EXTRACTA MATHEMATICAE Vol. **16**, Núm. 2, 153 – 210 (2001)

Characteristically Nilpotent Lie Algebras: A Survey

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(Survey paper presented by J.A. Jaramillo)

AMS *Subject Class.* (2000): 17B30, 17B56, 17B70 *Received January* 16, 2001

1. Introduction

The theory of characteristically nilpotent Lie algebras constitutes an independent research object since 1955. Until then, most studies about Lie algebras were oriented to the classical aspects of the theory, such as semisimple and reductive Lie algebras [92]. Though there exists a precedent in the theory of nilpotent Lie algebras; the Ph.D. thesis of K. Umlauf [95] in 1891, their structure was practically unknown and only classical results like Engel's theorem were known. From 1939 on, when Lie theorists were seeking from adequate presentations of the semisimple Lie algebras in terms of generators and relations, N. Jacobson proved that the exceptional complex simple Lie algebra G_2 of dimension 14 could be presented as the algebra of derivations of the Cayley algebra [47]. This result increased the interest in analyzing the derivations of an arbitrary Lie algebra. However, it was not until the fifties when the first determining results about derivations of nilpotent Lie algebras were obtained. It was proven that any nilpotent Lie algebra has an outer derivation, i.e., there exists at least one derivation which is not the adjoint operator for a vector of the algebra. Two years earlier, E. V. Schenkman [85] had published his derivation tower theorem for centerless Lie algebras, which described in a nice manner the derivation algebras. This theory was not applicable to the nilpotent algebras, as the adjoint representation is not faithful. This fact led to the assumption that the structure of derivations for nilpotent Lie algebras is much more difficult than for classical algebras. Again, Jacobson proved in 1955 that any Lie algebra over a field of characteristic zero which

has nondegenerate derivations is nilpotent. In the same paper [48] he asked for the converse. This result is assumed to be the origin of the theory of characteristically nilpotent Lie algebras. Dixmier and Lister [29] gave a negative answer to the converse of Jacobson's theorem. They defined a generalization of the central descending sequence and called the algebras satisfying the nullity of a power characteristically nilpotent. The example of Dixmier and Lister constituted the milestone for a new class of Lie algebras which seem, in appearance, to be scarce. The first paper about the structure of characteristically nilpotent Lie algebras, short CNLA, is due to Leger and Tôgô in 1959. They proved the equivalence of the sequence condition of Dixmier and Lister and the nilpotence of the Lie algebra of derivations. Although this paper does not give any additional example of such an algebra, it reduces the search to the class of nilpotent Lie algebras. On the other side, the deduced properties of a CNLA excluded the 2-step nilpotent or metabelian Lie algebras. The last author, S. Tôgô, published in 1961 an excellent work which contained much of the information known about derivation algebras of Lie algebras. Among others, he introduced special classes of algebras which were shown to be non CNLA [93]. The importance of CNLAs within the variety of nilpotent Lie algebra laws was soon recognized by the author, and he also formulated an interesting question which is nowadays not satisfactorily solved: the problem of Tôgô. He asked for the existence of CNLA of derivations, this is, algebras for which both the derivations and the derivations of these are nilpotent. Very little is known about the general structure of such Lie algebras, though its existence has been verified by various authors [8]. The deformations theory for algebraic structures of M. Gerstenhaber in 1964 [38], originally developed to study the rigidity of algebraic structures, has become since then a powerful tool to determine the nilpotence of derivations.

M. Vergne [96] applied in 1966 the cohomology theory of Lie algebras [60] to the study of the variety of nilpotent Lie algebras, obtaining in particular interesting results about its irreducible components. In particular, she showed the existence of only two naturally graded filiform Lie algebras, L_n and Q_n, the second existing only in even dimension. In particulñar, the first algebras has been a central research object for the last thirty years. Studying its deformations, lots of families of CNLA have been constructed [52]. In 1970 J. L. Dyer gave a nine dimensional example of CNLA [33], which was interesting in its own as it had an unipotent automorphism group. This property is not satisfied by the original example of Dixmier and Lister, and showed than even CNLA can have quite different behaviours. By that time, it was perfectly

known that such algebras could exist only from dimension 7 on, as a consequence of the classification in 1958 of the six dimensional algebras [75]. In 1972 G. Favre discovered the lowest dimensional CNLA known until then [35], which additionally was of the same nature as Dyer's example. At the same time, G. D. Leger and E. Luks investigated the metabelian Lie algebras and proved several results about their rank, and establishing that rank one algebras were given if the existence of a characteristic ideal containing the derived subalgebra is assured. These results can be interpreted as a constructive proof that the original example of 1957 is the known CNLA with lowest characteristic sequence. The last author applied in 1976 computational methods to prove the existence of CNLA in any dimension greater or equal to seven. Four years later, S. Yamaguchi constructed families of CNLA in arbitrary dimension, constructions that have been completed and generalized in later years [52]. The topological study of the variety of Lie algebra laws led R. Carles [18] to study the topological properties of CNLA. Among other results he states that the set of CNLA is constructible for $n \geq 7$. For the particular dimension 7, he also proves that CNLA do not form an open set. Recently [8] this result has been generalized to any dimension. Another interesting approach to the CNLA has been deformation theory applied to the Borel subalgebras of complex simple Lie algebras, like done by Y. B. Khakimdjanov in 1988 to prove that almost all deformations of the nilradical of Borel subalgebras of complex simple Lie algebras are characteristically nilpotent. This has shown that these algebras are in fact in abundance within the variety of nilpotent laws. M. Goze and the last cited author [40] proved, in 1994, that for any dimension $n \geq 9$ an irreducible component of the filiform variety $\mathcal{F}_n$ contains an open set consisting of CNLA.

Filiform Lie algebras, specifically the model filiform Lie algebra L_n, has been also the fundamental source for constructing families of CNLAs. In particular, its cohomology has been calculated, which has allowed to describe its deformations in a precise manner and characterize those deformations which are characteristically nilpotent [54]. Recently, we have turned our interest to nilpotent Lie algebras which structurally "look like Q_n". As known, this algebra cannot appear in odd dimension. This is a consequence of the so called centralizer property [22], which codifies information about the structure of the commutator subalgebra and the ideals of the central descending sequence. Now the centralizer property can be generalized to any naturally graded nilpotent Lie algebra, and defines a class of algebras which can be interpreted as those which are the "easiest nilpotent Lie algebras to deform for obtaining

CNLAs". The key to this is extension theory combined with deformation theory. This approach also leads to certain questions about the rigidity of a nilpotent Lie algebra. In 1970 Vergne postulated the nonexistence of nilpotent Lie algebras that are rigid in the variety $\mathfrak{L}^n$ for $n \neq 1$. In his study about the structure of rigid Lie algebras [18], Carles established that if a nilpotent Lie algebra is rigid, then it necessarily must be a CNLA. The strongness of this result seems to confirm the validity of the conjeture, although there is no known procedure to prove it.

Finally, we review some results about affine structures over Lie algebras. This kind of structures are of great importance not only for purposes of cohomology theory [15], but also for representation theory of nilpotent Lie algebras. The interesting point is that CNLA can admit an affine structure, such as it was proven for the example of Dixmier and Lister by Scheunemann [87] in 1974. Although practically nothing is known about CNLA with affine structures, the cohomological method developed by Burde in [15] could be an important source for studying these algebras.

2. Generalities

In this section we resume the elementary facts about Lie algebras that will be used thorughout the paper. Although it is often unnecessary to specify the base field, we will asume here that all Lie algebras are complex.

