CONTRIBUCIONES CIENTÍFICAS
EN HONOR DE MIRIAN ANDRÉS GÓMEZ
(Laureano Lambán, Ana Romero y Julio Rubio, editores),
Servicio de Publicaciones, Universidad de La Rioja,
Logroño, Spain, 2010.

MONOMIAL ITERATED MAPPING CONES

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Dedicado a la memoria de Mirian Andrés Gómez, con muchísimo cariño

Resumen. En este artículo revisamos el uso de conos iterados, una herramienta estándar para cálculos homológicos en álgebra conmutativa, en el contexto de ideales monomiales. Enumeramos los resultados obtenidos por diferentes autores en este tema y tratamos algunas de las cuestiones computacionales.

Abstract. We review the use of iterated mapping cones, a standard tool for homological computations in commutative algebra, in the context of monomial ideals. We list some results obtained by different authors on this topic and adress some of the computational issues.

1. Introduction

Iterated mapping cones are a standard tool in the computation of free resolutions of modules and ideals in the polynomial ring. They have been used both for the theoretical and computational aspects of the problem. In the monomial case, several well known resolutions arise as iterated mapping cones, e.g. the ones presented by Taylor [21], Eliahou-Kervaire [8], etc. Also, the minimal free resolution of some families of monomial ideals are of this form, e.g. stable, squarefree stable among others.

The technique of *Mayer-Vietoris trees* allows the study of the support of mapping cone resolutions in the monomial case. This leads to simple arguments to show that the resolution of certain families of ideals arise as iterated mapping cones, e.g. Ferrers, generalized k-out-of-n, series-parallel, etc [16, 17]. Working with these trees we can also perform some reductions on mapping cone resolutions in order to obtain minimal free resolutions of further families of ideals [18, 19].

Computationally, iterated mapping cones have had several problems due to the lack of simple methods to construct the neccesary elements of each step of the iterative process [11, 20]. The techniques of *effective* homology allow us to overcome this problem and produce effective algorithms to compute these cones [15].

Key words and phrases. Monomial ideal, iterated mapping cone, free resolution, Betti numbers.

2. Preliminaries and basic concepts

2.1. Minimal free resolutions and Betti numbers of monomial ideals. The general goal of a big part of the research in commutative algebra is to understand modules over the polynomial ring (recall that ideals are a special kind of modules). The easiest modules to understand are *free* modules, which are just direct sums of copies of the ring. A useful way to understand more complicated modules consists in assigning to them a particular collection of free modules and module morphisms, this is what is called a free resolution. In the case of graded (resp. multigraded) modules over the polynomial ring, we have the concept of graded (resp. multi graded) resolutions, and in this context, a concept of *minimal* (multi) graded resolution is available. The minimal free resolution of an ideal is a very good way to understand the ideal. Unfortunately, although very easy to define, minimal free resolutions are not always easy to obtain and this difficulty has led to a great amount of research on this very active topic.

The definition of minimal graded free resolution is the following:

Definition 1. *Let $R = \mathbf{k}[x_1, \dots, x_n]$ be the polynomial ring in n variables, and let $M = \bigoplus_{i\in\mathbb{N}} M_i$ a graded R-module. A minimal free resolution of M is an exact sequence*

$$\cdots \to F_{i+1} \stackrel{\phi_{i+1}}{\to} F_i \to \cdots \to F_1 \stackrel{\phi_1}{\to} F_0 \to M \to 0$$

Where:

- *All the F_k are finitely generated free modules of the form $F_k = \bigoplus_{i\in\mathbb{N}} R(-i)^{b_{i,k}}$, where $R(-i)$ is a copy of R with a shift in the graduation, i.e. the jth graded component of $R(-i)$ is R_{j-i}.*
- *All the morphisms ϕ_k are degree preserving maps.*
- *For each k, the homogeneous homomorphism ϕ_k maps the canonical basis of F_k to a minimal homogeneous system of generators of $\ker(\phi_{i-1})$.*

Minimal free resolutions exist for every graded module over the polynomial ring and are *unique* up to chain complex isomorphism, hence, the $b_{i,k}$'s in Definition 1 depend only on the module, they are called the graded *Betti numbers* of M (observe that only a finite number of them are non-zero). The kth *Betti number* of M is given by $\beta_k(M) = \sum_i b_{i,k}(M)$. In addition, for multigraded modules (such as monomial ideals, for instance) we can define multigraded minimal free resolutions and Betti numbers. See [7, 22] for details.

