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Representation of Operators Defined on the Space of Bochner Integrable Functions

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Introduction

The representation of linear operators, on the Banach space of Bochner integrable functions, has been the object of much study for the past fifty years. Dunford and Pettis began this investigation in 1940 with the representation of weakly compact and norm compact operators on $L_1(\mathbb{R})$ by a Bochner integral, see [6,8]. Andrews has extended their study to the case of the space $L_1(E)$, of E-valued, Bochner integrable functions, see [1,2]. The theory of liftings has also been used by Dinculeanu [7] and others to obtain a representation for the general linear operator on $L_1(E)$. It is worth noting that the representation is often related to the Radon-Nikodym property of a Banach space, see [2,11].

In this paper, we investigate the representation of operators on a space of Banach-valued Bochner integrable functions which are defined on a perfect measure space.

A summary of the paper follows. In the first section, we recall some definitions and notations that we need in the sequel. The second section points out three known results. The most important is the fact that a function defined on a compact Hausdorff space, with values in a Banach space E, is Bochner integrable for every regular Borel measure if it is continuous with respect to the weak topology $\sigma(E, E')$. In the third section, we give two representation theorems for an operator $T : L_1(E) \to D$, defined on the space of E-valued Bochner integrable functions on a perfect measure space, and with values in a Banach space D. In fact, we prove that, for such an operator $T : L_1(E) \to D$, there is a bounded and strongly integrable function g, which is continuous with respect to a weaker topology than the weak topology $\sigma(L(E,D),L(E,D)')$, and such that

$$T(f)=\int\langle f,g\rangle d\mu \qquad \forall\, f\in L_1(E)\,.$$

In the last section, we establish an isometry between the space of E-valued functions defined on a compact Hausdorff space Ω, which are continuous with respect to the weak topology $\sigma(E,E')$, and the space of equivalence classes of Bochner integrable E-valued functions defined on Ω, and that are essentially bounded.

1. Definitions and notations

Let $(X,\mathcal{A},\nu)$ be a finite positive measure space and E be a Banach space. The space of equivalence classes of E-valued Bochner integrable functions defined on X is denoted by $L_1(X,\mathcal{A},\nu,E)$, or by $L_1(E)$ if the context is clear enough. The symbol $\mathcal{L}_\infty(X,\mathcal{A},\nu,E)$ (or $\mathcal{L}_\infty(E)$) stands for the space of Bochner integrable E-valued functions which are essentially bounded and the symbol $L_\infty(X,\mathcal{A},\nu,E)$ (or $L_\infty(E)$) is the space of equivalence classes of functions in $\mathcal{L}_\infty(E)$. When there is no risk of confusion, we shall speak of functions in $L_\infty(E)$, or in $L_1(E)$.

If D is a second Banach space, we denote by $L(E,D)$ the space of linear continuous functions from E into D, and by $K(E,D)$ its subset of compact operators.

A function $g: X\to L(E,D)$ is strongly integrable if for each $t\in E$ the function $g(.)(t): X\to D$ is Bochner integrable. For $f\in L_1(E)$ and $g\in L_\infty(L(E,D))$, let us define the function $\langle f,g\rangle: X\to D$, $x\mapsto g(x)(f(x))$. It is a ν-Bochner integrable function ([6], [7, p. 102]), that is $\langle f,g\rangle\in L_1(D)$.

Let E be a Banach space, and E' its topological dual. The weak topology on E will be denote by $\sigma(E,E')$ and the weak star topology on E' will be denote by $\sigma(E',E)$. The weak-operator topology on $L(E,D)$ (see [10]) is the topology defined by the linear functionals

$$\begin{array}{rcl} L(E,D) & \to & \mathbb{R}\\ T & \mapsto & d'(Te)\end{array}$$

for $d'\in D'$, $e\in E$. The dual weak-operator topology (see [10]) is defined by the linear functionals

$$\begin{array}{rcl} L(E,D) & \to & \mathbb{R}\\ T & \mapsto & e''(T'(d'))\end{array}$$

for $d' \in D'$, $e'' \in E''$. The topology $\sigma(L(E,D), L(E,D)')$ is called the weak-Banach topology on $L(E,D)$.

Let Ω be a compact space, E be a Banach space, and τ be the norm topology on E, the weak topology $\sigma(E,E')$, or the weak-star topology on E, if E is a dual. The space of continuous functions from Ω into (E,τ) is denoted by $C(\Omega,(E,\tau))$. It will be normed by $\|f\| = \sup_{x\in\Omega} \|f(x)\|$, for $f \in C(\Omega,(E,\tau))$.

A Banach space E is a Grothendieck space if every $\sigma(E',E)$-convergent sequence in E' converges weakly in E'.