Definition. Let $\mathfrak{g}$ be a finite dimensional vectorial space over $\mathbb{C}$. A Lie algebra law over $\mathbb{C}^n$ is a bilinear alternated mapping $\mu \in \mathrm{Hom}\,(\mathbb{C}^n \times \mathbb{C}^n, \mathbb{C}^n)$ which satisfies the conditions

(1) $\mu\,(X, X) = 0$ for all $X \in \mathbb{C}^n$;

(2) $\mu\,(X, \mu\,(Y, Z)) + \mu\,(Z, \mu\,(X, Y)) + \mu\,(Y, \mu\,(Z, X)) = 0$ for all $X, Y, Z \in \mathbb{C}^n$ (Jacobi identity).

If μ is a Lie algebra law, the pair $\mathfrak{g} = (\mathbb{C}^n, \mu)$ is called Lie algebra. From now on we identify the Lie algebra with its law μ.

Remark. We say that μ is the law of $\mathfrak{g}$, and where necessary we use the bracket notation to describe the law:

$$[X, Y] = \mu\,(X, Y) \qquad \forall\, X, Y \in \mathfrak{g}\,.$$

The nondefined brackets are zero or obtained by symmetry.

DEFINITION. Given an ideal $\mathfrak{I}$ of $\mathfrak{g}$, we call centralizer of $\mathfrak{I}$ in $\mathfrak{g}$ to the subalgebra

$$C_{\mathfrak{g}}\mathfrak{I} = \{X \in \mathfrak{g} \mid \mu(X, \mathfrak{I}) = 0\}.$$

To any Lie algebra we can associate the two following sequences:

$$D^0\mathfrak{g} = \mathfrak{g} \supset D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \supset \cdots \supset D^k\mathfrak{g} = \left[D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}\right] \supset \cdots$$
$$C^0\mathfrak{g} = \mathfrak{g} \supset C^1\mathfrak{g} = D^1\mathfrak{g} \supset C^2\mathfrak{g} = \left[C^1\mathfrak{g}, \mathfrak{g}\right] \supset \cdots \supset C^k\mathfrak{g} = \left[C^{k-1}\mathfrak{g}, \mathfrak{g}\right] \supset \cdots$$

called respectively derived and descending central sequences of $\mathfrak{g}$.

DEFINITION. Let $\mathfrak{g}$ be a Lie algebra. We say that

(1) $\mathfrak{g}$ is solvable if there exists an integer $k \geq 1$ such that $D^k\mathfrak{g} = \{0\}$;
(2) $\mathfrak{g}$ is nilpotent if there exists an integer (called nilindex $n(\mathfrak{g})$ of $\mathfrak{g}$) $k \geq 1$ such that $C^k\mathfrak{g} = \{0\}$.

DEFINITION. An n-dimensional nilpotent Lie algebra is called filiform if

$$\dim C^k\mathfrak{g} = n - k - 1,\ 1 \leq k \leq n - 1.$$

Remark. Calling $p_i = \dim\left(\frac{C^{i-1}\mathfrak{g}}{C^i\mathfrak{g}}\right)$ for $1 \leq p_i \leq n(\mathfrak{g})$, the type of the nilpotent Lie algebra is the sequence $\{p_1, \ldots, p_r\}$. Then a filiform algebra corresponds to those of type $\{2, 1, \ldots, 1\}$ [97].

We recall the laws for the $(n+1)$-dimensional filiform Lie algebras L_n and Q_n, which are basically the only filiform Lie algebras we have to deal with here:

- L_n $(n \geq 3)$: $[X_1, X_i] = X_{i+1},\ 2 \leq i \leq n,$ over the basis $\{X_1, \ldots, X_{n+1}\}$.
- Q_{2m-1} $(m \geq 3)$: $[X_1, X_i] = X_{i+1},\ 2 \leq i \leq 2m-1,$ and $[X_j, X_{2m+1-j}] = (-1)^j X_{2m},\ 2 \leq j \leq m,$ over the basis $\{X_1, \ldots, X_{2m}\}$.

DEFINITION. A Lie algebra $\mathfrak{g}$ is graded over $\mathbb{Z}$ if it admits a decomposition

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$$

where the $\mathfrak{g}_k$ are $\mathbb{C}$-subspaces of $\mathfrak{g}$ which satisfy $[\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s}$, $r, s \in \mathbb{Z}$.

Observe that any graduation defines a sequence $S_k = F_k(\mathfrak{g}) = \bigoplus_{t\geq k} \mathfrak{g}_t$ with the properties

(1) $\mathfrak{g} = \bigsqcup S_k$,

(2) $[S_i, S_j] \subset S_{i+j} \ \forall i, j$,

(3) $S_i \subset S_j$ si $i > j$.

Definition. A family $\{S_i\}$ of subspaces of $\mathfrak{g}$ define a filtration (descending) over $\mathfrak{g}$ if it satisfies the above properties (1)-(3). The algebra is called filtered.

The construction can be reversed, i.e., any filtration defines a graduation by taking $\mathfrak{g}_k = \frac{S_{k-1}}{S_k}$ for $k \geq 1$. The graduation is called associated to the filtration $\{S_i\}$ and it defines a Lie algebra.

Definition. A nilpotent Lie algebra is called naturally graded if $\mathfrak{g} \simeq \mathfrak{gr}(\mathfrak{g})$, where $\mathfrak{gr}(\mathfrak{g})$ is the graduation associated to the filtration induced in $\mathfrak{g}$ by the central descending sequence.

It follows immediately that both L_n and Q_n are naturally graded. They are in fact the only filiform Lie algebras having this property [97].

Definition. A derivation f of a Lie algebra $\mathfrak{g}$ is a linear mapping $f : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[f(X), Y] + [X, f(Y)] - f[X, Y] = 0 \qquad \forall\ (X, Y) \in \mathfrak{g}^2 .$$

We denote by $\operatorname{Der}\mathfrak{g}$ the set of derivations of $\mathfrak{g}$. It is a Lie subalgebra of $\operatorname{End}\mathfrak{g}$.

Proposition. *For all X in $\mathfrak{g}$, the endomorphism adX is a derivation of $\mathfrak{g}$.*

Definition. The derivations f of $\mathfrak{g}$ which are of type $f = adX$ for $X \in \mathfrak{g}$ are called inner derivations.

2.1. Cohomology of Lie Algebras. There exists a general study of the cohomology of Lie algebra by considering the cohomology with values on a $\mathfrak{g}$-module. See for example references [60].

Let $\mathfrak{g}$ be a Lie algebra. A p-dimensional cochain of $\mathfrak{g}$ (with values in $\mathfrak{g}$) is a p-linear alternating mapping of $\mathfrak{g}^p$ in $\mathfrak{g}$ ($p \in \mathbb{N}^*$). A 0-cochain is a constant

function from $\mathfrak{g}$ to $\mathfrak{g}$. We denote by $C^p(\mathfrak{g},\mathfrak{g})$ as the space of the p-cochains and

$$C^*(\mathfrak{g},\mathfrak{g}) = \bigoplus_{p\geq 0} C^p(\mathfrak{g},\mathfrak{g}).$$

We can provide $C^p(\mathfrak{g},\mathfrak{g})$ of a $\mathfrak{g}$-module structure by putting

$$(X\Phi)(X_1,\ldots,X_p) = [X,\Phi(X_1,\ldots,X_p)] - \sum_{1\leq i\leq p} \Phi(X_1,\ldots,[X,X_i],\ldots,X_p)$$

for all $X_1,\ldots,X_p \in \mathfrak{g}$.