2.2. Mapping cone. The mapping cone is a construction that arises in topology, which has a direct analogue in algebra. Here we define the *algebraic mapping cone* of chain maps (chain complex morphisms).

Definition 2. *Let (C, ∂) and (C', ∂') be two chain complexes, and $f : C \to C'$ a chain map. The* algebraic mapping cone *of f, denoted $(M(f), \delta)$ is a chain complex defined as follows:*

$$M(f)_i = C_{i-1} \oplus C'_i$$

$$\delta_i(c, c') = (\partial_{i-1}c, \partial'_i c' + (-1)^i f_{i-1}c)$$

for $c \in C_{i-1}$, $c' \in C'_i$. The differential δ verifies $\delta^2 = 0$.

Applications of this construction to resolutions in the polynomial ring can be seen in [11, 20].

2.3. Iterated mapping cone resolutions. Mapping cones provide a recursive way to compute free resolutions of ideals in the polynomial ring. Briefly, the procedure is the following: Let $I = \langle f_1, \ldots, f_r \rangle \subseteq R$ be an ideal. Let $I_i = \langle f_1, \ldots, f_i \rangle$ be the subideal of I generated by the first i generators of I. There is a short exact sequence

$$0 \to R/(I_{i-1} : f_i) \xrightarrow{\phi} R/I_{i-1} \xrightarrow{\jmath} R/I_i \to 0 \tag{1}$$

for all $i \leq r$. Assume that free resolutions $\tilde{\mathcal{P}}$ and $\mathcal{P}'$ are known for $R/(I_{i-1} : f_i)$ and R/I_{i-1} respectively, then a resolution of R/I_i is obtained as the mapping cone of the chain complex morphism that lifts ϕ. Equivalently, the following sequence can also be used, which in some contexts is a more natural choice:

$$0 \to \tilde{I}_i \xrightarrow{\phi} I_{i-1} \oplus \langle f_i \rangle \xrightarrow{\jmath} I_i \to 0 \tag{2}$$

where $\tilde{I}_i = I_i \cap \langle f_i \rangle$.

3. Previous results

The construction of minimal free resolutions of monomial ideals is a problem that has received much attention in the last years. Two main directions have been followed: one direction focuses on the construction of non-minimal resolutions like [21, 13]; the other direction consists on the construction of the minimal free resolution for special types of ideals, starting with the seminal work of [8]. Also, the homological invariants associated to the minimal resolutions have been object of research [2, 3, 4].

3.1. Resolutions that arise as iterated mapping cones. Probably, the most used non-minimal free resolution of monomial ideals is the one given by D. Taylor in [21]. Despite this resolution is highly non-minimal in general, it is very important from a theoretical point of view. It is defined as follows:

Let I be a monomial ideal and $\{m_1, \ldots, m_r\}$ a generating set of I. For any subset $J = \{j_1, \ldots, j_s\} \subseteq \{1, \ldots, r\}$, let us denote $m_J = lcm(m_{j_1}, \ldots m_{j_s})$,and $J^i = \{j_1, \ldots, \widehat{j_i}, \ldots, j_s\}$. We can construct a resolution of R/I in the following way: Let T_s, $s \geq 0$ be a free R-module generated as a vector space by the set $\{u_J \, s.th. \, |J| = s\}$ and consider the R-linear differential

$$d(u_J) = \sum_{i \in J} (-1)^{i-1} \frac{m_J}{m_{J^i}} u_{J^i}$$

it is easy to verify that $d^2 = 0$. Moreover, this complex is acyclic and it is a resolution of R/I. The length of Taylor's resolution is given by the number of elements in the given generating set of the ideal (normally, we will assume that we have a minimal generating set for the ideal), which we denote by r. The rank of the i-th free module T_i is $\binom{r}{i}$, thus the sum of all these ranks is 2^r, this sum is known as the *size* of the resolution.

