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On the C-Farthest Points

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1. Introduction

Let X be a Banach space and let S be a bounded subset of X. We define a real valued function $r : X \longrightarrow \mathbb{R}$ by

$$r(x) = \sup\{\|x - z\| : z \in S\}$$

and call $r(x)$ the farthest distance from x to S. The function r is convex (as supremum of convex functions) and Lipschitz-continuous, in fact, $|r(x) - r(y)| \leq \|x - y\|$, for all $x, y \in X$. A point $z \in S$ is called a farthest point of S if there exists $x \in X$ such that $\|x - z\| = r(x)$. The existence of a farthest point of S is equivalent to the fact that the set

$$D = \{x \in X : (\exists z \in S)(\|x - z\| = r(x))\}$$

is nonempty.

Edelstein [4] showed that if X is uniformly convex Banach space, then the set D defined above is dense in X. Asplund [1] showed that if X is both reflexive and locally uniformly rotund, then the set D is dense in X. In several years ago, Lau [8] proved that if S is a weakly compact subset of a Banach space X, then the set D defined above contains a dense G_δ of X. In a few years ago Deville and Zizler [2] showed how to characterise weak compactness in terms of farthest points.

In the same way we define the nearest distance from x to a closed subset F of X by

$$N(x) = \inf\{\|x - z\| : z \in F\}$$

and a point $z \in F$ is called a nearest point of F, if there exists $x \in X$ such that $N(x) = \|x - z\|$. Lau [9] showed that if X is reflexive and the norm has the Kadec-Klee property (on the unit sphere, every weakly convergent sequence converges in norm) then the set of $X \setminus F$ with nearest point in F contains a dense G_δ subset of $X \setminus F$.

Let C be a convex subset in X with 0 in its interior. The Minkowski functional of C is $\rho : X \longrightarrow \mathbb{R}$ such that $\rho(x) = \inf \left\{\lambda > 0 \colon x\lambda^{-1} \in C\right\}$ (it is also called the gauge of C). Recently in [10], it was introduced the notion of $C-$nearest points. Let us fix a closed convex C such that 0 in its interior in a Banach space X. For a closed set F of X we define the $C-$nearest distance from x to F by

$$\rho(F, x) := \inf \{\rho(x - s) : s \in F\}$$

where ρ is the Minkowski functional of C. A point $z \in F$ is called a $C-$nearest point of F if there exists $x \in X \setminus \{x \in X \colon \rho(F, x) = 0\}$ such that $\rho(F, x) = \rho(x - z)$.

In [10] it is shown that if C has the Drop property (a closed convex set C has the Drop property if for every closed set F disjoint from C there exists an $a \in F$ such that $F \cap conv(C \cup \{a\}) = \{a\}$, see [7], [10] for more details) then for all closed subset F of X, the set of points of $X \setminus \{x \in X \colon \rho(F, x) = 0\}$ with $C-$nearest point in F contains a dense G_δ subset of $X \setminus \{x \in C \colon \rho(F, x) = 0\}$, thus extending a result of Lau [9].

In this work, we define the $C-$farthest distance. Let us fix a closed convex set C in Banach space X and assume $\text{int}C \neq \emptyset$, without loss of generality suppose the unit ball is contained in C. For a bounded subset S of X and x in X we define the $C-$farthest distance from x to S by

$$\tau(S, x) = \sup \{\rho(x - y) : y \in S\}$$

where ρ is the Minkowski gauge of C. A point $s \in S$ is called $C-$farthest point of S if there exists $x \in X$ such that

$$\tau(S, x) = \rho(x - s).$$

We prove that if S is weakly compact then the set of all points in X which have $C-$farthest points in X contains a dense G_δ of X, thus extending a result of Lau [8].

It is easy to see the $C-$farthest distance Lipschitz-continuous.

