systems of practices, and, therefore, different meanings (senses).

The distinction between conceptual and procedural knowledge, introduced in mathematics education in the work of Hiebert and Lefevre (1986), has been used by Serhan (2015), among other authors, in the case of learning the integral concept. We consider that the notion of pragmatic meaning proposed by the OSA for a concept, understood systemically as an onto-semiotic configuration, develops the notion of conceptual and procedural knowledge. This is because of the onto-semiotic configuration shows the elements that intervene in the conceptual and procedural components, as well as the relationships that exist between them. The onto-semiotic analysis that we have done in sections 4 and 5 shows, by means of two specific examples, the pieces of information about the integral that should be interconnected in order to affirm that a subject has conceptual and procedural knowledge. It is important to explain the argument component, as well as the processes of interpretation, representation, generalization, etc., that are put at stake in each case, which are implicit or poorly developed in the Hiebert and Lefevre model.

Final Thoughts

The onto-semiotic analysis applied to two meanings of the integral has revealed the complex network of objects and processes involved in mathematical activity, the knowledge that we consider useful for managing the teaching and learning of the integral. We have shown that the onto-semiotic approach to mathematical knowledge makes it possible to connect two complementary perspectives on mathematics: as a problem-solving activity for people, and as a system of objects and processes that regulate and emerge from this activity. It also considers the personal (cognitive) and institutional (epistemic) duality of mathematical knowledge, which is fundamental for the study of phenomena related to the teaching and learning of mathematics.

According to Serhan (2015), it is important for instructors to review the way definite integral is presented and taught in class. It is necessary to make more emphasis on the multiple representations, their connections, and how students may use the Riemann Sum to enhance their structural understanding of the definite integral.

Although it is a laborious analysis that requires a certain mastery of theoretical tools, we consider that researchers and teacher educators should be trained to carry out similar analysis of teaching content, at least the essential concepts and procedures of each subject. For each partial meaning of the object, and each resolution of the prototypical tasks that characterize it, it is necessary to identify the network of objects and processes involved, in order to be able to plan the teaching, manage interactions in the classroom, understand difficulties, and evaluate students' learning levels. The identification of the objects and processes involved in mathematical practices is a teaching competence that will allow us to understand the learning progression, manage the necessary institutionalization processes, and assess the students' mathematical competences.

It would be necessary to continue this research work with other meanings of the integral, in particular, the Fundamental Theorem of Calculus, which leads us to understand the integral as the incremental difference of the cumulative function. It is also necessary to broaden the study in three directions: deepen the articulation of the onto-semiotic analysis with the contributions of other theoretical frameworks, apply the OSA tools to the study of the students' personal meanings of the integral and analyze the impact of the use of the technology in mathematical practices for solving integral Calculus problems.

References

Attorps, I., Björk, K., Radic, M. & Tossavainen, T. (2013). Varied ways to teach the definite integral concept. *International Electronic Journal of Mathematics Education, 8*(2-3), 81-99.

Borji, V. & Font, V. (2019). Exploring students' understanding of integration by parts: a combined use of APOS and OSA. *EURASIA Journal of Mathematics, Science and Technology Education, 15*(7), em1721. https://doi.org/10.29333/ejmste/106166.

Bressoud, D., Ghedams, I., Martinez-Luances, V., & Törner, G. (2016). *Teaching and learning of Calculus.* Springer.https://doi.org/10.1007/978-3-319-32975-8_1

Brousseau, G. (1997). *Theory of didactical situations in Mathematics*.Kluwer.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques, 12*(1), 73-112.

Contreras, A., & Ordóñez, L. (2006). Complejidad ontosemiótica de un texto sobre la introducción a la integral definida. *Relime, 9*(1), 65-84.

Contreras, A., Ordóñez, L. & Wilhelmi, M. R. (2010). Influencia de las pruebas de acceso a la universidad en la enseñanza de la integral definida en el bachillerato. *Enseñanza de las Ciencias, 28*(3), 367-384. https://doi.org/10.5565/rev/ec/v28n3.63

Crisóstomo, E. (2012). *Idoneidad de procesos de estudio del cálculo integral en la formación de profesores de matemáticas: Una aproximación desde la investigación en didáctica del cálculo y el conocimiento profesional*. Tesis doctoral. Departamento de Didáctica de la Matemática. Universidad de Granada.Retrieved from: http://www.ugr.es/~jgodino/Tesis_doctorales/Edson_Crisostomo_tesis.pdf

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In Tall, D. (Eds.), *Advanced mathematical thinking* (pp. 95–123). Kluwer A. P. https://doi.org/10.1007/0-306-47203-1_7

Duval, R. (1995). *Sémiosis et pensée: registres sémiotiques et apprentissages intellectuels* [Semiosis and human thought. Semiotic registers and intellectual learning]. Peter Lang.

