

## SPECTRAL SEQUENCES: USES, PROBLEMS AND COMPUTATIONS

ANA ROMERO

*Dedicado a la memoria de Mirian Andrés, gran compañera y gran amiga.  
Cada día que pasa se te echa más de menos*

**RESUMEN.** Este artículo pretende ser un texto introductorio sobre sucesiones espectrales, una técnica de gran utilidad en Topología Algebraica que ha sido usada frecuentemente para calcular la homología y la homotopía de espacios. A través de algunos ejemplos, mostramos la utilidad de esta construcción y algunos de los problemas que aparecen de forma habitual cuando se trabaja con sucesiones espectrales. Además, presentamos un programa Common Lisp para calcular sucesiones espectrales; el algoritmo teórico que da lugar a nuestro programa está basado en el método de la homología efectiva.

**ABSTRACT.** This paper tries to be an introductory text to spectral sequences, a useful technique in Algebraic Topology which has been frequently used in order to compute homology and homotopy groups of spaces. By means of some examples we show the utility of this construction and some of the problems they do often appear when dealing with spectral sequences. Moreover, we present a Common Lisp program for computing spectral sequences; the theoretical algorithm supporting this program is based on the effective homology method.

### 1. INTRODUCTION

Some of the most important results Mirian obtained in her short research career were done in the field of Theorem Proving, a research subject consisting in proving mathematical theorems by a computer program. More concretely, she tried to demonstrate by means of the theorem prover ACL2 the correctness of some fragments of Kenzo, a Common Lisp program devoted to Symbolic Computation in Algebraic Topology. This is in fact one of the general research lines of the Programming and Symbolic Computation Team of Universidad de La Rioja, and many results have been obtained in this field. A different research line of our team consists in the development of new algorithms in Algebraic Topology and their implementation in Common Lisp, trying to enhance in this way the Kenzo system. In particular, we have worked with spectral sequences, a construction which was not implemented in Kenzo.

---

*Key words and phrases.* Symbolic Computation, Spectral Sequences, Serre Spectral Sequence, Eilenberg-Moore Spectral Sequence, Constructive Algebraic Topology, Common Lisp.

Partially supported by Ministerio de Educación y Ciencia, project MTM2009-13842-C02-01.

Spectral sequences are a useful technique in Algebraic Topology traditionally applied to calculate homology and homotopy groups of spaces (see [7] or [8]). The Serre spectral sequence [15], for example, gives information about the homology groups of the total space of a fibration when the homology groups of the base and fiber spaces are known. On the other hand, the Eilenberg-Moore spectral sequences [5] give information about the homology groups of the base space (resp. the fiber space) from the homologies of the total space and of the fiber (resp. base space). For the computation of homotopy groups, the spectral sequences of Adams [1] or Bousfield-Kan [2] can be used. And many other examples of spectral sequences can be found in the literature: Bockstein, Grothendieck, Hurewicz, Kenneth, Quillen, Van Kanpen, etc.

But the various classical spectral sequences pose a very important problem: they are not *algorithms*. A spectral sequence is a family of “pages”  $(E_{p,q}^r, d^r)_{r \geq 1}$  of differential bigraded modules, each page being made of the homology groups of the preceding one. Then, as expressed by John McCleary in his famous book [8]:

*It is worth repeating the caveat about differentials mentioned in Chapter 1: knowledge of  $E_{*,*}^r$  and  $d^r$  determines  $E_{*,*}^{r+1}$  but not  $d^{r+1}$ . If we think of a spectral sequence as a black box, then the input is a differential bigraded module, usually  $E_{*,*}^1$ , and, with each turn of the handle, the machine computes a successive homology according to a sequence of differentials. If some differential is unknown, then some other (any other) principle is needed to proceed. From Chapter 1, the reader is acquainted with several algebraic tricks that allow further calculation. In the nontrivial cases, it is often a deep geometric idea that is caught up in the knowledge of a differential.*

In most cases, it is in fact a matter of computability: the higher differentials of the spectral sequence are mathematically defined, but their definition is not constructive. In other words, the differentials are not computable with the usually provided information. Another different problem of spectral sequences is the extension problem at abutment: a spectral sequence gives one a filtration of the looked-for (homology or homotopy) groups, but then in some cases several solutions are possible.

The methods of *Effective Homology* [12] provide on the contrary *real algorithms* computing homology and homotopy groups of topological spaces. In particular, these techniques give to their user *algorithms* replacing some important spectral sequences such as those of Serre and Eilenberg-Moore: when the usual inputs of these spectral sequences are organized as *objects with effective homology*, general algorithms are produced computing for example the homology groups of the total space of a fibration, of an arbitrarily iterated loop space (Adams’ problem), of a classifying space, etc. But these spectral sequences are yet interesting and their structure can give useful informations about the involved constructions (for example, about the present transgressions); sometimes this information is more interesting than the final homology groups.

This paper tries to give a didactical description of spectral sequences, showing by means of some simple examples the usefulness of this construction and the problems which do usually appear when dealing with spectral sequences. On the

other hand, we also include here part of a work already published in [11], where we proved that is is possible to relate spectral sequences with the effective homology technique. More concretely, we will explain how the effective homology method can be used to produce algorithms computing the various components of some spectral sequences, higher differentials included.

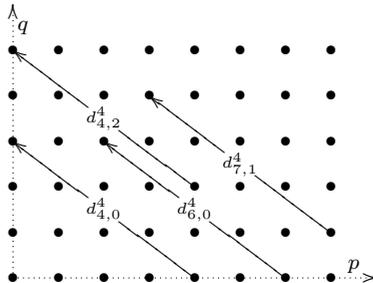
The paper is organized as follows. In Section 1, the definition and some important ideas about spectral sequences are introduced. In the two next sections we explain, by means of some examples, the utility of these constructions and the problems that they usually pose. Next, we present the effective homology method for the computation of homology groups; specifically, Subsection 5.1 includes some formal definitions and results, and in Subsection 5.2 some indications about the program Kenzo [4] (that implements this method) are given. In Section 6, we include a brief description of an algorithm we have developed which computes spectral sequences of filtered complexes with effective homology; then, in Section 7 we present some examples of application of our programs. The paper ends with a section of conclusions and further work.