PROPOSITION 1.1. *The function* $x \longrightarrow \tau(S,x)$ *is* $1-$*Lipschitz-continuous.*

Proof. Let $x, y \in X$. Then it is easy to see that

$$\tau(S,x) - \tau(S,y) = \tau(S, x-y+y) - \tau(S,y) \leq \rho(x-y)$$

By hypothesis, the unit ball is contained in C, then $\rho(y) \leq \|y\|$ for all $y \in X$.
Therefore

$$\tau(S,x) - \tau(S,y) \leq \rho(x-y) \leq \|x-y\|, \qquad \forall x, y \in X$$

then we deduce that τ is $1-$Lipschitz-continuous, and the proof is complete. ∎

For a convex function f and x in X with $f(x)$ finite, we define the subdifferential of f at x by

$$\partial^{-} f(x) = \{x^* \in X^* : \ < x^*, y-x > \leq f(y) - f(x), \quad \forall y \in X\}.$$

For $f \in X^*$, write $\rho_*(f) := \sup\{f(x) : x \in C\}$.

LEMMA 1.2. *Let* X *be a Banach space and let* S *be a bounded subset in* X. *Then for* $x \in X$, *each element* x^* *of* $\partial^{-}\tau(S,x)$ *we have:*
1) $< x^*, y > \leq \rho(y)$, *for all* $y \in X$
2) $\rho_*(x^*) \leq 1$.

Proof. We have

$$\begin{aligned}
\tau(S, x+ty) - \tau(S,x) &= \sup_{z \in S}\{\rho(x+ty-z)\} - \sup_{u \in S}\{\rho(x-u)\} \\
&\leq \sup_{z \in S}\{\rho(x-z)\} + \rho(ty) - \sup_{u \in S}\{\rho(x-u)\} \\
&= \rho(ty)
\end{aligned}$$

then for $t > 0$,

$$\frac{\tau(S, x+ty) - \tau(S,x)}{t} \leq \rho(y).$$

Let $x^* \in \partial^{-}\tau(S,x)$. Since the function $\tau(S,x)$ is convex (as supremum of such functions), then

$$< x^*, y-x > + \tau(S,x) \leq \tau(S,y), \quad \forall y \in X$$

which is equivalent to say that:

$$0 \leq \tau(S, x+ty) - \tau(S, x) - < x^{*}, x+ty-x >, \quad \forall y \in X, \quad \forall t \in \mathbb{R}$$

which implies that

$$\begin{aligned} 0 &\leq \frac{\tau(S, x+ty) - \tau(S, x)}{t} - < x^{*}, y >, \quad \forall t > 0, \quad \forall y \in X \\ &\leq \rho(y) - < x^{*}, y >, \quad \forall y \in X \end{aligned}$$

hence

$$< x^{*}, y > \leq \rho(y), \quad \forall y \in X,$$

consequently

$$\rho_{*}(x^{*}) := \sup\{< x^{*}, y >: y \in C\} \leq \sup\{\rho(y) : y \in C\} \leq 1$$

and the proof is complete. ∎

Proposition 1.3. *Let X be a Banach space and let S be a bounded subset in X. Let x in X and x^{*} in $\partial^{-}\tau(S, x)$. Then*

$$-\tau(S, x) \leq \inf_{y \in S} < x^{*}, y - x > .$$

Proof. Let x in X and let x^{*} in $\partial^{-}\tau(S, x)$. Then by Lemma 1.2

$$< x^{*}, y > \leq \rho(y), \quad \forall y \in X$$

then

$$< x^{*}, x - y > \leq \rho(x - y), \quad \forall y \in X$$

which is equivalent to :

$$-\rho(x - y) \leq < x^{*}, y - x > \quad \forall y \in X$$

therefore

$$-\sup_{y \in S} \rho(x - y) \leq \inf_{y \in S} < x^{*}, y - x >$$

concluding

$$-\tau(S, x) \leq \inf_{y \in S} < x^{*}, y - x >$$

and the proof is complete. ∎

2. $C-$FARTHEST POINTS

Recall that C be a norm closed, convex and $0 \in intC$ in Banach space X. Without loss of generality assume that the unit ball of X contained in C. Denote by ρ the Minkowski functional of $C: \rho(x) := \inf\left\{\lambda > 0 \colon x\lambda^{-1} \in C\right\}$. For a bounded subset S in X and $x \in X$ we define the $C-$farthest distance from x to S by $\tau(S, x) = \sup\{\rho(x-y) : y \in S\}$.