On the space $C^*(\mathfrak{g},\mathfrak{g})$ we define the endomorphism

$$\begin{array}{rcl} \delta : C^*(\mathfrak{g},\mathfrak{g}) & \to & C^*(\mathfrak{g},\mathfrak{g}) \\ \Phi & \to & \delta\Phi \end{array}$$

by putting $\delta\Phi(X) = X.\Phi$ if $\Phi \in C^0(\mathfrak{g},\mathfrak{g})$ and

$$\begin{aligned} \delta\Phi(X_1,\ldots,X_{p+1}) = & \sum_{1\leq s\leq p+1} (-1)^{s+1} (X_s.\Phi)(X_1,\ldots,\widehat{X}_s,\ldots,X_{p+1}) \\ & + \sum_{1\leq s\leq t\leq p+1} (-1)^{s+t} \Phi([X_s,X_t],X_1,\ldots,\widehat{X}_s,\ldots,\widehat{X}_{t,\ldots},X_{p+1}) \end{aligned}$$

if $\Phi \in C^p(\mathfrak{g},\mathfrak{g})$, $p \geq 1$.

By this definition, $\delta(C^p(\mathfrak{g},\mathfrak{g})) \subset C^{p+1}(\mathfrak{g},\mathfrak{g})$ and we can verify that

$$\delta \circ \delta = 0.$$

For all $p \geq 1$ we denote

$$Z^p(\mathfrak{g},\mathfrak{g}) = \operatorname{Ker} d\big|_{C^p(\mathfrak{g},\mathfrak{g})}, \qquad B^p(\mathfrak{g},\mathfrak{g}) = \operatorname{Im} d\big|_{C^p(\mathfrak{g},\mathfrak{g})}$$

and $H^p(\mathfrak{g},\mathfrak{g}) = Z^p(\mathfrak{g},\mathfrak{g})/B^p(\mathfrak{g},\mathfrak{g})$. This quotient space is called the cohomology space of $\mathfrak{g}$ of degree p (with values in $\mathfrak{g}$). For $p=0$ we put $B^0(\mathfrak{g},\mathfrak{g}) = \{0\}$ and $H^0(\mathfrak{g},\mathfrak{g}) = Z^0(\mathfrak{g},\mathfrak{g})$. This last space can be identified to the space of all $\mathfrak{g}$-invariant elements that is $\{X \in \mathfrak{g} \mid adY(X) = 0 \quad \forall\ Y \in \mathfrak{g}\}$. Then $Z^0(\mathfrak{g},\mathfrak{g}) = Z(\mathfrak{g})$ (the center of $\mathfrak{g}$).

2.1.1. The Space $H^1(\mathfrak{g},\mathfrak{g})$. We have $Z^1(\mathfrak{g},\mathfrak{g}) = \{f : \mathfrak{g} \to \mathfrak{g} \mid \delta f = 0\}$. But $\delta f(X,Y) = [f(X),Y] + [X,f(Y)] - f[X,Y]$. Then $Z^1(\mathfrak{g},\mathfrak{g})$ is nothing but the algebra of derivations of $\mathfrak{g}$:

$$Z^1(\mathfrak{g},\mathfrak{g}) = \operatorname{Der}\mathfrak{g}\,.$$

It is the same for:

$$B^1(\mathfrak{g},\mathfrak{g}) = \{adX \mid X \in \mathfrak{g}\}\,.$$

Thus the space $H^1(\mathfrak{g},\mathfrak{g})$ can be interpreted as the set of the outer derivations of the Lie algebra $\mathfrak{g}$.

Let I be an ideal of $\mathfrak{g}$. We consider the cochains $\varphi : I^p \to \mathfrak{g}$ on I with values in $\mathfrak{g}$. For these cochains we can also define, by restriction, the coboundary operator δ. As I is an ideal of $\mathfrak{g}$, $H^1(\mathfrak{g},\mathfrak{g})$ is a $\mathfrak{g}$-module. So we can consider the cohomology space $H^*(I,\mathfrak{g})$.

A p-cochain φ of $C^p(I,\mathfrak{g})$ is $\mathfrak{g}$-invariant if it satisfies:

$$\begin{aligned} X\varphi(X_1,\ldots,X_p) = {} & [X,\varphi(X_1,\ldots,X_p)] \\ & - \sum_{1\le i\le p} \varphi(X_1,\ldots,X_{i-1},[X,X_i]],\ldots,X_p) = 0\,. \end{aligned}$$

We denote by $C^*(I,\mathfrak{g})^{\mathfrak{g}}$ the set of cochains on I which are $\mathfrak{g}$-invariant and $H^*(I,\mathfrak{g})^{\mathfrak{g}}$ the correspondent cohomology space. Each element $\overline{\varphi}$ of $H^p(I,\mathfrak{g})^{\mathfrak{g}}$ has a representative which is the restriction to I of a cochain Ψ in $C^p(\mathfrak{g},\mathfrak{g})$ such that $d\psi \in \left(\mathfrak{g}/I,\mathfrak{g}^I\right)$ where $\mathfrak{g}^I = \{X \in \mathfrak{g} \mid [X,Y] = 0 \quad \forall\, Y \in I\}$. This element $d\psi$ does not depend upon the choice of the representative of $\overline{\varphi}$. Let t_{p+1}be the homomorphism so defined:

$$t_{p+1} : H^p(I,\mathfrak{g})^{\mathfrak{g}} \to H^{p+1}\left(\mathfrak{g}/I,\mathfrak{g}^I\right).$$

We define an exact sequence:

$$\begin{aligned} 0 \to H^p\left(\mathfrak{g}/I,\mathfrak{g}^I\right) \xrightarrow{l_p} H^p(\mathfrak{g},\mathfrak{g}) \xrightarrow{r_p} H^p(I,\mathfrak{g})^{\mathfrak{g}} \\ \xrightarrow{r_{p+1}} H^{p+1}\left(\mathfrak{g}/I,\mathfrak{g}^I\right) \to H^{p+1}(\mathfrak{g},\mathfrak{g}) \end{aligned}$$

where r_p is the homomorphism restriction and l_p is defined by looking upon the cochains of $\mathfrak{g}/I$ in $\mathfrak{g}^I$ as cochain of $\mathfrak{g}$ in $\mathfrak{g}$.

Example. We suppose that $\operatorname{codim}(I) = 1$. Then $\dim \mathfrak{g}/I = 1$ and $C^p(\mathfrak{g}/I,\mathfrak{g}) = 0$ for $p \ge 2$. Thus $0 \to 0 \to H^2(\mathfrak{g},\mathfrak{g}) \to H^2(I,\mathfrak{g})^{\mathfrak{g}} \to 0$ and we have

$$H^2(\mathfrak{g},\mathfrak{g}) = H^2(I,\mathfrak{g})^{\mathfrak{g}}\,.$$

2.2. The Spaces $H^2(\mathfrak{g},\mathbb{C})$. Recall that the space $H^2(\mathfrak{g},\mathbb{C}^p)$ can be interpreted as the space of classes of p-dimensional central extensions of the Lie algebra $\mathfrak{g}$. We recall the elementary facts:

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra with law μ_0. A central extension of $\mathfrak{g}$ by $\mathbb{C}^p$ is an exact sequence of Lie algebras

$$0 \to \mathbb{C}^p \to \bar{\mathfrak{g}} \to \mathfrak{g} \to 0$$

such that $\mathbb{C}^p \subset Z(\bar{\mathfrak{g}})$. Let α be a cocycle of the De Rham cohomology $Z^2(\mathfrak{g},\mathbb{C}^p)$. This gives the extension

$$0 \to \mathbb{C}^p \to \mathbb{C}^p \oplus \mathfrak{g} \to \mathfrak{g} \to 0$$

with associated law $\mu = \mu_0 + \alpha$ defined by

$$\mu((a,x),(b,y)) = (\alpha\mu_0(x,y), \mu_0(x,y)).$$

In the following we are only interested in extensions of $\mathbb{C}$ by $\mathfrak{g}$, i.e, extensions of degree one. It is well known that the space of 2-cocycles $Z^2(\mathfrak{g},\mathbb{C})$ is identified with the space of linear forms over $\bigwedge^2 \mathfrak{g}$ which are zero over the subspace Ω:

$$\Omega := \langle \mu_0(x,y) \wedge z + \mu_0(y,z) \wedge x + \mu_0(z,x) \wedge y \rangle_{\mathbb{C}} .$$

The extension classes are defined modulus the coboundaries $B^2(\mathfrak{g},\mathbb{C})$. This allows to identify the cohomology space $H^2(\mathfrak{g},\mathbb{C})$ with the dual of the space $\frac{\operatorname{Ker}\lambda}{\Omega}$, where $\lambda \in \operatorname{Hom}(\bigwedge^2 \mathfrak{g}, \mathfrak{g})$ is defined as

$$\lambda(x \wedge y) = \mu_0(x,y) \;\; x,y \in \mathfrak{g}.$$

In fact we have $H_2(\mathfrak{g},\mathbb{C}) = \frac{\operatorname{Ker}\lambda}{\Omega}$ for the 2-homology space, and as $H^2(\mathfrak{g},\mathbb{C}) = \operatorname{Hom}_{\mathbb{C}}(H_2(\mathfrak{g},\mathbb{C}),\mathbb{C})$ the assertion follows.

Notation. Let $\varphi_{ij} \in H^2(\mathfrak{g},\mathbb{C})$ the cocycles defined by

$$\varphi_{ij}(X_k, X_l) = \delta_{ik}\delta_{jl}.$$

Observe that a cocycle φ can be written as a linear combination of the preceding cocycles. We have:

Lemma. *$\sum a^{ij}\varphi_{ij} = 0$ if and only if $\sum a^{ij}(X_i \wedge X_j) \in \Omega$.*

Let $\mathfrak{g}$ be an n-dimensional nilpotent Lie algebra. The subspace of central extensions is noted by $E_{c,1}(\mathfrak{g})$. It has been shown that this space is irreducible and constructible. However, for our purpose this space is too general. We only need certain cohomology classes of this space.

NOTATION. For $k \geq 2$ let

$$H_k^{2,t}(\mathfrak{g},\mathbb{C}) = \left\{\varphi_{ij} \in H^2(\mathfrak{g},\mathbb{C}) \mid i+j = 2t+1+k\right\}, \quad 1 \leq t \leq \left[\frac{n-3}{2}\right],$$

$$H_k^{2,\frac{t}{2}}(\mathfrak{g},\mathbb{C}) = \left\{\varphi_{ij} \in H^2(\mathfrak{g},\mathbb{C}) \mid i+j = t+1+k\right\}, \quad \begin{array}{l} t \in \{1,\ldots,\left[\frac{n-3}{2}\right]\} \\ t \equiv 1 \,(\mathrm{mod}\ 2) \end{array}.$$

These cocycles are essential to determine the central extensions which are additionally naturally graded. If $\mathbf{E}_{c,1}(\mathfrak{g})$ denotes the central extensions that are naturally graded, we consider the subspaces

$$\mathbf{E}_{c,1}^{t,k_1,\ldots,k_r}(\mathfrak{g}) = \left\{\mu \in \mathbf{E}_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum \varphi_{ij}^{k_i}\right),\ \varphi_{ij}^{k_i} \in H_{k_i}^{2,t}(\mathfrak{g},\mathbb{C})\right\},$$

$$\mathbf{E}_{c,1}^{\frac{t}{2},k_1,\ldots,k_r}(\mathfrak{g}) = \left\{\mu \in \mathbf{E}_{c,1}(\mathfrak{g}) \mid \mu = \mu_0 + \left(\sum \varphi_{ij}^{k_i}\right),\ \varphi_{ij}^{k_i} \in H_{k_i}^{2,\frac{t}{2}}(\mathfrak{g},\mathbb{C})\right\},$$

where $0 \leq k_j \in \mathbb{Z}$, $j = 1,\ldots,r$.

Given a basis $\{X_1,\ldots,X_n,X_{n+1}\}$ of μ belonging to any of these spaces, the Lie algebra law is defined by:

$$\mu(X_i,X_j) = \mu_0(X_i,X_j) + \left(\sum \varphi_{ij}^k\right) X_{n+1},\ 1 \leq i,j \leq n\ \ (X_i,X_j) \in \mathfrak{g}^2.$$

LEMMA. *As vector spaces, the following identity holds:*

$$\mathbf{E}_{c,1}(\mathfrak{g}) = \sum_{t,k} \mathbf{E}_{c,1}^{t,k_1,\ldots,k_r}(\mathfrak{g}) + \mathbf{E}_{c,1}^{\frac{t}{2},k_1,\ldots,k_r}(\mathfrak{g}).$$

This follows easily. Observe that, though t is bounded by the dimension, $k \geq 2$ has no restrictions. However, the sum is finite, for the spaces $\mathbf{E}_{c,1}^{t,k_1,\ldots,k_r}$ are zero for almost any choice $(k_1,\ldots,k_r)$.

Given the Lie algebra $\mathfrak{g} = (\mathbb{C}^n,\mu_0)$, we have the associated graduation $\mathfrak{gr}(\mathfrak{g}) = \sum_{i=1}^{n(\mathfrak{g})} \mathfrak{g}_i$, where $\mathfrak{g}_i = \frac{C^{i-1}\mathfrak{g}}{C^i\mathfrak{g}}$ and $n(\mathfrak{g})$ is the nilindex of $\mathfrak{g}$. Independently of $\mathfrak{g}$ being naturally graded or not, any vector X has a fixed position in one of the graduation blocks. The study of the central extensions which preserve a graduation is reduced to the study of the position of the adjoined vector X_{n+1}. Note that in this sense the cocycles $\varphi_{ij} \in H_k^{2,t}(\mathfrak{g},\mathbb{C})$ codify this information.

2.3. The Algebraic Variety $\mathfrak{L}^n$. A n-dimensional complex Lie algebra can be seen as a pair $\mathfrak{g} = (\mathbb{C}^n, \mu)$ where μ is a Lie algebra law on $\mathbb{C}^n$, the underlying vector space to $\mathfrak{g}$ is $\mathbb{C}^n$ and μ the bracket of $\mathfrak{g}$. We will note by $\mathfrak{L}^n$ the set of Lie algebra laws on $\mathbb{C}^n$. It is a subset of the vectorial space of alternating bilinear mappings on $\mathbb{C}^n$.

Definition. Two laws μ and $\mu' \in \mathfrak{L}^n$ are said isomorphic, if there is $f \in Gl(n, \mathbb{C})$ such that

$$\mu'(X, Y) = f * \mu(X, Y) = f^{-1}(\mu(f(X), f(Y)))$$

for all $X, Y \in \mathbb{C}^n$.

In this case, the Lie algebras $\mathfrak{g} = (\mathbb{C}^n, \mu)$ and $\mathfrak{g}' = (\mathbb{C}^n, \mu')$ are isomorphic. The set $\mathcal{O}(\mu)$ of the laws isomorphic to μ is called the orbit of μ.