## 2. BASIC DEFINITIONS AND RESULTS

We include in this section some basic definitions and properties about spectral sequences, which have been mostly extracted from [7]. A more complete reference is of course [8].

**Definition.** Let  $R$  be a ring, a *bigraded  $R$ -module* is a family of  $R$ -modules  $E = \{E_{p,q}\}_{p,q \in \mathbb{Z}}$ . A *differential*  $d : E \rightarrow E$  of bidegree  $(-r, r - 1)$  is a family of morphisms of  $R$ -modules  $d_{p,q} : E_{p,q} \rightarrow E_{p-r,q+r-1}$  for each  $p, q \in \mathbb{Z}$ , such that  $d_{p-r,q+r-1} \circ d_{p,q} = 0$ . The pair  $(E, d)$  is called a *differential bigraded module*.

The following figure represents a bigraded module, with  $r = 4$ . Only some differential maps have been drawn.



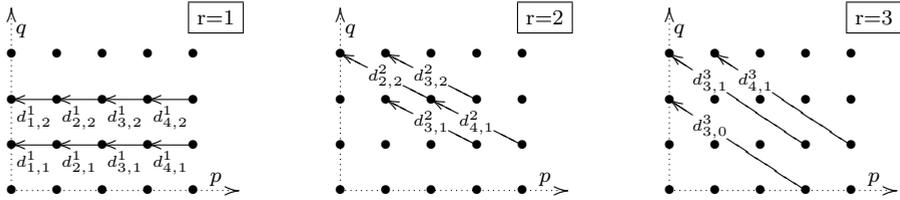
The relations  $d_{p-r,q+r-1} \circ d_{p,q} = 0$  allow us to define the *homology* of  $E$  as the bigraded  $R$ -module  $H(E, d) \cong H(E) = \{H_{p,q}(E)\}_{p,q \in \mathbb{Z}}$  with

$$H_{p,q}(E) = \frac{\text{Ker } d_{p,q}}{\text{Im } d_{p+r,q-r+1}}$$

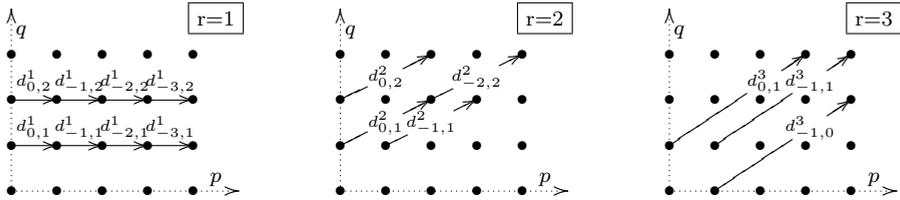
**Definition.** A *spectral sequence*  $E = (E^r, d^r)_{r \geq 1}$  is a sequence of bigraded  $R$ -modules  $E^r = \{E_{p,q}^r\}_{p,q \in \mathbb{Z}}$ , each provided with a differential  $d^r = \{d_{p,q}^r\}_{p,q \in \mathbb{Z}}$  of bidegree  $(-r, r - 1)$  and with isomorphisms  $H(E^r, d^r) \cong E^{r+1}$  for every  $r \geq 1$ .

**Definition.** A spectral sequence  $E = (E^r, d^r)_{r \geq 1}$  is a *first quadrant spectral sequence* if for all  $r \geq 1$   $E^r_{p,q} = 0$  when  $p < 0$  or  $q < 0$ . A *second quadrant spectral sequence*  $E$  is one with  $E^r_{p,q} = 0$  if  $p > 0$  or  $q < 0$ .

If  $E$  is a first quadrant spectral sequence, it is useful to represent the bigraded modules  $E^r = \{E^r_{p,q}\}_{p,q \in \mathbb{Z}}$  at the lattice points of the first quadrant of the plane. In the figures that follow we consider the levels  $r = 1, 2$ , and 3, but only some differential maps  $d^r_{p,q}$  are included.



Similarly, in the case of a second quadrant spectral sequence, the bigraded modules  $E^r = \{E^r_{p,q}\}_{p,q \in \mathbb{Z}}$  can be displayed at the lattice points of the second quadrant of the  $(p, q)$ -plane. However, we consider more convenient to represent them also in the first quadrant. To this aim, we simply change the sign of the first index  $p$ , that is to say, we represent the module  $E^r_{p,q}$  at the point  $(-p, q)$  (which is in the first quadrant). In this way the differential maps have shift  $(r, r - 1)$ .



A spectral sequence  $E$  can be presented as a tower

$$0 = B^0 \subseteq B^1 \subseteq B^2 \subseteq \dots \subseteq Z^2 \subseteq Z^1 \subseteq Z^0 = E^1$$

of bigraded submodules of  $E^1$ , where  $E^{r+1} = Z^r/B^r$  and the differential  $d^{r+1}$  can be taken as a mapping  $Z^r/B^r \rightarrow Z^r/B^r$ , with kernel  $Z^{r+1}/B^r$  and image  $B^{r+1}/B^r$ .

We say that the module  $Z^{r-1}$  is the set of elements that *live till stage*  $r$ , while  $B^{r-1}$  is the module of elements that *bound by stage*  $r$ . Let  $Z^\infty = \bigcap_r Z^r$  be the submodule of  $E^1$  of elements that *survive forever* and  $B^\infty = \bigcup_r B^r$  the submodule of those elements which *eventually bound*. It is clear that  $B^\infty \subseteq Z^\infty$  and therefore the spectral sequence determines a bigraded module  $E^\infty = \{E^\infty_{p,q}\}_{p,q \in \mathbb{Z}}$  given by

$$E^\infty_{p,q} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q}}$$

which is the bigraded module that remains after the computation of the infinite sequence of successive homologies. The modules  $E^\infty_{p,q}$  are called the *final* or *target modules* of the spectral sequence  $E$ .