In this section we shall prove the following :

THEOREM 2.1. *Let X be a Banach space. Let S be a weakly compact subset in X. Then the set $\{x \in X \colon \rho(x-z) = \tau(S,x) \text{ for some } z \in S\}$ contains a dense G_δ of X. Furthermore, the set of $C-$farthest points of S is nonempty.*

Thus, in the particular case when C is the unit ball the set

$$\{x \in X \colon \|x-z\| = \sup\{\|x-y\| \colon y \in S\} \text{ for some } z \in S\}$$

contains a dense G_δ of X, this is exactly Lau's theorem [8].

It is well known that every bounded, weakly closed subset S in reflexive Banach space X is weakly compact. Then the set

$$D := \{x \in X \colon \rho(x-z) = \tau(S,x) \text{ for some } z \in S\}$$

contains a dense G_δ subset of X and hence the set of $C-$farthest points of S is nonempty. For the same reasons, the set

$$A = \{x \in X \colon \|x-z\| = \sup\{\|x-y\| \colon y \in S\} \text{ for some } z \in S\}$$

contains a dense G_δ subset of X and hence the set of farthest points of S is nonempty.

We like to extend Asplund [1] result. In this direction we give the following definition :

DEFINITION 2.2. 1) The Minkowski function ρ is said to be locally uniformly rotund at x, if $\lim \rho(x-x_n) = 0$ whenever (x_n) is a sequence in X is such that $\lim \rho(x_n) = \rho(x)$ and $\lim \rho(x+x_n) = 2\rho(x)$. If ρ is locally uniformly rotund at each point of X, we call ρ locally uniformly rotund or locally uniformly convex (L.U.R. for short).

2) A point x in a closed convex set F is said to be strongly $C-$exposed provided there exists $x^* \neq 0$ in X^*, such that $x^*(x) = \sup_F x^*$ and for each $(x_n) \subset F$,

$$< x^*, x_n > \longrightarrow \sup_F x^* \quad \text{implies} \quad \rho(x_n - x) \longrightarrow 0.$$

Noting that the definition of local uniform rotundity for the gauge has been introduced independently in [5].

Remark 2.3. Assume now that the gauge ρ is L.U.R. and let S be a bounded set in X and $x \in X$ has a $C-$farthest point in S, i.e. there is $s \in S$ such that $\rho(x-s) = \sup\{\rho(x-y) : y \in S\}$. Then s is strongly $C-$exposed point of the set $x - \rho(x-s)C$ and s is thus a strongly $C-$exposed point of $S \subset x - \rho(x-s)C$. In an other hand, by using the separation theorem, one can show that any point of ∂C is a strongly $C-$exposed point of C.

Corollary 2.4. *Let X be a reflexive Banach space. We assume that ρ is locally uniformly convex. Then for every bounded, closed subset S in X, the set*

$$A = \{x \in X : \rho(x-z) = \tau(S,x) \text{ for some } z \in S\}$$

contains a dense G_δ subset of X and hence the set of $C-$farthest points of S is nonempty.

Proof. By Remark 2.3 we deduce that, each $C-$farthest point of $\overline{conv}S$ is a strongly $C-$exposed point of $\overline{conv}S$ and hence is contained in S. So that, the sets of $C-$farthest points of S and $\overline{conv}S$ coincide. We apply now Theorem 2.1 on $\overline{conv}S$ and the proof is complete. ∎

In the particular case where C is the unit ball we have Asplund [1] result:

Corollary 2.5. *Let X be a reflexive Banach space with locally uniformly convex norm. Then for every bounded, closed subset S in X, the set*

$$A = \{x \in X : \|x-z\| = \sup\{\|x-y\| : y \in S\} \text{ for some } z \in S\}$$

contains a dense G_δ subset of X and hence the set of farthest points of S is nonempty.