Let us fix a basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{C}^n$. The structural constants of $\mu \in \mathfrak{L}^n$ are the complex numbers C_{ij}^k given by

$$\mu(e_i, e_j) = \sum_{k=1}^{n} C_{ij}^k e_k.$$

As the basis is fixed, we can identify the law μ with its structural constants. These constants satisfy:

$$\text{(I)} \quad \begin{cases} C_{ij}^k = -C_{ji}^k\,, \quad 1 \le i < j \le n\,, \quad 1 \le k \le n\,, \\ \sum_{l=1}^{n} C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{jl}^s = 0\,, \;\; 1 \le i < j < k \le n\,, \; 1 \le s \le n\,. \end{cases}$$

Then $\mathfrak{L}^n$ appears as an algebraic variety embedded in the linear space of alternating bilinear mapping on $\mathbb{C}^n$, isomorphic to $\mathbb{C}^{\frac{n^3 - n^{\acute{e}}}{2}}$.

Let be $\mu \in \mathfrak{L}^n$ and consider the Lie subgroup G_μ of $Gl(n, \mathbb{C})$ defined by

$$G_\mu = \{f \in Gl(n, \mathbb{C}) \mid f * \mu = \mu\}.$$

Its Lie algebra is the Lie algebra of derivations of μ. Let be $\mathcal{O}(\mu)$ the orbit of μ respect the action of $Gl(n, \mathbb{C})$. It is isomorphic to the homogeneous space $Gl(n, \mathbb{C})/G_\mu$. Then it is a $\mathcal{C}^\infty$ differential manifold of dimension

$$\dim \mathcal{O}(\mu) = n^2 - \dim \mathrm{Der}(\mu).$$

It is not difficult to see that the orbit $\mathcal{O}(\mu)$ of μ is a differentiable manifold [96] embedded in $\mathfrak{L}^n$ defined by

$$\mathcal{O}(\mu) = \frac{Gl(n, \mathbb{C})}{G_\mu}\,.$$

We consider a point μ' close to μ in $\mathcal{O}(\mu)$. There is $f \in Gl(n,\mathbb{C})$ such that $\mu' = f*\mu$. Suppose that f is close to the identity: $f = Id+\varepsilon g$, with $g \in gl(n)$. Then

$$\begin{aligned}\mu'(X,Y) = \mu(X,Y) &+ \varepsilon[-g(\mu(X,Y)) + \mu(g(X),Y) + \mu(X,g(Y))] \\ &+ \varepsilon^2[\mu(g(X),g(Y)) - g(\mu(g(X),Y) + \mu(X,g(Y)) - g\mu(X,Y)]\,.\end{aligned}$$

Then

$$\frac{\mu'(X,Y) - \mu(X,Y)}{\varepsilon} \longrightarrow_{\varepsilon\to 0} \delta_\mu g(X,Y)\,.$$

Among the possible orbits, those which are open are specially important for the study of the variety, as we will see later.

DEFINITION. Let μ be a law such that the orbit $\mathcal{O}(\mu)$ is open in $\mathfrak{L}^n$. Then μ is called a rigid law.

PROPOSITION. *The tangent space to the orbit $\mathcal{O}(\mu)$ at the point μ is the space $B^2(\mu,\mu)$ of the 2-cocycles of the Chevalley cohomology of μ.*

Let μ be in $\mathfrak{L}^n$ and consider a bilinear alternating mapping $\mu' = \mu + t\varphi$ where t is a small parameter. Then $\mu' \in \mathfrak{L}^n$ for all t if and only if we have:

$$\left.\begin{array}{l}\delta\varphi = 0 \\ \varphi \in \mathfrak{L}^n\end{array}\right\}.$$

PROPOSITION. *A straight line Δ passing throught μ is a tangent line in μ to $\mathfrak{L}^n$ if its direction is given by a vector of $Z^2(\mu,\mu)$.*

Suppose that $H^2(\mu,\mu) = 0$. Then the tangent space to $\mathcal{O}(\mu)$ at the point μ is the set of the tangent lines to $\mathfrak{L}^n$ at the point μ. Thus the tangent space to $\mathfrak{L}^n$ exists in this point and it is equal to $B^2(\mu,\mu)$. The point μ is a nonsingular point. We deduce of this that the inclusion $\mathcal{O}\,(\mu) \hookrightarrow \mathfrak{L}^n$ is a local homeomorphism. This property is valid for all points of $\mathcal{O}(\mu)$, then $\mathcal{O}(\mu)$ is open in $\mathfrak{L}^n$ (for the induced metric topology).

PROPOSITION. *Let $\mu \in \mathfrak{L}^n$ such that $H^2(\mu,\mu) = 0$. If the algebraic variety $\mathfrak{L}^n$ is provided with the metric topology induced by $\mathbb{C}^{\frac{n^3-n^2}{2}}$, then the orbit $\mathcal{O}(\mu)$ is open in $\mathfrak{L}^n$.*

This geometrical approach shows the problems underlying to the existence of singular points in the algebraic variety $\mathfrak{L}^n$ [21].

2.4. Formal Deformations. Let be $\varphi, \psi \in C^2(\mathbb{C}^n, \mathbb{C}^n)$ two skew-symmetric bilinear maps on $\mathbb{C}^n$. We define the trilinear mapping $\varphi \circ \psi$ on $\mathbb{C}^n$ by

$$\begin{aligned}\varphi \circ \psi(X,Y,Z) = \ & \varphi(\psi(X,Y),Z) + \varphi(\psi(Y,Z),X) + \varphi(\psi(Z,X),Y) \\ & + \psi(\varphi(X,Y),Z) + \psi(\varphi(Y,Z),X) + \psi(\varphi(Z,X),Y)\end{aligned}$$

for all $X, Y, Z \in \mathbb{C}^n$. Using this notation, the Lie bracket is written $\mu \circ \mu = 0$.

Let be $\mu_0 \in L^n$ and $\varphi \in C^2(\mathbb{C}^n, \mathbb{C}^n)$. Then $\varphi \in Z^2(\mu_0, \mu_0)$ if and only if

$$\mu_0 \circ \varphi + \varphi \circ \mu_0 = \delta_{\mu_0}\varphi = 0.$$

Definition. A (formal) deformation of a law $\mu_0 \in L^n$ is a formal sequence with parameter t

$$\mu_t = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_i$$

where the φ_i are skew-symmetric bilinear maps $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ such that μ_t satisfies the formal Jacobi identity $\mu_t \circ \mu_t = 0$.

Let us develop this last equation

$$\begin{aligned}\mu_t \circ \mu_t = \mu_0 \circ \mu_0 + t\delta_{\mu_0}\varphi_1 + t^2(\varphi_1 \circ \varphi_1 + \delta_{\mu_0}\varphi_2) \\ + t^3(\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 + \delta_{\mu_0}\varphi_3) + \cdots,\end{aligned}$$

and the formal equation $\mu_t \circ \mu_t = 0$ is equivalent to the infinite system

$$\left.\begin{array}{rcl} \mu_0 \circ \mu_0 & = & 0 \\ \delta_{\mu_0}\varphi_1 & = & 0 \\ \varphi_1 \circ \varphi_1 & = & -\delta_{\mu_0}\varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 & = & -\delta_{\mu_0}\varphi_3 \\ & \vdots & \\ \varphi_p \circ \varphi_p + \sum_{1 \le i \le p-1} \varphi_i \circ \varphi_{2p-i} + \varphi_{2p-i} \circ \varphi_i & = & -\delta_{\mu_0}\varphi_{2p} \\ \sum_{1 \le i \le p} \varphi_i \circ \varphi_{2p+1-i} + \varphi_{2p+1-i} \circ \varphi_i & = & -\delta_{\mu_0}\varphi_{2p+1} \\ & \vdots & \end{array}\right\}. \qquad \text{(II)}$$

Then the first term φ_1 of a deformation μ_t of a Lie algebra law μ_0 belongs to $Z^2(\mu_0, \mu_0)$. This term is called the infinitesimal part of the deformation μ_t of μ_0.