Once the notion and some important observations about spectral sequences have been defined, a natural question comes to the mind: what are spectral sequences used for? The following section tries to answer this question.

### 3. WHY ARE SPECTRAL SEQUENCES USEFUL?

Since their definition at the beginning of the fifties, spectral sequences have appeared many times in the literature. They are often used with a basic goal: it is desired to compute a graded module  $H_*$ , which can be the homology or cohomology of some space or some other graded algebraic invariant associated with some object. In any case,  $H_*$  is frequently difficult to obtain.

In order to proceed, we suppose that  $H_*$  is *filtered*, that is,  $H_*$  comes equipped with a sequence of subobjects

$$\{0\} \subseteq \cdots \subseteq F_{p-1}H_* \subseteq F_pH_* \subseteq F_{p+1}H_* \subseteq \cdots \subseteq H_*$$

If we now consider  $q = n - p$  and  $H_{p,q} = F_pH_{p+q}/F_{p-1}H_{p+q}$ , we obtain a bigraded module  $\{H_{p,q}\}$ . And this could be the target of a spectral sequence! This allows us to formalize the notion of *convergence* of spectral sequences.

**Definition.** Let  $H_* = \{H_n\}_{n \in \mathbb{N}}$  be a graded module. A spectral sequence  $(E^r, d^r)_{r \geq 1}$  is said to *converge* to  $H_*$  (denoted by  $E^1 \Rightarrow H_*$ ) if there is a filtration  $F$  of  $H_*$  and for each  $p$  one has isomorphisms of graded modules

$$E_{p,*}^\infty \cong \frac{F_pH_*}{F_{p-1}H_*}$$

The collection  $H_* = \{H_n\}_{n \in \mathbb{N}}$  is called the *abutment* of the spectral sequence.

In the case of a free graded module (that is,  $H_n$  is torsion free for each  $n$ ),  $H_*$  can be recovered up to isomorphism from the bigraded module by taking direct sums, that is

$$H_n \cong \bigoplus_{p=0}^{\infty} E_{p,n-p}^\infty$$

If  $H_*$  is an arbitrary graded module, however, there could be extension problems that prevent one from reconstructing  $H_*$  from the associated bigraded module. But in any case, we can take the groups  $E_{p,q}^\infty$  as a first approximation to  $H_*$ .

It follows that our basic goal  $H_*$  is approximated, if we can find a spectral sequence converging to  $H_*$ . And this has been done for many different  $H_*$ ; let us recall some of the most famous examples of spectral sequences:

- the Serre spectral sequence converges to the homology groups of the total space of a fibration;
- the Eilenberg-Moore spectral sequence converges to the homology groups of the loop space of a simplicial set;
- the Adams spectral sequence converges to the homotopy groups of a simplicial set  $X$ .

The theorems expressing these results and, in general, all theorems expressing the existence of any spectral sequence, have the following generic form.





the multiplication by  $q + 1$ . Thus, it can be deduced that all the groups  $E_{3,2q}^3$  die and the only non-null final groups are  $E_{0,0}^\infty \cong \mathbb{Z}$  and  $E_{0,2q}^\infty \cong \mathbb{Z}_q$  for  $q \geq 2$ .

On account of the isomorphisms  $E_{p,q}^\infty \cong F_p H_{p+q}(X_4) / F_{p-1} H_{p+q}(X_4)$ , in this case the Serre spectral sequence entirely gives the homology groups of the total space  $X_4$ :  $H_0(X_4) \cong \mathbb{Z}$ ,  $H_{2n}(X_4) \cong \mathbb{Z}_n$  for  $n \geq 2$ , and the other  $H_n(X_4)$  are null. Furthermore, the Hurewicz Theorem and the long exact sequence of homotopy (see [9] for details) imply that  $\pi_4(S^3) \cong \pi_4(X_4) \cong H_4(X_4) \cong \mathbb{Z}_2$ .

Then, a new fibration

$$G_3 \hookrightarrow X_5 \rightarrow X_4$$

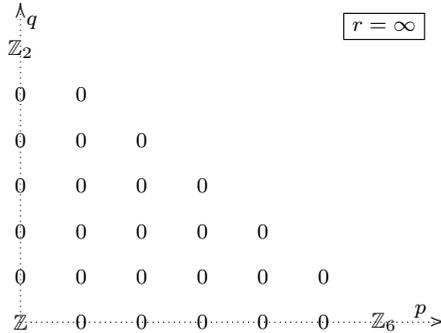
should be considered to determine  $\pi_5(S^3)$ , where  $G_3 = K(\mathbb{Z}_2, 3)$  is chosen because  $\pi_4(X_4) \cong \mathbb{Z}_2$ . In this case Serre was also able to obtain all the necessary ingredients to compute the maps  $d_{p,q}^r$  which play an important role in the beginning of the associated spectral sequence. The main tool (extra information) is again the multiplicative structure in cohomology and more generally the module structure with respect to the Steenrod algebra  $\mathcal{A}_2$  [16].

For  $p + q = 5$ , there is only one non-null group,  $E_{0,5}^\infty \cong \mathbb{Z}_2$ , and therefore  $H_5(X_5) \cong \mathbb{Z}_2$ . Again the Hurewicz theorem and the long homotopy exact sequence imply  $\pi_5(S^3) \cong \pi_5(X_4) \cong \pi_5(X_5) \cong H_5(X_5) \cong \mathbb{Z}_2$ ; it was the first important result obtained by Serre.

In order to compute  $\pi_6(S^3)$ , we consider a new fibration

$$G_4 \hookrightarrow X_6 \rightarrow X_5$$

with  $G_4 = K(\mathbb{Z}_2, 4)$ . The final groups  $E_{p,q}^\infty$  (with  $p + q \leq 6$ ) of this spectral sequence are shown in the following figure.