Proof. The unit ball of X is convex with nonempty interior and $\|.\|$ is just its Minkowski gauge. The corollary is now a particular case of the Corollary 2.4. ∎

Now its time to give the proof of Theorem 2.1. For $n \in \mathbb{N}$, put

$$F_n = \left\{x \in X, \inf_{y \in S} < x^*, y - x > \geq -\tau(S,x) + \frac{1}{n}, \text{ for some } x^* \in \partial^- \tau(S,x)\right\}$$

PROPOSITION 2.6. *F_n is a closed subset of X.*

Proof. Let (x_m) be a sequence in F_n which converges to an x in X. For each m, choose $x_m^* \in \partial^- \tau(S, x_m)$ such that $\inf_{z \in S} < x_m^*, z - x_m > \geq -\tau(S, x_m) + \frac{1}{n}$ and by Lemma 1.2: $\rho_*(x_m^*) \leq 1$.

Since the unit ball is contained in C, then

$$\rho(x) \leq \|x\|, \quad \forall x \in X \quad \text{and} \quad \|x_m^*\|_{X^*} \leq \rho_*(x_m^*) \leq 1. \tag{1}$$

Therefore, without loss of generality, we assume that (x_m^*) converges weak* to x^*.

Let $y \in X$. Then :

$$\begin{gathered} | < x_m^*, y - x_m > - < x^*, y - x > | \leq \\ |< x_m^*, y - x_m > - < x_m^*, y - x >| + |< x_m^*, y - x > - < x^*, y - x >| = \\ |< x_m^*, x - x_m >| + |< x_m^* - x^*, y - x >| \leq \\ \rho_*(x_m^*)\, \rho(x_m - x) + |< x_m^* - x^*, y - x >| \leq \\ \|x_m - x\| + |< x_m^* - x^*, y - x >| \end{gathered}$$

(the last inequality is by (1)).

This shows that the sequence $(< x_m^*, y - x_m >)$ converges to $< x^*, y - x >$.

In the other hand, we have $x_m^* \in \partial^- \tau(S, x_m)$, then

$$< x_m^*, y - x > + \tau(S, x_m) \leq \tau(S, y), \quad \forall y \in X,$$

hence it follows that (by passing to the limit) :

$$< x^*, y - x > + \tau(S, x) \leq \tau(S, y), \quad \forall y \in X,$$

which implies that

$$x^* \in \partial^- \tau(S, x). \tag{2}$$

In the other hand, by definition of x_m, we have

$$< x_m^*, z - x_m > \geq -\tau(S, x_m) + \frac{1}{n}, \quad \forall z \in S$$

hence

$$< x^*, z - x > \geq -\tau(S, x) + \frac{1}{n}, \quad \forall z \in S. \tag{3}$$

By (2) and (3) we deduce that $x \in F_n$, and F_n is a closed subset of X. ∎

For $x \in X$, write $r(x) = \sup\{\|x - z\| : z \in S\}$.

Proposition 2.7. *For each n, F_n has empty interior.*

Proof. Suppose that for some n, F_n has nonempty interior, then there is a ball U centered at $y_0 \in F_n$ of radius $2\lambda r(y_0)$ (where $r(y_0) = \sup\{\|x - y_0\| : x \in S\}$) for some $\lambda > 0$ such that $U \subset F_n$. Let $\varepsilon = \frac{\lambda}{4(1+\lambda)n} \min\{r(y_0), 1\}$. Choose z_0 in S such that

$$\tau(S, y_0) \geq \rho(y_0 - z_0) > \tau(S, y_0) - \varepsilon > \tau(S, y_0)/2 \tag{1}$$

and put

$$x_0 = y_0 + \lambda(y_0 - z_0). \tag{2}$$

Choose x_1 in the segment $[x_0, y_0]$ such that

$$\|x_0 - x_1\| = \varepsilon. \tag{3}$$

We have $\|x_0 - y_0\| = \lambda\|y_0 - z_0\| \leq \lambda r(y_0) < 2\lambda r(y_0)$. So x_0 and thus $x_1 \in U \subset F_n$. We have $x_1 \in F_n$, therefore by definition of F_n, there exists $x_1^* \in \partial^- \tau(S, x_1)$ such that