DEFINITION. A formal deformation of μ_0 is called linear deformation if it is of lenght one, that is of the type $\mu_0 + t\varphi_1$ with $\varphi_1 \in Z^2(\mu_0, \mu_0)$.

For a such deformation we have necessarily $\varphi_1 \circ \varphi_1 = 0$ that is $\varphi_1 \in L^n$.

Now consider $\varphi_1 \in Z^2(\mu_0, \mu_0)$ for $\mu_0 \in L^n$. It is the infinitesimal part of a formal deformation of μ_0 if and only if there are $\varphi_i \in C^2(\mu_0, \mu_0)$, $i \geq 2$, such that the system (II) is satisfied.

PROPOSITION. *If $H^3(\mu_0, \mu_0) = 0$ then every $\varphi_1 \in Z^2(\mu_0, \mu_0)$ is an infinitesimal part of a formal deformation of μ_0.*

In fact, if $\varphi_1 \in Z^2(\mu_0, \mu_0)$ then $\varphi_1 \circ \varphi_1 \in Z^3(\mu_0, \mu_0)$. If $H^3(\mu_0, \mu_0) = 0$, then it exits $\varphi_2 \in C^2(\mu_0, \mu_0)$ such that $\varphi_1 \circ \varphi_1 = \delta\varphi_2$. In this case $\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 \in Z^3(\mu_0, \mu_0)$. It exits $\varphi_3 \in C^2(\mu_0, \mu_0)$ such that

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = \delta\varphi_3.$$

As this we can solve step by step all the equations of the system (II).

Let us consider two formal deformations μ_t^1 and μ_t^2 of a law μ_0. They are called equivalent if there exits a formal linear isomorphism Φ_t of $\mathbb{C}^n$ of the following form:

$$\Phi_t = Id + \sum_{i \geq 1} t^i g_i$$

with $g_i \in gl(n, \mathbb{C})$ such that

$$\mu_t^2(X, Y) = \Phi_t^{-1}(\mu_t^1(\Phi_t(X), \Phi_t(Y))$$

for all $X; Y \in \mathbb{C}^n$.

DEFINITION. A deformation μ_t of μ_0 is called trivial if it is equivalent to μ_0.

Let $\mu_t^1 = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_i$ and $\mu_t^2 = \mu_0 + \sum_{t=1}^{\infty} t^i \psi_i$ be two equivalent deformation of μ_0. It is easy to see that

$$\varphi_1 - \psi_1 \in B^2(\mu_0, \mu_0).$$

Thus we can consider that the set of infinitesimal parts of deformations is parametrized by $H^2(\mu_0, \mu_0)$.

2.5. Characteristic Sequence of a Nilpotent Lie Algebra. Let $\mathfrak{n}$ be a complex finite dimensional Lie algebra. Consider the derived subalgebra $C^1\mathfrak{n}$. Let $Y \in \mathfrak{n} - C^1\mathfrak{n}$ be a vector of $\mathfrak{n}$ which does not belong to the derived subalgebra. Consider the ordered sequence

$$c(Y) = (h_1, h_2, \cdots,)$$

$h_1 \geq h_2, \ldots, \geq h_p$, where h_i is the dimension of the i^{th} Jordan block of the nilpotent operator adY. As Y is necessary an eigenvector of adY, then $h_p = 1$. Let Y_1 and Y_2 be in $\mathfrak{n} - C^1\mathfrak{n}$. Let be $c(Y_1) = (h_1, \ldots, h_{p_1} = 1)$ and $c(Y_2) = (k_1, \ldots, k_{p_2} = 1)$ the corresponding sequences. We have $h_1 \geq h_2 \geq \cdots \geq h_{p_1}$ and $k_1 \geq k_2 \geq \cdots \geq k_{p_2}$ with $h_1 + \cdots + h_{p_1} = k_1 + \cdots + k_{p_2} = n = \dim \mathfrak{n}$. We will say that $c(Y_1) \geq c(Y_2)$ if there is i such that $h_1 = k_1$, $h_2 = k_2$, $\ldots$, $h_{i-1} = k_{i-1}$, $h_i > k_i$. This defines a total order relation on the set of sequences $c(Y)$ (lexicografic order) and we can consider the maximum of these sequences.

Definition. The characteristic sequence of the nilpotent Lie algebra $\mathfrak{n}$ is the following sequence:

$$c(\mathfrak{n}) = \sup \left\{ c(Y) \mid Y \in \mathfrak{n} - C^1\mathfrak{n} \right\}.$$

It is an invariant up to isomorphism of $\mathfrak{n}$, fintroduced by Ancochea and Goze in [5]. A vector $X \in \mathfrak{n} - C^1\mathfrak{n}$ such that $c(X) = c(\mathfrak{n})$ is called a characteristic vector of $\mathfrak{n}$.

This invariant is well adapted for study the deformations of nilpotent Lie algebras. In fact let $\mathfrak{n}$ and $\mathfrak{n}'$ be two n-dimensional complex nilpotent Lie algebras and μ and μ' the corresponding laws. Suppose that μ' is a perturbation of μ, that is, in a fixed basis, the structural constant of μ' are close of those of μ. In this case, the linear operator $ad_{\mu'}Y$ is a perturbation (in the classical sense) of the linear operator $ad_\mu Y$. As these two operators are nilpotent, the restriction of $ad_{\mu'}Y$ to the first Jordan block $J_{h_1}$of $ad_\mu Y$ satisfies $(ad_{\mu'}Y \mid J_{h_1})^{h_1-2} \neq 0$. Then, the first Jordan block of $ad_{\mu'}Y$ has a dimension greater or equal than h_1. This proves that

$$c(\mathfrak{n}') \geq c(\mathfrak{n}).$$

Proposition. *If $\mathfrak{n}$ and $\mathfrak{n}'$ are two n-dimensional complex nilpotent Lie algebras such that $\mathfrak{n}'$ is a perturbation of $\mathfrak{n}$, then*

$$c(\mathfrak{n}') \geq c(\mathfrak{n}).$$

This last property allowed to determine, for example, all the irreducible components of the algebraic variety of n-dimensional nilpotent Lie algebras for $n \leq 8$.

3. Characteristically nilpotent Lie algebras

In studying the varieties of laws, the characteristically nilpotent algebras have shown their importance in the determination of irreducible components. For example, in dimension 7 there are two components, the first formed by filiform Lie algebras and the second generated by the orbit closure of a family of characteristically nilpotent Lie algebras [8].

The main problem in the study of characteristically nilpotent Lie algebras is the determination of conditions for an algebra of derivations to be nilpotent: for an arbitrary nilpotent Lie algebra the structure of the algebra of derivations can variate from representations of the special linear algebras $\mathfrak{sl}_n$ to certain nilpotent Lie algebras.

The origin of all this is the cited result of Jacobson [48].