A subresolution of Taylor's was given in [13] and it is known as the *Lyubeznik resolution*. It is defined as follows: For a given subset $J \subseteq \{1 \dots r\}$ and an integer $1 \leq s \leq r$, let $J_{>s} = \{j \in J | j > s\}$; then the Lyubeznik resolution is generated by those basis elements u_J such that for all $1 \leq s \leq r$ one has that m_s does not divide $m_{J_{>s}}$. It is clear that, unlike Taylor's, Lyubeznik resolution depends on the ordering in which the generators of the ideal are given.

Example 1. *Let us consider the following monomial ideal in three variables: $I = \langle x^2y, xy^3, xz, yz \rangle$, the Taylor resolution of I has length 4, size 16 and the differentials are given by*

$$d_1 = \begin{pmatrix} x^2y & xy^3 & xz & yz \end{pmatrix} \quad d_2 = \begin{pmatrix} y^2 & z & z & 0 & 0 & 0 \\ -x & 0 & 0 & z & z & 0 \\ 0 & -xy & 0 & -y^3 & 0 & y \\ 0 & 0 & -x^2 & 0 & -xy^2 & -x \end{pmatrix}$$

$$d_3 = \begin{pmatrix} -z & -z & 0 & 0 \\ y^2 & 0 & -1 & 0 \\ 0 & y^2 & 1 & 0 \\ -x & 0 & 0 & -1 \\ 0 & -x & 0 & 1 \\ 0 & 0 & -x & y^2 \end{pmatrix} \qquad d_4 = \begin{pmatrix} 1 \\ -1 \\ y^2 \\ -x \end{pmatrix}$$

Lyubeznik's resolution is in this case equal to Taylor's if we keep the ordering $m_1 = x^2y$, $m_2 = xy^3$, $m_3 = xz$, $m_4 = yz$. On the contrary if we change the order to $m_1 = xz$, $m_2 = yz$, $m_3 = x^2y$, $m_4 = xy^3$, then the Lyubeznik resolution is generated by u_1, u_2, u_3, u_4, u_{12}, u_{13}, u_{14}, u_{34} and u_{134} i.e. the size of this resolution is 10. In this case, the Lyubeznik resolution is minimal. The differentials are given by

$$\delta_1 = \begin{pmatrix} xz & yz & x^2y & xy^3 \end{pmatrix} \quad \delta_2 = \begin{pmatrix} y^2 & z & 0 & 0 \\ -x & 0 & z & 0 \\ 0 & -xy & -y^3 & y \\ 0 & 0 & 0 & -x \end{pmatrix} \quad \delta_3 = \begin{pmatrix} -z \\ y^2 \\ -x \\ 0 \end{pmatrix}$$

To see that Taylor's resolution arises as a mapping cone, consider sequence (2) using as generating system for $I_i \cap \langle m_i \rangle$ the monomials $\{\operatorname{lcm}(m_1, m_i), \dots, \operatorname{lcm}(m_{i-1}, m_i)\}$, which in general do not form a minimal generating set of the corresponding ideal, hence the non-minimality of this resolution.

We also have the following

Proposition 1. *Lyubeznik resolution arises as an iterated mapping cone.*

Proof: Using again sequence (2), assume that there exists $1 \leq k \leq i$ such that m_k divides m_{ji} for some j then, it is clear that $k < j$ and m_{ki} divides m_{ji}, hence, removing m_{ji} from the generating set of $I \cap \langle m_i \rangle$ does not affect the ideal i.e. this is a redundant generator. $\square$

3.2. Monomial ideals whose minimal free resolution is obtained as an iterated mapping cone. Many authors have studied the minimal free resolution of particular families of monomial ideals. We give here an account on several of these families, the minimal resolution of which arises as an iterated mapping cone.