$$\inf_{y \in S}\{< x_1^*, y - x_1 >\} \geq -\tau(S, x_1) + \frac{1}{n}, \tag{4}$$

since $x_1^* \in \partial^- \tau(S, x_1)$, by Lemma 1.2

$$\rho_*(x_1^*) \leq 1, \tag{5}$$

by (1) we have $\tau(S, y_0) < \rho(y_0 - z_0) + \varepsilon$. So

$$\tau(S, y_0) - \tau(S, x_1) < \rho(y_0 - z_0) + \varepsilon - \tau(S, x_1), \tag{6}$$

by (2) we have

$$x_0 - z_0 = (1 + \lambda)(y_0 - z_0), \text{ then } \frac{\rho(x_0 - z_0)}{1 + \lambda} = \rho(y_0 - z_0). \tag{7}$$

Combining (3), (4), (5), (6) and (7) we show :

$$\begin{array}{rcl}\tau(S,y_0)-\tau(S,x_1) & < & \frac{1}{1+\lambda}\rho(x_0-z_0)+\varepsilon-\tau(S,x_1)\\ & \leq & \frac{1}{1+\lambda}\tau(S,x_0)+\varepsilon-\tau(S,x_1)\\ & = & \frac{1}{1+\lambda}\sup\{\rho(x_0-z)\,;z\in S\}+\varepsilon-\tau(S,x_1)\\ & \leq & \frac{1}{1+\lambda}\left[\tau(S,x_1)+\rho(x_0-x_1)\right]+\varepsilon-\tau(S,x_1)\\ & \leq & \frac{1}{1+\lambda}\tau(S,x_1)+\frac{\|x_0-x_1\|}{1+\lambda}+\varepsilon-\tau(S,x_1)\\ & < & \frac{1}{1+\lambda}\tau(S,x_1)+2\varepsilon-\tau(S,x_1)\\ & = & \frac{-\lambda}{1+\lambda}\tau(S,x_1)+2\varepsilon\\ & \leq & \frac{\lambda}{1+\lambda}\left[<x_1^*,z_0-x_1>-\frac{1}{n}\right]+2\varepsilon\\ & \leq & \frac{\lambda}{1+\lambda}\left[<x_1^*,z_0-x_0>-\frac{1}{n}\right]+3\varepsilon\\ & = & <x_1^*,y_0-x_0>-\frac{\lambda}{(1+\lambda)\,n}+3\varepsilon\\ & \leq & <x_1^*,y_0-x_1>-\frac{\lambda}{(1+\lambda)\,n}+4\varepsilon\\ & \leq & <x_1^*,y_0-x_1>.\end{array}$$

Hence $\tau(S,y_0) < \tau(S,x_1)+ < x_1^*,y_0-x_1 >$ and this contradicts $x_1^*\in\partial^-\tau(S,x_1)$. Thus concluding the proof of the proposition. ∎

Now we have all tools to give the proof of Theorem 2.1.

3. Proof of Theorem 2.1

Proof. For each $n\in\mathbb{N}$, put

$$F_n=\left\{x\in X,\inf_{y\in S}<x^*,y-x>\geq-\tau(S,x)+\frac{1}{n},\quad\text{for some } x^*\in\partial^-\tau(S,x)\right\}$$

and

$$F=\bigcup_n F_n.$$

Let $D = X \backslash F = X \backslash \bigcup_n F_n = \bigcap_n (X \backslash F_n)$

By Propositions 2.6 and 2.7, for all n, F_n is closed and has empty interior, then for all $n, X \backslash F_n$ is open and a dense subset in X. Hence D is a dense G_δ in X.