In this case there are three non-null groups  $E_{p,q}^\infty$  for  $p + q \leq 6$ :  $E_{0,0}^\infty \cong \mathbb{Z}$ ,  $E_{0,6}^\infty \cong \mathbb{Z}_2$ , and  $E_{6,0}^\infty \cong \mathbb{Z}_6$ . In degree 6 we obtain a short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_6(X_6) \rightarrow \mathbb{Z}_6 \rightarrow 0$$

but there are two possible extensions (the trivial one  $\mathbb{Z}_2 \oplus \mathbb{Z}_6$  and the twisted one  $\mathbb{Z}_{12}$ ), and the Serre spectral sequence does not give any information that allows one to make the correct choice. In this way, Serre proved  $\pi_6(S^3)$  has 12 elements, but he was unable to choose between the two possible options  $\mathbb{Z}_{12}$  and  $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ . Two years later, Barratt and Paechter [3] proved that the group  $\pi_6(S^3)$  has an element of order 4, and consequently  $\pi_6(S^3) \cong \mathbb{Z}_{12}$  is the correct answer.

These examples illustrate the fact that the computation of spectral sequences is not an easy task and in some situations some other information is needed to overcome the ambiguities in the spectral sequence itself. In other cases, this computation is in fact *not possible*, since some differential maps  $d_{p,q}^r$  cannot be determined by any other means (we do not have the necessary extra information). Furthermore, even if we can determine all the stages of the spectral sequence and the final groups  $E_{p,q}^\infty$  are reached, extension problems must be solved in order to determine the graded module  $H_*$ .

## 5. THE EFFECTIVE HOMOLOGY METHOD

As seen before, spectral sequences are a useful tool in Algebraic Topology but they cannot be determined in general, only in some elementary cases. On the contrary, the effective homology method provides real algorithms for the computation of homology groups. In fact, the program Kenzo (that will be presented in Subsection 5.2) uses the notion of *object with effective homology* to compute homology groups of spaces, and has obtained the homology groups of some complicated spaces related to the most common spectral sequences, those of Serre and Eilenberg-Moore.

**5.1. Definitions and fundamental results.** In this section we present the main definitions and results of the effective homology method. See [12] and [13] for more details.

**Definition.** A *reduction*  $\rho$  between two chain complexes  $C_* = (C_n, d_{C_n})_{n \in \mathbb{N}}$  and  $D_* = (D_n, d_{D_n})_{n \in \mathbb{N}}$  (which is denoted  $\rho : C_* \Rightarrow D_*$ ) is a triple  $(f, g, h)$  where: (a) the components  $f$  and  $g$  are chain complex morphisms  $f : C_* \rightarrow D_*$  and  $g : D_* \rightarrow C_*$ ; (b) the component  $h$  is a homotopy operator  $h : C_* \rightarrow C_{*+1}$  (a graded group morphism of degree +1); (c) the following relations are satisfied: (1)  $fg = \text{Id}_D$ ; (2)  $d_C h + h d_C = \text{Id}_C - g f$ ; (3)  $fh = 0$ ; (4)  $hg = 0$ ; (5)  $hh = 0$ .

These relations express that  $C_*$  is the direct sum of  $D_*$  and an acyclic complex. This decomposition is simply  $C_* = \text{Ker } f \oplus \text{Im } g$ , with  $\text{Im } g \cong D_*$  and  $H_*(\text{Ker } f) = 0$ . In particular, this implies that the graded homology groups  $H_*(C_*)$  and  $H_*(D_*)$  are canonically isomorphic.

**Definition.** A (*strong chain*) *equivalence*  $\varepsilon$  between two chain complexes  $C_*$  and  $D_*$ , denoted by  $\varepsilon : C_* \Leftrightarrow D_*$ , is a triple  $(B_*, \rho_1, \rho_2)$  where  $B_*$  is a chain complex, and  $\rho_1$  and  $\rho_2$  are reductions  $\rho_1 : B_* \Rightarrow C_*$  and  $\rho_2 : B_* \Rightarrow D_*$ .

**Remark.** We must use the notion of *effective* chain complex: it is essentially a free chain complex  $C_*$  where each group  $C_n$  is finitely generated, and a provided algorithm returns a (distinguished)  $\mathbb{Z}$ -basis in each degree  $n$ ; in particular, its homology groups are elementarily computable (for details, see [12]).

**Definition.** An *object with effective homology*  $X$  is a quadruple  $(X, C_*(X), HC_*, \varepsilon)$  where  $C_*(X)$  is a chain complex canonically associated with  $X$  (which allows us to study the homological nature of  $X$ ),  $HC_*$  is an effective chain complex, and  $\varepsilon$  is an equivalence  $\varepsilon : C_*(X) \Leftrightarrow HC_*$ .

It is important to understand that in general the  $HC_*$  component of an object with effective homology is *not* made of the homology groups of  $X$ ; this component  $HC_*$  is a free  $\mathbb{Z}$ -chain complex of finite type, in general with a non-null differential, whose homology groups  $H_*(HC_*)$  can be determined by means of an elementary algorithm. From the equivalence  $\varepsilon$  one can deduce the isomorphism  $H_*(X) := H_*(C_*(X)) \cong H_*(HC_*)$ , which allows one to compute the homology groups of the initial space  $X$ .

In this way, the notion of object with effective homology makes it possible to compute homology groups of complicated spaces by means of homology groups of effective complexes. The effective homology technique is based on the following idea: given some topological spaces  $X_1, \dots, X_n$ , a topological constructor  $\Phi$  produces a new topological space  $X$ . If effective homology versions of the spaces  $X_1, \dots, X_n$  are known, then one should be able to build an effective homology version of the space  $X$ , and this version would allow us to compute the homology groups of  $X$ .

A typical example of this kind of situation is the loop space constructor. Given a 1-reduced simplicial set  $X$  with effective homology, it is possible to determine the effective homology of the loop space  $\Omega(X)$ , which in particular allows one to compute the homology groups  $H_*(\Omega(X))$ . Moreover, if  $X$  is  $m$ -reduced, this process may be iterated  $m$  times, producing an effective homology version of  $\Omega^k(X)$ , for  $k \leq m$ . Effective homology versions are also known for classifying spaces or total spaces of fibrations, see [13] for more information.