The best known is the minimal free resolution of *stable* ideals, due to S. Eliahou and M. Kervaire [8]. The resolution of Eliahou and Kervaire arises as an iterated mapping cone, as noted in [5], where the minimal resolution of more general types of monomial ideals are given as iterated mapping cones, namely *lex-seg with holes* and *lex-seg plus s-powers* ideals. Based on the methods of Eliahou and Kervaire, the minimal free resolutions of several other families of ideals have been described in [1, 5]. These types of ideals have received attention due to their relation to some important problems in commutative algebra. Stable ideals play a fundamental role at the interplay of Hilbert functions and Betti numbers of homogeneous ideals and also in the context of generic initial ideals. Lex-segment and lex-segment with powers are related to some conjectures in the field [14]. Connected to these ideals are prime, primary and related ideals, the minimal free resolution of which can also be obtained as iterated mapping cones [16, 18].

Some monomial ideals appearing in a context of algebraic geometry be resolved by iterated mapping cones. In his paper [23], G. Valla studies a type of non-stable monomial ideals to obtain the graded Betti numbers of two general points in $\mathbf{P}^3$. The minimal free resolutions of such ideals can be easily obtained as an iterated mapping cone, as was shown in [16, 18]. C. A. Francisco uses iterated mapping cones in [9] to study the Betti numbers of ideals associated to two fat points in $\mathbf{P}^n$ and at most $n+1$ general double points in $\mathbf{P}^n$.

In the context of the algebraic analysis of system reliability every coherent system, e.g. a network, has a monomial ideal associated to it, and the knowledge of the Betti numbers and Hilbert series of the ideal is used to compute the reliability of the system [10]. Some of the most relevant systems in reliability theory were studied in [17]. The Betti numbers of ideals of k-out-of-n systems and some variants, including the consecutive k-out-of-n model, were computed using iterated mapping cones. Also, the minimal free resolution of the ideals corresponding to series-parallel systems are obtained as iterated mapping cones.

In all these contexts, the knowledge of the minimal free resolution of the studied ideals provide closed form formulas for the Betti numbers of the ideal and other numerical invariants. In some cases, even recursive formulas for the Betti numbers of ideals in the family can be obtained and one can study the asymptotical behaviour of these ideals. A very promising area of work in this respect is that of edge ideals, i.e. monomial ideals associated to graphs. This is work in progress by several authors.

4. A short account on some computational aspects

The computation of minimal free resolutions of ideals in the polynomial ring is a computationally hard task, even in the monomial case. The procedure based on iterated mapping cones gives us a natural way to reach the minimal free resolution. On a first step, we compute the mapping cone resolution and then we minimize it using standard methods (see [6] for instance). However, this procedure presents two different problems. The first one is at the core of the problem: when our ideal is big enough (and such ideals occur in actual applications) the resolution is a huge object that cannot be handled by a computer. In this case, we might prefer to obtain a description of the main features of the resolution without actually computing it. We treat this approach in Section 4.1.

On the contrary, sometimes we want an explicit description of the full resolution of the ideal. Using iterated mapping cones can be useful in this respect, but it is not easy to use the method directly. However, the use of the techniques of *effective homology* [15] provides an explicit way to use the iterated mapping cone procedure. We report briefly on this method in Section 4.2.

4.1. Computing Betti numbers without computing minimal free resolutions. In some cases, only the Betti numbers or other numerical invariants, such as the Hilbert series or Castelnuovo-Mumford regularity, are needed. Therefore, one can attempt to compute the multigraded Betti numbers of monomial ideals without computing their minimal free resolution. One way to attack this problem is the use of Mayer-Vietoris trees, see [16], which provide the multigraded support of a resolution based on iterated mapping cones and tools to perform reductions on it. Other partial computations such as multigraded Hilbert series or Castelnuovo-Mumford regularity can be performed in an efficient way using this tool.