We shall call a finite positive measure space $(\Omega, \mathcal{BO}, \mu)$ perfect if

1. Ω is an extremally disconnected compact Hausdorff space (of finite measure);
2. $\mathcal{BO}$ is the Borel algebra of Ω;
3. μ is a regular Borel measure on $\mathcal{BO}$ with the following properties:
   - every nonempty clopen (closed and open) set has positive measure,
   - every nowhere dense Borel set has measure zero,
   - for every Borel set B there exists a clopen set C such that the symmetric difference $B \Delta C$ has measure zero.

2. Some important results

The following theorem shows that we may always work with perfect measure spaces when we analyse properties of the corresponding Bochner space.

Theorem 1. ([4]) *For all finite measure spaces $(X, \mathcal{A}, \nu)$, there exists a perfect measure space $(\Omega, \mathcal{BO}, \mu)$, such that for all Banach spaces E and for $p \in [1,\infty]$ there is a linear surjective isometry*

$$\Phi_{p,E} : L_p(\Omega, \mathcal{BO}, \mu, E) \to L_p(X, \mathcal{A}, \nu, E)\,.$$

Theorem 2 is the starting point in the proofs of the representation theorems.

Theorem 2. ([3]) *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space, and E be a Banach space. The function*

$$\begin{array}{rcl} C(\Omega,(E',\sigma(E',E))) & \to & L_1(\Omega,\mathcal{BO},\mu,E)' \\ g & \mapsto & \psi_g : f \mapsto \psi_g(f) = \int \langle f, g\rangle d\mu \end{array}$$

is a surjective linear isometry.

Theorem 3 gives a key fact and will be used repeatedly.

Theorem 3. ([13]) *Let Ω be a compact Hausdorff space, E be a Banach space, and $f : \Omega \to E$. If f is continuous with respect to the weak topology, $\sigma(E, E')$, on E, then it is μ-integrable with respect to each finite regular Borel measure μ on Ω.*

3. Representation theorems

The proof of the following theorem extends ideas of the short proof of Dunford-Pettis' theorem in [12].

Theorem 4. *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space and let E and D be two Banach spaces. For each weakly compact operator $T : L_1(E) \to D$ there is a bounded function $g : \Omega \to L(E, D)$, which is continuous with respect to the weak-operator topology on $L(E, D)$, strongly integrable and such that*

$$T(f) = \int \langle f, g \rangle d\mu \qquad \forall\, f \in L_1(E)\,.$$

Moreover, if E is a Grothendieck space and if $\operatorname{Im} g \subseteq K(E, D)$, then g is sequentially continuous with respect to the weak-Banach topology on $L(E, D)$.

Proof. By theorem 2, it is possible to define a linear surjective isometry

$$\begin{array}{rcl} H : L_1(E)' & \to & C(\Omega, (E', \sigma(E', E))) \\ \psi & \mapsto & H(\psi) \end{array}$$

such that

$$\int \langle f, H(\psi) \rangle d\mu = \psi(f) \qquad \forall\, f \in L_1(E)\,.$$

By virtue of the Davis-Figiel-Johnson-Pelczynski factorization theorem, we may assume that D is reflexive. Thus, the adjoint operator

$$\begin{array}{rcl} T' : D' & \to & L_1(E)' \\ s' & \mapsto & T'(s') = s' \circ T \end{array}$$

is weakly compact. Moreover, by [5, p. 21], the function $T'' : L_1(E)'' \to D''$ is continuous with respect to the weak-star topology $\sigma(L_1(E)'', L_1(E)')$, and the weak topology $\sigma(D'', D''')$, and its range, $\operatorname{Im} T''$, is contained in D.

For $t \in E$, let us define

$$\begin{array}{rcl} \delta^t : \Omega & \to & C(\Omega, (E', \sigma(E', E)))' \\ k & \mapsto & \delta^t(k) : f \mapsto \delta^t(k)(f) = f(k)(t)\,. \end{array}$$

One can see easily that this function is well-defined and continuous with respect to the weak-star topology on $C(\Omega, (E', \sigma(E', E)))'$.

Since D is reflexive, the canonical mapping J_D is bijective. We can thus define for $t \in E$,

$$g_t = J_D^{-1} \circ T'' \circ H' \circ \delta^t\,.$$

The functions g_t are continuous with respect to the weak topology $\sigma(D, D')$, and therefore μ-integrable.