Theorem 1. *Let $\mathfrak{g}$ be a Lie algebra and suppose that it admits a nondegenerate derivation f. Then $\mathfrak{g}$ is a nilpotent Lie algebra.*

According to our convention, the Lie algebra is defined over a the field of complex numbers. Otherwise the assertion would be false, as it has been verified that this result fails when the characteristic of the base field is nonzeo.

The example of Dixmier and Lister, appearing as the first known characteristically nilpotent Lie algebra, was the response to the validity question of Jacobson's theorem of 1955. This algebra is very interesting in many aspects; it is one of the few known CNLA of nilindex 3, which is the lowest possible nilindex such an algebra can have. We find this intriguing; the authors not only gave the first example to a new class of nilpotent Lie algebras, that also developed an "extreme" example in that sense. Unfortunately, we do not know how Dixmier and Lister came to this algebra.

The construction is of an eight dimensional Lie algebra $\mathfrak{g}_0$ defined over the basis $\{X_1, \ldots, X_8\}$ and law

$$
\begin{aligned}
&[X_1, X_2] = X_5\,; \quad [X_1, X_3] = X_6\,; \quad [X_1, X_4] = X_7\,; \quad [X_1, X_5] = -X_8\,;\\
&[X_2, X_3] = X_8\,; \quad [X_2, X_4] = X_6\,; \quad [X_2, X_6] = -X_7\,; \quad [X_3, X_4] = -X_5\,;\\
&[X_3, X_5] = -X_7\,; \quad [X_4, X_6] = -X_8\,.
\end{aligned}
$$

Let us define the following generalization of the central descending sequence for a Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}^{[1]} = \mathrm{Der}\,(\mathfrak{g})\,(\mathfrak{g}) = \{X \in \mathfrak{g} \mid X = f\,(Y)\,,\ f \in \mathrm{Der}\,(\mathfrak{g})\,,\ Y \in \mathfrak{g}\}$$

and

$$\mathfrak{g}^{[k]} = \mathrm{Der}\,(\mathfrak{g})\left(\mathfrak{g}^{[k-1]}\right), \qquad k > 1\,.$$

The main result about this algebra is the following:

THEOREM 2. *If f is a derivation of $\mathfrak{g}_0$ then $f\,(\mathfrak{g}_0) \subset C^1\mathfrak{g}_0$; hence any derivation if nilpotent.*

The proof of this is strongly related with the fact that the algebra $\mathfrak{g}_0$ annihilates a power of the preceding sequence. For this reason, they defined characteristically nilpotent Lie algebras as follows:

DEFINITION. A Lie algebra $\mathfrak{g}$ is called characteristically nilpotent if there exists an integer m such that $\mathfrak{g}^{[m]} = 0$.

The listed algebra has a twelve dimensional Lie algebra of derivations, from which six correspond to inner derivations. Now, any linear operator sending the algebra $\mathfrak{g}_0$ into its center, which is generated by the vectors X_7 and X_8, is easily seen to be a derivation of $\mathfrak{g}_0$. The ideal uf these derivations has dimension eight, having a two dimensional subspace in common with the space of inner derivations. This fact can be interpreted in the sense that the dimension of the cohomology space $H^1\,(\mathfrak{g}_0, \mathfrak{g}_0)$ is as small as possible. Dixmier and Lister asked if the algebras of this type, which satisfy the generalization of the central descending sequence, were more treatable than ordinary nilpotent Lie algebras. In certain aspects this is true, as the topological properties of CNLA show, but on the other their determination and classification is a rather difficult question, and one can hardly say that it constitutes a simplification. However, CNLA have undoubtly contributed to a better understanding of the geometry of the variety $\mathcal{N}^n$. The theorem above proves in fact much more than the characteristic nilpotence of the listed algebra: $\mathfrak{g}_0$ is not the derived subalgebra of any Lie algebra. Thus one can pose the question: if $\mathfrak{g}$ is a CNLA, is it true that $\mathfrak{g}$ cannot be the derived subalgebra of a Lie algebra? A first condition is given above, as the nilpotence of $\mathfrak{g}$ and the fact that any derivation maps the algebra into its derived subalgebra ensures that it cannot be a commutator algebra. Leger and Tôgô found out other conditions to assure the nonexistence of an algebra containing a given CNLA as derived subalgebra [66]:

PROPOSITION. *Let $\mathfrak{g}$ be a CNLA. If* $\mathrm{Der}\,(\mathfrak{g})$ *annihilates the center* $Z\,(\mathfrak{g})$ *of* $\mathfrak{g}$, *then* $\mathfrak{g}$ *is not a derived algebra.*

This proposition is based on the fact that for a CNLA we have

$$[\mathfrak{g}, Z_i] \subset Z_{i-2}$$

where $Z_0 = (0)$ and Z_i is the largest subspace of $\mathfrak{g}$ such that $\mathrm{Der}\,(\mathfrak{g})\, Z_i \subset Z_{i-1}$ for $i \geq 1$. The existence of an index such that $\mathfrak{g} = Z_r$ follows immediately. The authors also deduce an interesting numerical condition, also based on this inductively defined sequence:

THEOREM 3. *Let $\mathfrak{g}$ be a CNLA, and n and m be the smallest integers for which $C^{n-1}\mathfrak{g} = 0$ and $\mathfrak{g}^{[m]} = 0$. If $2(m-1) > n+1$, then $\mathfrak{g}$ is not a derived subalgebra.*

In particular, it follows that $\mathfrak{g}$ is no derived subalgebra if $\mathrm{Der}\,(\mathfrak{g})\,\mathfrak{g} \subset C^1\mathfrak{g}$, which recovers the property of Dixmier and Lister's algebra, or $\mathfrak{g}^{[4]} = 0$.

Now, for the general case E. Luks [69] proved in 1976 that a CNLA can appear as derived subalgebra of a Lie algebra. This remarkable fact divides in fact the CNLA into two classes: those being commutators of others and those not. The algebra L of Luks has dimension 16, and is defined over the basis $\{X_1, \ldots, X_{16}\}$ by the law:

$$\begin{aligned}
&[X_1, X_i] = X_{5+i}\,,\ i = 2,3,4,5\,; \quad [X_1, X_6] = X_{13}\,; \quad [X_1, X_i] = X_{8+i}\,,\ i = 7,8\,;\\
&[X_2, X_i] = X_{i+8}\,,\ i = 3,4,6\,; \quad [X_2, X_5] = X_{15}\,; [X_2, X_7] = -X_{16}\\
&[X_3, X_4] = -X_{13} - \frac{9}{5}X_{15}\,; \quad [X_3, X_5] = -X_{14}\,; \quad [X_3, X_6] = -X_{16}\,;\\
&[X_4, X_5] = 2X_{16}\,.
\end{aligned}$$

The procedure used is the following: consider the transporter of an ideal I into J, where both ideals are characteristic. Then the following statements are verified:

1. I_3 is the transporter of C^1L to 0;
2. I_4 is the transporter of L to $Z(L)$;
3. I_2 is the transporter of C^1L to $[I_4, I_4]$;
4. I_6 is the transporter of L to $[I_2, I_2]$;
5. I_5 is the transporter of I_2 to $[L, I_6]$.

where I_i is the ideal generated by the vectors $\{X_j\}_{j\geq i}$ (for $1 \leq i \leq 16$). Clearly $I_2, \ldots, I_6$ are characteristic ideals, and it follows that L is a CNLA. For the second part, consider the derivations f and g defined respectively by:

$$
\begin{aligned}
&f(X_3) = X_7\,; \quad f(X_4) = 2X_8\,; \quad f(X_5) = 3X_9 + 2X_{11}\,;\\
&f(X_6) = 4X_{10} + 5X_{12}\,; \quad f(X_8) = X_{15}\,; \quad f(X_9) = 2X_{16}\,; \quad f(X_{11}) = -X_{16}\,;\\
&g(X_i) = X_{i+1}\,,\ i = 2,3,4,5,7,11,13\,; \quad g(X_8) = X_9 + X_{11}\,;\\
&g(X_9) = X_{10} + X_{12}\,; \quad g(X_{10}) = X_{13} + X_{15}\,;\\
&g(X_{12}) = -X_{13} - \frac{4}{5}X_{15}\,; \quad g(X_{14}) = -X_{16}\,.
\end{aligned}
$$

If one considers the brackets $[f, g]$ in $\mathrm{Der}\,(L)$, it gives $ad\,(X_1)$ as result. Thus we can extend the algebra to the semidirect product $\overline{L} = \{f', g'\} + L$, where the brachets in L are the same and the action of $\{f', g'\}$ over L is given by $[f', X_i] = f\,(X_i)\,,\ [g', X_i] = g\,(X_i)$ for all i and $[f', g'] = X_1$. It follows at once that $\left[\overline{L}, \overline{L}\right] = L$.