Font, V., Godino, J. D. & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*, 97–124. https://doi.org/10.1007/s10649-012-9411-0

Godino, J. D. & Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in Mathematics Education. In: A. Sierpinska & J. Kilpatrick (Ed.), *Mathematics education as a research domain: A search for identity* (pp. 177-195). Dordrecht: Kluwer, A. P.

Godino, J. D., Batanero, C. & Font, V. (2019). The onto-semiotic approach: implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39* (1), 37- 42. EID: 2-s2.0-85073318621

Greefrath, G, Oldenburg, R., Siller, H. S., Ulm, V. & Weigand, H. G. (2016). Aspects and “Grundvorstellungen” of the concepts of derivative and integral subject matter-related didactical perspectives of concept formation. *Journal Mathematik Didaktik,Suppl 1*, 99-129. DOI 10.1007/s13138-016-0100-x

Grundmeier, T. A., Hansen, J., & Sousa, E. (2006). An exploration of definition and procedural fluency in integral Calculus. *PRIMUS, 16*(2), 178-191.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education, 32*, 195–222.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Lawrence Erlbaum Associates.

Jones, S. R. (2013). Understanding the integral: Students’ symbolic forms. *Journal of Mathematical Behavior, 32*(2), 122–141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S, R. (2015). Areas, anti-derivatives, and adding up pieces: definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behaviour, 38*, 9–28. https://doi.org/10.1016/j.jmathb.2015.01.001

Kouropatov, A., & Dreyfus, T. (2013). Constructing the integral concept on the basis of the idea of accumulation: Suggestion for a high school curriculum. *International Journal of Mathematical Education in Science and Technology, 44,* 641–651.

Kouropatov, A. & Dreyfus, T (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM Mathematics Education, 46*, 533–548.

Marton, F., Runesson, U., & Tsui, A. (2004). The space of learning. In F. Marton and A.Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-40). Lawrence Erlbaum Associates, INC Publishers.

Mateus, E. (2016). Análisis didáctico a un proceso de instrucción del método de integración por partes. *Bolema, 30*(55), 559-585. https://doi.org/10.1590/1980-4415v30n55a13.

Ordoñez, L. (2011). *Restricciones institucionales en las matemáticas de 2º de bachillerato en cuanto al significado del objeto integral definida*. Tesis doctoral. Departamento de Didáctica de las Ciencias. Universidad de Jaén.Retrieved from: http://enfoqueontosemiotico.ugr.es/documentos/Tesis_doctoral_Lourdes_Ordo%C3%B1ez.pdf

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics. 14*, 1-18. https://doi.org/10.1007/bf00704699

Peirce, Ch. S. (1958). *Collected papers of Charles Sanders Peirce. 1931-1935*. Harvard UP.

Pino-Fan, L., Font, V., Gordillo, W., Larios, V., & Breda, A. (2018). Analysis of the meanings of the antiderivative used by students of the first engineering courses. *International Journal of Science and Mathematics Education, 16*(6), 1091–1113. https://doi.org/10.1007/s10763-017-9826-2

Rasslan, S. & Tall, D. (2002). Definitions and images for the definite integral concept. In: A. D. Cockburn & E. Nardi (eds.) *Proceedings of the 26th Conference PME*, Norwich, 4, 89-96.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *Journal of Mathematical Behavior, 33*, 230– 245. https://doi.org/10.1016/j.jmathb.2013.12.002

Serhan, D. (2015). Students' understanding of the definite integral concept. *International Journal of Research in Education and Science,* 1(1), 84-88. https://doi.org/10.21890/ijres.00515

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press. https://doi.org/10.1017/cbo9780511499944

Starbird, M. (2006). *Change and motion: Calculus made clear*, 2nd Edition. The Teaching Company.

Stewart, J. (2016). *Calculus. Early transcendentals*. Cengage Learning.

Tall, D. O. (2004). Building theories: the three worlds of mathematics. *For the learning of mathematics, 24* (1), 29-32. https://doi.org/10.1017/cbo9781139565202.011

Tall, D., O. & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169. https://doi.org/10.1007/bf00305619

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in Calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Mathematical Association of America. https://doi.org/10.5948/upo9780883859759.005.

Wittgenstein, L. (1953). *Philosophical investigations*. The MacMillan Company.

Maria Burgos is senior lecturer at the Granada University, Spain.

Seydel Bueno is lecturer at Camaguey University, Cuba.

Juan D. Godino is retired university lecturer of mathematics education at Granada University, Spain.

Olga Pérez is lecturer at Camaguey University, Cuba.

Contact Address: Direct correspondence concerning this article, should be addressed to the author. **Postal Address:** Facultad de Ciencias de la Educación, Campus Universitario de Cartuja C.P. 18071 (Granada) Granad**a Email:** mariaburgos@ugr.es