**5.2. The Kenzo program.** The Kenzo program [4] is a Lisp 16,000 lines program devoted to Symbolic Computation in Algebraic Topology. It works with rich and complex algebraic structures (chain complexes, differential graded algebras, simplicial sets, simplicial groups, morphisms between these objects, reductions, etc.) and has obtained some results (for example homology groups of iterated loop spaces of a loop space modified by a cell attachment, components of complex Postnikov towers, etc.) which had never been determined before.

The fundamental idea of the Kenzo system for the computation of homology groups is the notion of object with effective homology. Specifically, to obtain the homology groups of a space  $X$ , the program proceeds in the following way: if the complex is effective, then its homology groups can be determined by means of diagonalization of matrices. Otherwise, the program uses the effective homology of the space, which is located in one of its slots.

To explain roughly the general style of Kenzo computations, let us firstly consider a didactical example. The “minimal” simplicial model of the Eilenberg-MacLane space  $K(\mathbb{Z}, 1)$  is defined by  $K(\mathbb{Z}, 1)_n = Z^1(\Delta^n, \mathbb{Z}) = \mathbb{Z}^n$ ; an infinite number of simplices is required in every dimension  $\geq 1$ . This does not prevent such an object from being installed and handled by the Kenzo program.

```
> (setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
```

The `k-z` Kenzo function constructs the standard simplicial Eilenberg-MacLane space and this object is assigned to the symbol `kz1`. In ordinary mathematics

notation, a 3-simplex of `kz1` could be for example  $[3|5| - 5]$ , denoted by  $(3\ 5\ -5)$  in Kenzo. The faces of this simplex can be computed:

```
> (dotimes (i 4)
  (print (face kz1 i 3 '(3 5 -5))))
<AbSm - (5 -5)>
<AbSm - (8 -5)>
<AbSm 1 (3)>
<AbSm - (3 5)>
NIL
```

You recognize the bar construction faces; in particular the face of index 2 is degenerated:  $\partial_2[3| - 5|5] = \eta_1[3]$ . “Local” (in fact simplex-wise) computations are so possible, we say this object is *locally effective*. But no global information is available. For example if we try to obtain the list of non-degenerate simplices in dimension 3:

```
> (basis kz1 3)
Error: The object [K1 Abelian-Simplicial-Group] is locally-effective.
```

This basis in fact is  $\mathbb{Z}^3$ , an infinite set whose element *list* cannot be explicitly stored nor displayed! So that the homology groups of `kz1` cannot be elementarily computed. But it is well known  $K(\mathbb{Z}, 1)$  has the homotopy type of the circle  $S^1$ ; the Kenzo program knows this fact, reachable as follows. We can ask for the effective homology of  $K(\mathbb{Z}, 1)$ :

```
> (efhm kz1)
[K22 Homotopy-Equivalence K1 <= K1 => K16]
```

A reduction  $K_1 = K(\mathbb{Z}, 1) \Rightarrow K_{16}$  is constructed by Kenzo. What is  $K_{16}$ ?

```
> (orgn (k 16))
(CIRCLE)
```

What about the basis of this circle in dimensions 0, 1 and 2?

```
>(dotimes (i 3)
  (print (basis (k 16) i)))
(*)
(S1)
NIL
NIL
```

`NIL =  $\emptyset$`  and the second `NIL` is “technical” (independently produced by the iterative `dotimes`). The basis are available, the boundary operators too:

```
> (? (k 16) 1 'S1)
-----{CMBN 0}
-----
```

The boundary of the unique non-degenerate 1-simplex is the null combination of degree 0. So that the homology groups of  $K(\mathbb{Z}, 1)$  are computable through the *effective* equivalent object  $K_{16}$ :

```
> (homology kz1 0 3)
Homology in dimension 0 :
Component Z
---done---
```

Homology in dimension 1 :  
 Component Z  
 ---done---

Homology in dimension 2 :  
 ---done---

This mechanism for computing homology groups of a chain complex through its effective homology can also be used to determine some types of spectral sequences. We explain the relation between both techniques in the following section.

### 6. AN ALGORITHM COMPUTING SPECTRAL SEQUENCES OF FILTERED COMPLEXES

Many spectral sequences (including those of Serre and Eilenberg-Moore) are defined by means of *filtered chain complexes*. Given a filtered chain complex, one can construct a spectral sequence which under certain good conditions converges to its homology groups. In this case there exists a formal expression for *all* groups  $E_{p,q}^r$  and differential maps of the spectral sequence, given by the following theorem.

**Theorem.** [7] *Let  $F$  be a filtration of a chain complex  $C_* = (C_n, d_n)_{n \in \mathbb{N}}$ . There exists a spectral sequence  $E = E(C_*, F) = (E^r, d^r)_{r \geq 1}$ , defined by*

$$E_{p,q}^r = \frac{Z_{p,q}^r \cup F_{p-1}C_{p+q}}{d_{p+q+1}(Z_{p+r-1,q-r+2}^{r-1}) \cup F_{p-1}C_{p+q}}$$

where  $Z_{p,q}^r$  is the submodule  $Z_{p,q}^r = \{a \in F_p C_{p+q} \mid d_{p+q}(a) \in F_{p-r} C_{p+q-1}\} \subseteq F_p C_{p+q}$ , and  $d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$  is the morphism induced on these subquotients by the differential map  $d_{p+q} : C_{p+q} \rightarrow C_{p+q-1}$ . Furthermore, if  $F$  is bounded, then the spectral sequence converges to  $H_*(C_*)$ .