Using recursively the exact sequences (1) and (2) helps in the computation of the multigraded Betti numbers of $I = \langle m_1, \ldots, m_r \rangle$. We use here sequence (2). The involved ideals can be displayed as a tree, the root of which is I and every node J has as *children* $\tilde{J}$ on the left and J' on the right (if J is generated by r monomials, $\tilde{J}$ denotes $\tilde{J}_r$ and J' denotes J_{r-1}). This is what we call a *Mayer-Vietoris Tree* of the monomial ideal I, denoted $\mathrm{MVT}(I)$. Each node in a Mayer-Vietoris tree has a *position*: the root has position 1 and the left and right children of the node in position p have respectively, positions $2p$ and $2p+1$. The node of $\mathrm{MVT}(I)$ in position p is denoted $\mathrm{MVT}_p(I)$. We call *relevant nodes* to those in an even position or in position 1. We also assign a *dimension* to each node: the root has dimension 0 and the left and right children of any node of dimension d have dimension $d+1$ and d respectively.

The properties of Mayer-Vietoris trees allow us to perform homological computations on monomial ideals. The following propositions are proved in [16] together with other features of Mayer-Vietoris trees.

Proposition 2. *If $\beta_{i,\alpha}(I) \neq 0$ for some i, then x^α is a generator of some node J in any Mayer-Vietoris tree* $\mathrm{MVT}(I)$.

Proposition 3. *If x^α appears only once as a generator of a relevant node J in* $\mathrm{MVT}(I)$ *then there exists exactly one $i \in \mathbb{N}$ such that $\beta_{i,\alpha}(I) = 1$ and $\beta_{j,\alpha}(I) = 0$ for all $i \neq j$.*

The homological degree i to which relevant multidegrees contribute is the dimension of the node of the Mayer-Vietoris tree in which it appears.

Example 2. *Let us consider the ideal $I = \langle xy^2, xyz^3, y^5, z^6 \rangle \subseteq \mathbf{k}[x,y,z]$. A Mayer-Vietoris tree of this ideal is shown in Figure 1: Every node is given by*

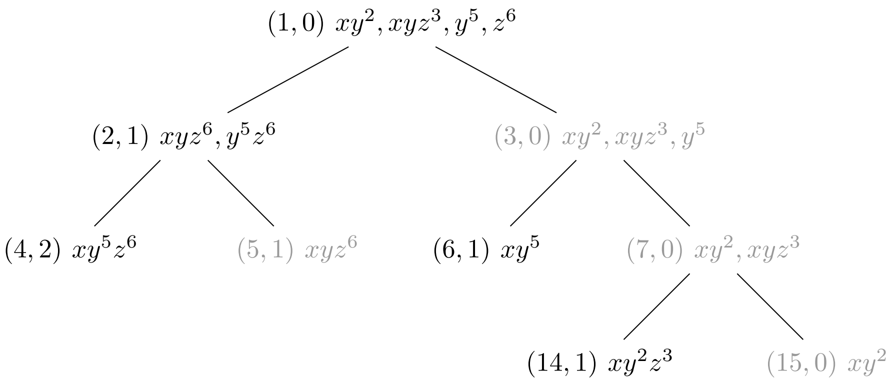

FIGURE 1. A Mayer-Vietoris tree of $\langle xy^2, xyz^3, y^5, z^6\rangle$.

a triple `(position, dimension) ideal` *and the relevant nodes are the ones in strong black color. Observe that this tree has no repeated multidegree in the relevant nodes, therefore the multigraded Betti numbers of I are just read from the tree. In this case we have $\beta_0(I) = 4$, $\beta_1(I) = 4$ and $\beta_2(I) = 1$. The Betti multidegrees are those of the generators of the relevant nodes in the tree.*

This way to display the support of the iterated mapping cone allows to reduce the iterated mapping cone resolution whose support is given by the corresponding Mayer-Vietoris tree. The basic definition and result are the following:

Definition 3. *Let b_1 and b_2 be two binary numbers. We can say that b_1 and b_2 have the following form: $b_1 = u_1 \dots u_k 0 \tilde{u}_1 \dots \tilde{u}_{l_1}$, $b_2 = u_1 \dots u_k 1 u'_1 \dots u'_{l_2}$. We say that b_1 and b_2 are* compatible *if the number of zeros in $\tilde{u}_1 \dots \tilde{u}_{l_1}$ and $u'_1 \dots u'_{l_2}$ are equal. Observe that the total number of zeros of two compatible binary numbers differs by one.*