Let us define

$$\begin{array}{rcl} g : \Omega & \to & L(E, D) \\ k & \mapsto & g(k) : t \mapsto g(k)(t) = g_t(k)\,. \end{array}$$

This function is bounded by $\|J_D^{-1} \circ T'' \circ H'\|$. Moreover, it is continuous with respect to the weak-operator topology on $L(E, D)$. Indeed, one has, for $s' \in D', t \in E$ and $k \in \Omega$,

$$\begin{aligned} s'(g_t(k)) &= s'(J_D^{-1} \circ T'' \circ H' \circ \delta^t(k)) = (T''(H'(\delta^t(k))))(s') \\ &= (\delta^t(k) \circ H \circ T')(s') = (H(T'(s')))(k)(t)\,. \end{aligned}$$

It remains to prove that

$$T(f) = \int \langle f, g \rangle d\mu \qquad \forall\, f \in L_1(E)\,.$$

By the density of simple functions in $L_1(E)$, it suffices to prove this formula for $f = t\chi_B$ with $t \in E$ and $B \in \mathcal{BO}$. But, if $s' \in D'$, then we have

$$s' \int_B g_t(k) d\mu(k) = \int_B (H(T'(s'))(k)(t) d\mu(k) = (T'(s'))(t\chi_B) = s'(T(t\chi_B))\,.$$

The second part of the theorem follows from [10, p. 269]. ∎

A key fact for the proof of the preceding theorem was the theorem of Cambern-Greim. By means of this theorem, we have been enable to take a linear surjective isometry

$$\begin{array}{rcl} H : L_1(E)' & \to & C(\Omega, (E', \sigma(E', E))) \\ \psi & \mapsto & H(\psi) \end{array}$$

such that

$$\int \langle f, H(\psi)\rangle d\mu = \psi(f) \qquad \forall\, f \in L_1(E)\,.$$

By using the same idea, we shall now give a second theorem of representation by a kernel that will be continuous with respect to the dual weak-operator topology, which is finer than the weak-operator topology. For this purpose, we need the following definition, which recall the Cambern-Greim's theorem.

Definition. Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space. We shall say that a Banach space E satisfies the Bochner-isomorphism property if there is a surjective linear isometry

$$\begin{array}{rcl} H : L_1(E)' & \to & C(\Omega, (E', \sigma(E', E''))) \\ \psi & \mapsto & H(\psi) \end{array}$$

such that

$$\int \langle f, H(\psi)\rangle d\mu = \psi(f) \qquad \forall\, f \in L_1(E)\,.$$

Example. Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space. By Cambern-Greim's theorem, any Banach space E for which are equal the spaces $C(\Omega, (E', \sigma(E', E)))$ and $C(\Omega, (E', \sigma(E', E'')))$ satisfies the Bochner isomorphism property. In particular, a reflexive Banach space satisfies this property.

We now prove a result similar to theorem 4, using the dual weak-operator topology. We then investigate the Bochner-isomorphism property in the next section.

Theorem 5. *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space, E be a Banach space satisfying the Bochner-isomorphism property, and D be a Banach space. For each weakly compact operator $T : L_1(E) \to D$ there is a bounded function $g : \Omega \to L(E, D)$, which is continuous with respect to the dual weak-operator topology on $L(E, D)$, strongly integrable and such that*

$$T(f) = \int \langle f, g\rangle d\mu \qquad \forall\, f \in L_1(E)\,.$$

Proof. It suffices to follow the scheme of proof of the preceding theorem and to use the following function :

$$\begin{array}{rcl} g : \Omega & \to & L(E, D) \\ k & \mapsto & g(k) : t \mapsto g(k)(t) = (J_D^{-1} \circ T'' \circ H' \circ \delta^{J_E(t)})(k)\,, \end{array}$$

where, for $t'' \in E''$, we define

$$\begin{array}{rcl} \delta^{t''} : \Omega & \to & C(\Omega, (E', \sigma(E', E'')))' \\ k & \mapsto & \delta^{t''}(k) : f \mapsto \delta^{t''}(k)(f) = t''(f(k)) \,. \quad \blacksquare \end{array}$$

4. Bochner isomorphism property

THEOREM 6. *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space, E be a Banach space, and τ be the norm topology on E, or the weak topology $\sigma(E, E')$. The function*

$$\begin{array}{rcl} [.]_\tau : C(\Omega, (E, \tau)) & \to & L_\infty(E) \\ f & \mapsto & [f]_\tau = \{g \in \mathcal{L}_\infty(E) \mid g = f \;\; \mu - \text{a. e.}\} \end{array}$$

is a linear isometry. It is surjective if the set of (equivalence classes of) functions of the form $\sum_{i=1}^{\infty} r_i \chi_{B_i}$, with $\{r_i\}_{i \in \mathbb{N}}$ relatively τ-compact in E and $\{B_i\} \subseteq \mathcal{BO}$ pairwise disjoint, is dense in $L_\infty(E)$.