One can easily observe that if the initial filtered complex  $C_*$  is finitely generated (in other words, it is effective), this expression allows one to determine the groups  $E_{p,q}^r$  and the differential maps  $d_{p,q}^r$  of the spectral sequence by means of elementary operations with matrices. But in the general situation the subgroups  $Z_{p,q}^r$  which appear in the formula are not necessarily of finite type and in many cases one cannot calculate them. Thus, this formal expression is not always sufficient to *compute* the spectral sequence, which can only be directly determined in the case the filtered complex is effective. However, if the chain complex has effective homology and some natural conditions are satisfied, we have proved [11] the different components of the spectral sequence can also be computed by means of an analogous spectral sequence deduced of an appropriate filtration on the associated effective complex, which is isomorphic to the spectral sequence of the initial complex after some level  $k$ .

Our results lead to the following algorithm.

**Algorithm.**

*Input:* a filtered chain complex  $C_*$  with effective homology, such that some “natural conditions” are satisfied.

*Output:* an integer  $k$  (which is usually equal to 1 or 2) and all the components of the Serre spectral sequence associated with the filtered chain complex after level  $k$ , that is to say:

- the groups  $E_{p,q}^r$  for each  $p, q \in \mathbb{Z}$  and  $r \geq k$ ,
- the differential maps  $d_{p,q}^r$  for every  $p, q \in \mathbb{Z}$  and  $r \geq k$ ,
- the convergence level for each degree  $n \in \mathbb{N}$ ,
- the filtration of the homology groups  $H_*(C_*)$ , in other words, the groups  $F_p H_n(C_*)$  for each degree  $n \in \mathbb{N}$  and filtration index  $p \in \mathbb{Z}$ .

This algorithm has been implemented in Common Lisp, enhancing the Kenzo system, and allows one to compute spectral sequences associated with bicomplexes, and also the Serre and Eilenberg-Moore spectral sequences when the spaces involved in the constructions are objects with effective homology.

This work has been extended trying to produce algorithms computing other types of spectral sequences. In particular, we have considered the Bousfield-Kan spectral sequence [2], which is related with the computation of homotopy groups of spaces, and we have developed some algorithms for computing the groups and the differential maps of this spectral sequence for levels  $r = 1$  and 2. The computation of higher levels of the spectral sequence is much more complicated and is being studied yet. See [10] for details.

## 7. EXAMPLES AND COMPUTATIONS

As explained before, spectral sequences are a useful construction in Algebraic Topology but they are not algorithms, in many cases some extra information is necessary in order to determine the different components. However, in the case of spectral sequences associated with filtered complexes, the algorithm we have developed makes it possible to determine the groups and the differential maps of every level if the filtered complex is provided with effective homology.

In this section we include two examples of application of our new programs. In the first case, the spectral sequence is well known and can be obtained without using a computer; we propose it as didactic example for a better understanding of the new functionality. The second example shows a more complicated calculation.

**7.1. Serre spectral sequence.** We consider the twisted Cartesian product  $S^2 \times_\tau K(\mathbb{Z}, 1)$  for a twisting operator  $\tau : S^2 \rightarrow K(\mathbb{Z}, 1)$  with  $\tau(\mathbf{s}2) = [1]$ . We use here the standard simplicial description of the 2-sphere, with a unique non-degenerate simplex  $\mathbf{s}2$  in dimension 2. A principal fibration is then defined by a unique 1-simplex of the simplicial structural group. The result in this case is the Hopf fibration, the total space  $S^2 \times_\tau K(\mathbb{Z}, 1)$  being a simplicial model for the 3-sphere  $S^3$ . The same example could be processed with  $\tau(\mathbf{s}2) = [2]$ , the total space then being the real projective space  $P^3\mathbb{R}$ . Let us remark that, since  $K(\mathbb{Z}, 1)$  is not effective, the space  $S^2 \times_\tau K(\mathbb{Z}, 1)$  is not effective either, and therefore the effective homology is necessary to determine its spectral sequence.

The twisted product  $S^2 \times_{\tau} K(\mathbb{Z}, 1)$  is built in Kenzo in the following way:

```
>(setf s2 (sphere 2))
[K23 Simplicial-Set]
>(setf kz1 (k-z 1))
[K1 Abelian-Simplicial-Group]
>(setf tau (build-smmr
           :sorc s2
           :trgt kz1
           :degr -1
           :sintr #'(lambda (dmns gmsm) (absm 0 '(1)))
           :orgn '(s2-tw-kz1)))
[K28 Fibration K23 -> K1]
>(setf s2-tw1-kz1 (fibration-total tau))
[K34 Simplicial-Set]
```

The object `tau` implements the twisting operator  $\tau : S^2 \rightarrow K(\mathbb{Z}, 1)$  as a simplicial morphism of degree  $-1$  that sends the unique non-degenerate simplex `s2` of dimension 2 to the 1-simplex (1) of the simplicial set `kz1`. The function `fibration-total` builds the total space of the fibration defined by the twisting operator `tau` (this operator contains as source and target spaces the base and the fibre spaces of the fibration respectively), which is a twisted Cartesian product of the base and fibre.

Since  $K(\mathbb{Z}, 1)$  is not effective, the new space  $S^2 \times_{\tau} K(\mathbb{Z}, 1)$  is not effective either, but using the effective homologies of  $S^2$  and  $K(\mathbb{Z}, 1)$  Kenzo computes automatically the effective homology of `s2-tw1-kz1` and stores it in the slot `efhm`.

```
> (efhm s2-tw1-kz1)
[K109 Homotopy-Equivalence K34 <= K99 => K95]
```

The right bottom chain complex in this equivalence is the (effective) chain complex  $C_*(S^2) \otimes_t C_*(S^1)$ , obtained from the *usual* tensor product  $C_*(S^2) \otimes C_*(S^1)$  by adding an appropriate perturbation of the differential map.

```
> (setf s2-twtp1-s1 (rbcc (efhm s2-tw1-kz1)))
[K95 Chain-Complex]
> (orgn s2-twtp1-s1)
(add [K74 Chain-Complex] [K93 Morphism (degree -1): K74 -> K74])
> (orgn (k 74))
(tnsr-prdc [K23 Simplicial-Set] [K16 Chain-Complex])
> (orgn (k 23))
(sphere 2)
> (orgn (k 16))
(circle)
```