For finishing the proof, we need to show that the set D is contained in the set

$$\{x \in X : \rho (x - z) = \tau (S, x) \quad \text{for some} \quad z \in S\}.$$

Let $x \in D$ and let $x^* \in \partial^- \tau (S, x)$. Then x is not in F_n for all n. Which means that $\inf \{< x^*, z - x >: z \in S\} = -\tau (S, x)$. Since S is a weakly compact subset in X, there exists a point z_0 in S such that $-\tau (S, x) =< x^*, z_0 - x >$. Therefore

$$\tau (S, x) = \sup_{y \in S} \rho (x - y) \geq \rho (x - z_0) \geq < x^*, x - z_0 >= \tau (S, x)$$

Then, there is z_0 in S such that $\tau (S, x) = \rho (x - z_0)$, and the theorem is proved. ∎

The following definition it was introduced in [6]:

DEFINITION 3.1. We say that the gauge ρ has the $C-$intersection property, if for every bounded, closed and convex subset S in X and x_0 in $X \backslash S$, there exists $r > 0$ and x_1 in X such that $\rho (x_1, r) \supset C$ and $x_0 \notin \rho (x_1, r)$ where $\rho (x_1, r) = \{x \in X : \rho (x_1 - x) \leq r\}$

Remark 3.2. Repeating the same technics as in ([3] p. 55) one can prove the following: If the gauge ρ of C is Fréchet differentiable in $X \setminus \{0\}$, then ρ has the $C-$intersection property.

In the particular case where C is the unit ball this property was introduced by Mazur [11] and shown to hold for all Banach spaces having a Fréchet differentiable norm ([3] p. 55).

PROPOSITION 3.3. *Let X be a Banach space. We assume that the Minkowski gauge ρ of C has the $C-$intersection property. Let S be a weakly compact convex subset in X. Then S is a closed convex hull of points which are $C-$farthest points in S to some points in X.*

Therefore we deduce that, when C is the unit ball of X, then S is a closed convex hull of points which are farthest points in S to some points in X, this is exactly the Edelstein's theorem [4].

We give now the proof of Proposition 2.10.

Proof. Let $Far\,(S)$ be the set of all points in S which are $C-$farthest to some points in X. We shall show that $S = \overline{conv} Far\,(S)\,.$

By absurd, assume that there is $s \in S \backslash \overline{conv}\,(Far\,(S))\,.$ By hypothesis the Minkowski gauge ρ has the $C-$intersection property, then there is $x_0 \in X$ and $r > 0$ such that $S \subset \rho\,(x_0, r)$ and $s \notin \rho\,(x_0, r)$ where $\rho\,(x_0, r) = \{x \in X \colon \rho\,(x_0 - x) \leq r\}\,.$ Then $\rho\,(x_0 - s) > r$. Let $\varepsilon = \frac{\rho(x_0 - s) - r}{4} > 0$. Now Theorem 2.1 confirm that there exists $x_1 \in X$ such that

$$\|x_0 - x_1\| < \varepsilon =: \frac{\rho\,(x_0 - s) - r}{4}, \tag{1}$$

and x_1 has a $C-$farthest point $s_1 \in Far\,(S)\,.$ Since $s \in S$, then

$$\rho\,(x_1 - s) \leq \rho\,(x_1 - s_1) \tag{2}$$

combining (1), (2) and the fact that $\rho\,(x) \leq \|x\| \quad \forall x \in X$, we show that

$$\begin{aligned}
\rho\,(x_0 - s_1) &= \rho\,(x_0 - x_1 + x_1 - s_1) \\
&\geq \rho\,(x_1 - s_1) - \rho\,(x_0 - x_1) \\
&\geq \rho\,(x_1 - s) - \rho\,(x_0 - x_1) \\
&= \rho\,(x_1 - x_0 + x_0 - s) - \rho\,(x_0 - x_1) \\
&\geq \rho\,(x_0 - s) - \rho\,(x_1 - x_0) - \rho\,(x_0 - x_1) \\
&\geq \rho\,(x_0 - s) - 2\,\|x_0 - x_1\| \\
&\geq \rho\,(x_0 - s) - 2\varepsilon \\
&= 4\varepsilon + r - 2\varepsilon \\
&= 2\varepsilon + r > r
\end{aligned}$$

a contradiction with $S \subset \rho\,(x_0, r) = \{x \in X \colon \rho\,(x_0 - x) \leq r\}\,.$ Thus $S = \overline{conv}\,(Far\,(S))$ and the proof is complete. ∎

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