We say that two positive integers $n_1, n_2 \in \mathbb{N}$ are compatible *if their corresponding binary expressions are compatible. We say that two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{N}$ are compatible if every pair (a, b), $a \in \mathcal{A}, b \in \mathcal{B}$ is compatible.*

Proposition 4. *Let I be a monomial ideal, $\mu \in \mathbb{N}^n$ a multidegree appearing in the relevant nodes of positions p_1 and p_2 of a Mayer-Vietoris tree of I. Let e_1 and e_2 be their corresponding generators in the associated Mayer-Vietoris resolution of I.*

If e_1 and e_2 are a reduction pair then p_1 and p_2 are compatible.

The resulting reduced tree supports a resolution of I that is not minimal in general, but the minimal free resolution of further families of ideals can be found in this way [19].

4.2. Effective computation of mapping cone resolutions. [1] The main computational difficulty of the procedure that uses iterated mapping cones to construct resolutions in the polynomial ring is, as pointed in [11], the construction of

[1]This section is joint work with F. Sergeraert.

the necessary chain complex morphisms, even in the (easier) monomial case. Specific methods for monomial ideals with linear quotients were given in [11] but they cannot be applied in general. On the other hand, an actual algorithm for the construction of these morphisms and hence for the construction of iterated mapping cone resolutions was given in [20]; it is an algorithm that uses syzygy computations for the construction of the morphisms, which is computationally expensive, and is not particularly efficient in the context of monomial ideals. To overcome this difficulty one can use the techniques of *effective homology* (see [15] for an introduction to effective homology which contains applications to commutative algebra). Using *effective* versions of the short exact sequences above and of the resolutions used in the iterated mapping cone procedure, we obtain conceptually simple yet efficient algorithms for the computation of free resolutions of ideals in the polynomial ring. To be able to perform explicit computations on mapping cones, in particular to construct the necessary morphisms, we need a truly constructive version of the resolutions involved. To achieve this goal we are concerned with the computation of *effective* resolutions.

Definition 4. *A resolution $(\mathcal{P}, d)$ of an R-module M is said to be effective if we have explicit homotopy operators $h : \mathcal{P}_{i-1} \to \mathcal{P}_i$ for all i, such that $hd + dh = 1$. Observe that h is required to be only a morphism of k-vector spaces.*

Theorem 1 ([15], Theorem 62). *Let $\rho = (f, g, h) : \mathcal{C}_* \Rrightarrow \mathcal{D}_*$ and $\rho' = (f', g', h') : \mathcal{C}'_* \Rrightarrow \mathcal{D}'_*$ be two reductions and $\phi : \mathcal{C}'_* \to \mathcal{C}_*$ a chain complex morphism. Then these data define a canonical reduction [2]:*

$$\rho'' = (f'', g'', h'') : \mathrm{Cone}(\phi) \Rrightarrow \mathrm{Cone}(f\phi g')$$

The algorithm makes use of sequences of type (1) for the iterative computation of resolutions of R/I_1 up to $R/I_r = R/I$, but equivalently, sequences of type (2) can be used. On each iteration of the algorithm we have the following data:

- (R_1, d_1) is an effective resolution of $R/(I_{s-1} : m_s)$.
- (R_2, d_2) is an effective resolution of R/I_{s-1}.
- The morphism ϕ of Sequence (1) will denote the multiplication map $\times m_s$.

The result is the mapping cone of ϕ, which can be structured as an effective resolution of R/I_s, allowing the process to be iterated.

Working with effective resolutions is the key point allowing us to lift the given multiplication map ϕ to a chain complex morphism. If the lifting of ϕ is known as ϕ_i at degree i, then the next component ϕ_{i+1} is obtained as $\phi_{i+1} = h_{2,i}\phi_i d_{1,i+1}$ where d_1 is the differential of R_1 and h_2 the *known* contracting homotopy operator of the *effective* resolution R_2.