This effective homology allows us to determine the homology groups of the total space  $S^2 \times_{\tau} K(\mathbb{Z}, 1)$ .

```
> (homology s2-tw1-kz1 0 5)
Homology in dimension 0 :
Component Z
---done---
```

```

Homology in dimension 1 :
---done---
Homology in dimension 2 :
---done---
Homology in dimension 3 :
Component Z
---done---
Homology in dimension 4 :
---done---
nil

```

In order to compute the Serre spectral sequence of our fibration it is necessary to change the space  $S^2 \times_{\tau} K(\mathbb{Z}, 1)$  into a filtered complex. The filtration in a twisted Cartesian product  $B \times_{\tau} G$  is defined through the degeneracy degree with respect to the base space: a generator  $(x_n, y_n) \in C_n(B \times G)$  has a filtration degree less or equal to  $q$  if  $\exists \bar{x}_q \in B_q$  such that  $x_n = s_{i_{n-q}} \cdots s_{i_1} \bar{x}_q$ . Such a filtration has been implemented by a function called `twpr-flin`.

```

>(change-chcm-to-flcc s2-tw1-kz1 crpr-flin '(crpr-flin))
[K34 Filtered-Complex]

```

A filtration is also needed in the effective complex,  $C_*(S^2) \otimes_t C_*(S^1)$ , which is filtered by the base dimension. In general, for a twisted tensor product:  $F_p(C(B) \otimes_t C(G)) = \bigoplus_{m \leq p} C(B)_m \otimes C(G)$ . The implementation in Common Lisp is done by means of a function `tnpr-flin`.

```

> (change-chcm-to-flcc s2-twtp1-s1 tnpr-flin '(tnpr-flin))
[K95 Filtered-Complex]

```

Once the filtrations are defined, the new programs can be used to compute the spectral sequence of the twisted product  $S^2 \times_{\tau} K(\mathbb{Z}, 1)$ , which is isomorphic in every level to that of the effective complex  $C_*(S^2) \otimes_t C_*(S^1)$ . For instance, the groups  $E_{2,0}^2$  and  $E_{0,1}^2$  are equal to  $\mathbb{Z}$ :

```

> (spsq-group s2-tw1-kz1 2 2 0)
Spectral sequence E^2_{2,0}
Component Z
> (spsq-group s2-tw1-kz1 2 0 1)
Spectral sequence E^2_{0,1}
Component Z

```

The differential function in a group  $E_{p,q}^r$  can be computed using the function `spct-sqn-dffr`. The last argument must be a list that represents the coordinates of the element we want to apply the differential to (with regard to the generators of the subgroup in the numerator). In the example that follows, the differential  $d_{2,0}^2$  is applied to the unique generator of the group  $E_{2,0}^2 \cong \mathbb{Z}$ , and therefore the list of coordinates must be (1).

```

> (spct-sqn-dffr s2-tw1-kz1 2 2 0 '(1))
(1)

```

The obtained list (1) shows that the result of applying  $d_{2,0}^2$  to the generator of the group  $E_{2,0}^2$  is the combination  $1 * g_{0,1}^2$ , where  $g_{0,1}^2$  is the generator of the group  $E_{0,1}^2 \cong \mathbb{Z}$ . This last result means that the differential map  $d_{2,0}^2 : E_{2,0}^2 \rightarrow E_{0,1}^2$  is an

isomorphism. Since the next stage in the spectral sequence  $E^3$  is isomorphic to the bigraded homology group of  $E^2$ ,  $E_{p,q}^3 \cong H_{p,q}(E^2) = \text{Ker } d_{p,q}^2 / \text{Im } d_{p+2,q-1}^2$ , it is clear that the groups  $E_{0,1}^3$  and  $E_{2,0}^3$  must be null:

```
> (spsq-group s2-tw1-kz1 3 0 1)
Spectral sequence E^3_{0,1}
NIL
> (spsq-group s2-tw1-kz1 3 2 0)
Spectral sequence E^3_{2,0}
NIL
```

It is also possible to obtain, for each degree  $n$ , the level  $r$  at which the convergence of the spectral sequence has been reached, that is, the smallest  $r$  such that  $E_{p,q}^\infty = E_{p,q}^r$  for all  $p, q$  with  $p + q = n$ . For instance, for  $n = 0$  and  $n = 1$  the convergence levels are 1 and 3 respectively:

```
>(spsq-cnvg s2-tw1-kz1 0)
1
>(spsq-cnvg s2-tw1-kz1 1)
3
```

Thus, we can obtain the groups  $E_{p,q}^\infty$  with  $p + q = 0$  or  $p + q = 1$  by computing the corresponding groups  $E_{0,0}^1$ ,  $E_{0,1}^3$ , and  $E_{1,0}^3$ :

```
> (spsq-group s2-tw1-kz1 1 0 0)
Spectral sequence E^1_{0,0}
Component Z
> (spsq-group s2-tw1-kz1 3 0 1)
Spectral sequence E^3_{0,1}
NIL
> (spsq-group s2-tw1-kz1 3 1 0)
Spectral sequence E^3_{1,0}
NIL
```

Finally, we can determine the filtration of the homology groups  $H_*(S^2 \times_\tau K(\mathbb{Z}, 1))$  induced by the filtration of the chain complex. For instance, for  $H_3 \cong \mathbb{Z}$  we obtain  $F_0H_3 = F_1H_3 = 0 \subset F_2H_3 = F_3H_3 = H_3 \cong \mathbb{Z}$ .

```
> (hmlg-fltr s2-tw1-kz1 3 0)
Filtration F_0 H_3
nil
> (hmlg-fltr s2-tw1-kz1 3 1)
Filtration F_1 H_3
nil
> (hmlg-fltr s2-tw1-kz1 3 2)
Filtration F_2 H_3
Component Z
> (hmlg-fltr s2-tw1-kz1 3 3)
Filtration F_3 H_3
Component Z
```

**7.2. Eilenberg-Moore spectral sequence.** The programs presented here can also be used to determine the Eilenberg-Moore spectral sequence between a simplicial set  $X$  and its loop space  $\Omega(X)$ . If the space  $X$  is an 1-reduced simplicial set with effective homology, the program Kenzo determines the effective homology of its loop space  $\Omega(X)$  using the cobar construction on a coalgebra. Moreover, if  $X$  is  $m$ -reduced, this process may be iterated  $m$  times, producing an effective homology version of  $\Omega^k(X)$ ,  $k \leq m$ . The effective homology of the loop space together with the natural filtration defined on the cobar construction allows the computation of the spectral sequence between  $H_*(X)$  and  $H_*(\Omega(X))$  for every level  $r$ .