Proof. For the first part, it suffices to show that $[.]_{\sigma(E,E')}$ is an isometry, i.e., for $f \in C(\Omega, (E, \sigma(E, E'))$, we have

$$\sup_{k \in \Omega} \|f(k)\| = \|[f]\|_{L_\infty} = \inf\{v \mid \|f(k)\| \le v \;\; \mu\text{-a.e.}\} \,.$$

It is obvious that $\sup_k \|f(k)\| \ge \inf\{v \mid \|f(k)\| \le v \;\; \mu\text{-a.e.}\}$.

In the other hand, we shall prove that

$$\|f(k)\| \le v \;\; \mu\text{-a.e.} \; \forall\, v \in \mathbb{R} \quad \Rightarrow \quad \sup_k \|f(k)\| \le v \,.$$

We shall work by contraposition. Suppose thus that there is $k \in \Omega$ and $x' \in E'$ such that $\|x'\| = 1$ and $|x' f(k)| > v$. By the continuity of f and the fact that Ω is totally disconnected, there is a clopen set $B_{x'}$ in Ω, containing k and such that $|x' f(l)| > v \;\; \forall\, l \in B_{x'}$. Since μ is a perfect measure, $\mu(B_{x'}) > 0$.

Therefore, if $C \in \mathcal{BO}$ is of null measure, then there is $t \in B_{x'}$ and $t \notin C$. Consequently, we get

$$\begin{array}{c} \forall C \in \mathcal{BO} \text{ with } \mu(C) = 0 \;\; \exists\, t \in \Omega \setminus C \text{ and } \exists\, x' \in E' \\ \text{such that } \|x'\| = 1 \text{ and } |x' f(t)| > v \,. \end{array}$$

It remains to prove the surjectivity. For this, it suffices to prove that each function $f = \sum_{i=1}^{\infty} r_i \chi_{B_i}$, with $\{r_i\}_{i \in \mathbb{N}}$ relatively τ-compact in E and

$\{B_i\} \subseteq \mathcal{BO}$ pairwise disjoint, is equal μ-almost everywhere to a τ-continuous function. By [9, p. 821], the function f is norm-continuous on a dense open subset O of Ω. Let us denote by f_O the restriction of f to O. Since the range of f_O is relatively τ-compact in E and Ω is the Stone-Cech compactification of O, we can extend this function continuously to a function f', belonging to $C(\Omega, (E, \tau))$. But, the complement $\Omega \backslash O$ is a nowhere dense Borel set and thus, it is of measure zero. Consequently, f is equal to f' μ-almost everywhere. ∎

In particular, if E is reflexive, then the function $[.]_{\sigma(E,E')}$, defined in the preceding theorem, is an isomorphism of Banach spaces. More generally,

PROPOSITION 7. *Let $(X, \mathcal{A}, \nu)$ be a finite positive measure space and E be a Banach space. The set of (equivalence classes of) functions of the form $\sum_{i=1}^{\infty} r_i \chi_{B_i}$, with $\{r_i\}_{i \in \mathbb{N}}$ relatively weakly compact in E and $\{B_i\} \subseteq \mathcal{A}$ pairwise disjoint, is dense in $L_\infty(E)$ if the set of relatively weakly compact sequences in E is dense in the Banach space $l_\infty(E)$, of bounded sequences in E.*

COROLLARY 8. *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space and E be a Banach space. If the set of relatively weakly compact sequences in E is dense in the Banach space $l_\infty(E)$, of bounded sequences in E, then the function*

$$\begin{array}{rcl} [.]_\sigma : C(\Omega, (E, \sigma(E, E'))) & \rightarrow & L_\infty(E) \\ f & \mapsto & [f]_\sigma = \{g \in \mathcal{L}_\infty(E) \mid g = f \ \ \mu - \text{a. e.}\} \end{array}$$

is a linear surjective isometry.

COROLLARY 9. *Let $(\Omega, \mathcal{BO}, \mu)$ be a perfect measure space and E be a Banach space such that its dual E' satisfies the Radon-Nikodym property. If the set of relatively weakly compact sequences in E' is dense in the Banach space $l_\infty(E')$, of bounded sequences in E', then E satisfies the Bochner-isomorphism property.*

Proof. By [6, p. 98], E' has the Radon-Nikodym property with respect to μ if and only if the function

$$\begin{array}{rcl} V : L_\infty(E') & \rightarrow & L_1(E)' \\ g & \mapsto & V(g) : f \mapsto V(g)(f) = \int \langle f, g \rangle d\mu \end{array}$$

is a linear surjective isometry. ∎

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