This process starts at level 0:

[2]For the concept of reduction and other basic concepts in effective homology, see [15].

$$\begin{tikzcd}[column sep=huge]
\downarrow & \downarrow \\
R_{1,0} \arrow[r, dashed, "\phi_0"] \arrow[d, "d_{1,0}"'] & R_{2,0} \arrow[d, shift right, "d_{2,0}"'] \\
R/(I_{s-1} : m_s) \arrow[r, "\phi"'] & R/I_s \arrow[u, shift right, "h_{2,-1}"']
\end{tikzcd}$$

where $h_{2,-1}$ is a lifting of the surjection $d_{2,0} : R_{2,0} \to R/I_s$. Such a lifting, as all the $h_{2,i}$ components, cannot be an R-module morphism; it is only a k-vector space morphism.

The looked-for resolution R for the quotient module R/I_{s+1} is the cone of $\overline{\phi} : R_1 \to R_2$ with $\overline{\phi} = (\phi_i)$. The differential D of this cone and its contracting homotopy H are defined by the matrices:

$$D_i = \begin{bmatrix} -d_{1,i-1} & 0 \\ \phi_{i-1} & d_{2,i} \end{bmatrix} \qquad H_i = \begin{bmatrix} -h_{1,i-1} & 0 \\ h_{2,i}\phi_i h_{1,i_1} & h_{2,i} \end{bmatrix}$$

The algorithm has the following basic steps:

1. Update the ideal: We just add the new generator m_s to I_{s-1} to obtain the ideal I_s for which the output of the algorithm is a resolution.
2. Construct the skeleton of the mapping cone: Each module in Cone(ϕ) is given by $R1_i \oplus R2_{i+1}$, therefore the modules in the resolutions are straightforward to compute. We denote by $R_{i,j}$the nodes in this skeleton.
3. Copy the nodes of R_2 in the skeleton: These will remain unchanged when seen in Cone(ϕ).
4. Update the node $R_{0,0}$: While this node is just a copy of R the differential d_0 must be changed so that $im(d_0) = R/I_s$.
5. Copy the nodes of R_1: If we use Sequence 2 these nodes remain also unchanged when embedded in R. Note that since we use Sequence 1 then the multidegree in which these modules are generated must be multiplied by m_s.
6. Update the nodes of R_1: In this moment we just change the signs of the differentials and homotopies coming from R_1, which form the first part of the differential and homotopies in the cone.
7. Compute the lifting of the connection morphism. The formulas given by the application of Theorem 1 for the new differential and homotopy are used here. To compute ϕ_i we need $d_{1,i}$, which is already given in the data, ϕ_{i-1} which has already been computed in the previous iteration, and $h_{2,i-1}$ which is computed by a recursive algorithm that is described below.

Each step of the algorithm computes then the mapping cone of ϕ plus a collection of k-vector morphisms, which are the decisive pieces to compute all the necessary homotopies. These two components together with an algorithm that is actually able to compute the homotopies are enough to allow the iterative procedure to construct the mapping cone at the next step.

This way to proceed avoids the main difficulty in [11] allowing an explicit constructive process based on iterated mapping cones. An actual implementation of this procedure has been built by F. Sergeraert in the `KENZO` system [12].

5. Conclusions

We have reported on the use of algebraic mapping cones to perform homological computations on monomial ideals. This tool has been used by different authors to obtain explicit resolutions of several families of ideals or even to obtain actual Betti numbers in a wide variety of examples. On the other hand, the recursive nature of the procedure makes it suitable to construct actual algorithms and programs to perform such computations in computer algebra systems. However, we believe that the utility of this method has not yet been sufficiently exploited. There are several big classes of important monomial ideals that can be defined recursively or depending on some parameters. Is to these kinds of ideals to which the iterated mapping cone techniques can be succesfully applied. The first attempts of this approach, in the context of system reliability have already given very promising results. The application of these techniques in other contexts, such as edge ideals is certainly of interest.

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