The Eilenberg-Moore spectral sequence has been traditionally considered to be an important tool for obtaining homotopic information of a space, by means of its relation with its loop space. In particular, it can be used for the study of the effect of the attachment of a disk to an space of infinite dimension, especially a loop space, a problem which seems to be very difficult in general as explained in [6]. Our programs have determined the different elements of this (second quadrant) spectral sequence for some spaces constructed in this way that, up to now, have not appeared in the literature. In Figures 1 and 2 we present the groups  $E_{p,q}^\infty$  (for  $q - p \leq 8$ ) of the spectral sequences for the spaces  $\Omega(S^3)$  and  $\Omega(S^3) \cup_2 D^3$  (the last one obtained from  $\Omega(S^3)$  by attaching a 3-disk by a map  $\gamma : S^2 \rightarrow \Omega(S^3)$  of degree 2). The first space and its loop space have been extensively considered by theoretical methods and a lot of results about them are known. However, for our second example, the attachment of the 3-disk increases the difficulty of the calculation of the Eilenberg-Moore spectral sequence between  $\Omega(S^3) \cup_2 D^3$  and its loop space that, up to our knowledge, had not been determined before. See Figures 1 and 2 for the calculated  $E_{p,q}^r$ 's.

## 8. CONCLUSIONS

In this paper, we have introduced spectral sequences as a useful tool of Algebraic Topology. Although they are not general algorithms, the effective homology method makes it possible to compute them when the spaces involved are filtered chain complexes with effective homology. In this way, we have developed some programs computing spectral sequences of filtered complexes, enhancing the Kenzo system. These programs can be applied to compute, for instance, spectral sequences of double complexes, the Serre spectral sequence, the Eilenberg-Moore spectral sequences...

In a different work [10] we have developed similar algorithms and programs trying to determine the Bousfield-Kan spectral sequence, related with the computation of homotopy groups of spaces, which is not defined by means of a filtered chain complex. For the moment we have developed some algorithms for computing the groups and the differential maps of this spectral sequence for levels  $r = 1$  and 2, but the computation of higher levels of the spectral sequence is being studied yet. To follow our work, we think it could be convenient to consider the notion of "effective exact couple" [14], recently developed by Sergeraert, and try to apply it to our problem.



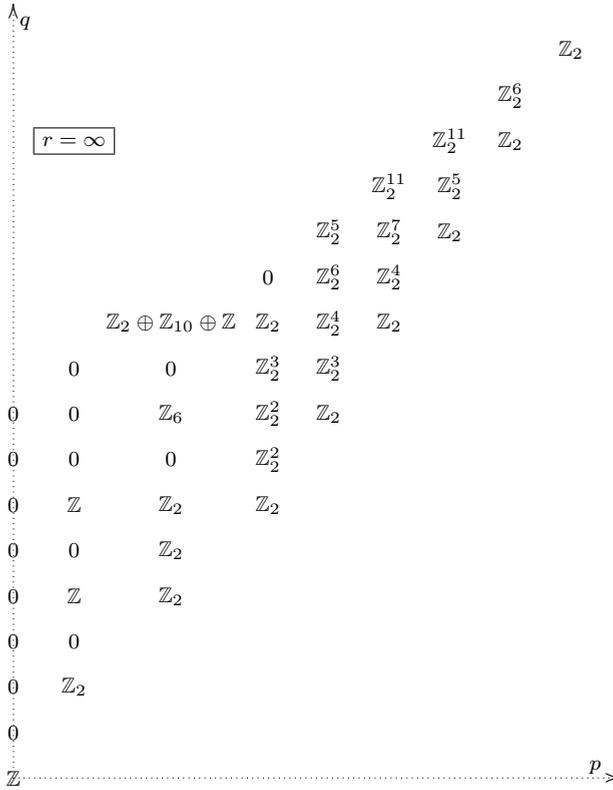


FIGURE 2. Groups  $E_{p,q}^{\infty}$  of the Eilenberg-Moore spectral sequence between  $\Omega(S^3) \cup_2 D^3$  and  $\Omega(\Omega(S^3) \cup_2 D^3)$ .

- [11] A. ROMERO, J. RUBIO, F. SERGERAERT. Computing spectral sequences. *Journal of Symbolic Computation* **41**(10), 1059–1079, 2006.
- [12] J. RUBIO, F. SERGERAERT. Constructive Algebraic Topology. *Bulletin des Sciences Mathématiques* **126**(5), 389–412, 2002.
- [13] J. RUBIO, F. SERGERAERT. *Constructive Homological Algebra and Applications, Lecture Notes Summer School on Mathematics, Algorithms, and Proofs*. University of Genova, 2006. <http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Genova-Lecture-Notes.pdf>.
- [14] F. SERGERAERT. *Effective Exact Couples*. Preprint. <http://www-fourier.ujf-grenoble.fr/~sergerar/Papers/Exact-Couples-2-2.pdf>
- [15] J. P. SERRE. Homologie singulière des espaces fibrés. *Annals of Mathematics* **54**(3), 425–505, 1951.
- [16] N. E. STEENROD. *Cohomology operations*. Princeton University Press, 1962.

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, SPAIN  
 E-mail address: ana.romero@unirioja.es