This algebra not only gives a surprising response to the question of Dixmier and Lister, it moreover gives an idea of how the algebra has to be structured for being candidate for derived algebra. Observe that the key of the preceding cosntruction is the existence of two derivations (whose rest class modulo $B^1\,(L, L)$ is nonzero) whose composition equals the adjoint operator of the characteristic vector X_1. Thus, a necessary condition is deduced immediately, namely the existence of derivations in the algebra such that their composition is in the linear subspace generated by the adjoint operators $ad\,(X_i)\,,\ i = 1, \ldots, k$, where these vectors are generators of the nilpotent Lie algebra.

Using a similar argumentation one can show that Luks' algebra also has an unipotent automorphism group. One may ask if there is a connection between the property of being a commutator algebra and an automorphism group of this kind. L. Auslander remarked in [33] that Dixmier and Lister's example has not an unipotent automorphism group. Now, a CNLA which is additionally a derived algebra and posseses a nonunipotent automorphism group could be constructed by considering the direct sum of two Lie algebras which satisfy the two first conditions [66]. Inspite of this result, there are wide known classes of CNLA which cannot appear as a commutator algebra. This concerns the filiform Lie algebras. It can be shown that if a filiform Lie algebra $\mathfrak{g}$ is the derived algebra of $\mathfrak{g}'$, then it suffices to consider the case where $\dim \mathfrak{g}' = \dim \mathfrak{g} + 1$. This has been done in [24]. The reduction is not difficult to prove, and using it the assertion that $\mathfrak{g}$ is not a derived algebra follows at once. In fact, this reduction can be seen as a conseuence of a more general

result due to M. Goze and Y. B. Khakimdjanov [40], where they analyze in detail the maximal tori of derivations of an arbitrary filiform Lie algebra. The following result can be interpreted as a characterization of those filiform Lie algebras which are not characteristically nilpotent:

THEOREM 4. *Let $\mathfrak{g}$ be an $(n+1)$-dimensional filiform Lie algebra which has a nontrivial semisimple derivation f. There exists a basis $\{X_0, \dots, X_n\}$ adapted to f such that the brackets of $\mathfrak{g}$ satisfy one of the following cases:*

(i) $\mathfrak{g} = L_n$;
$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-1$;

(ii) $\mathfrak{g} = A^r_{n+1}(\alpha_1, \dots, \alpha_t), \quad 1 \le r \le n-3, \quad t = \left[\frac{n-r-1}{2}\right]$;
$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-1$;
$[X_i, X_j] = \Big(\sum_{k=1}^{t} \alpha_k (-1)^{k-i} C^{k-1}_{j-k-1}\Big) X_{i+j+r},$
$1 \le i, j \le n, \quad i+j+r \le n$;

(iii) $\mathfrak{g} = Q_n, \quad n = 2m-1$;
$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-2$;
$[X_i, X_{n-i}] = (-1)^i X_n, \quad 1 \le i \le n-1$;

(iv) $\mathfrak{g} = B^r_{n+1}(\alpha_1, \dots, \alpha_t), \quad n = 2m+1, \quad 1 \le r \le n-4, \quad t = \left[\frac{n-r-2}{2}\right]$;
$[X_i, X_j] = \Big(\sum_{k=1}^{t} \alpha_k (-1)^{k-i} C^{k-1}_{j-k-1}\Big) X_{i+j+r},$
$1 \le i, j \le n-1, \quad i+j+r \le n-1$;
$[X_0, X_i] = X_{i+1}, \ 1 \le i \le n-2$;

(v) $\mathfrak{g} = C_{n+1}(\alpha_1, \dots, \alpha_t), \quad n = 2m+1, \quad t = m-1$;
$[X_0, X_i] = X_{i+1}, \quad 1 \le i \le n-2$;
$[X_i, X_{n-i}] = (-1)^i X_n, \quad 1 \le i \le n-1$;
$[X_i, X_{n-i-2k}] = (-1)^i \alpha_k X_n, \quad 1 \le k \le m-1, \quad 1 \le i \le n-2k-1$;

where $(\alpha_1, \dots, \alpha_t)$ are parameters satisfying the polynomial relations given by the Jacobi relations over this basis.

4. Structural properties of CNLA

After the example of Dixmier and Lister in 1957, Leger and Tôgô iniciated the structural study of CNLA. Their paper [66] does not provide additional examples, but it is of considerable significance for later work. At first, they observe that the property of being characteristically nilpotent does not depend

on the ground field. More precisely: if the Lie algebra $\mathfrak{g}$ is characteristically nilpotent as F-algebra (here it is not necessary to suppose that it has characteristic zero) and $K\backslash F$ is a field extension, then $\mathfrak{g}$ is also a CNLA as K-algebra. However, the structural properties deduced by the authors are more important, as they give an idea of which algebras have to be avoided in the search after CNLA:

LEMMA. *If $\mathfrak{g}$ is characteristically nilpotent, then*

(i) *the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is contained in the derived subalgebra $C^1\mathfrak{g}$;*

(ii) $C^2\mathfrak{g} \neq 0$.

The first condition makes reference to the nonexistence of direct summands in $\mathfrak{g}$ which constitute of central vectors. Thus the study of characteristically nilpotent Lie algebras reduces to nonsplit nilpotent Lie algebras. The second condition has a more important consequence: it tells that for a Lie algebra being characteristically nilpotent, the nilindex must be at least three (observe that this is the index for the algebra of Dixmier and Lister). This fact is remarkable, as it shows the incompatibility of being as nilpotent as possible (as it occurs for the 2-step nilpotent or metabelian Lie algebras) and having all its derivations nilpotent. Metabelian Lie algebras and their derivations have been deeply studied by Leger and Luks [64], where they proved that its rank is always greater than one, the equality given only under certain conditions. Recently Galitski and Timashev [37] have used invariant theory to classify these algebras up to dimension nine. The preceding lemma leads to the question wheter a CNLA can be a direct sum. The following result is also from [66]:

LEMMA. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. If $\mathfrak{g}$ is the direct sum of two nontrivial ideals, one of which is central, then it posseses at least nontrivial semisimple derivation.*

These two lemmas give the following reinterpretation of the sequence $\mathfrak{g}^{[k]}$ introduced earlier:

THEOREM 5. *Let $\mathfrak{g}$ be a Lie algebra and* $\mathrm{Der}(\mathfrak{g})$ *its Lie algebra of derivations. Then $\mathfrak{g}$ is characteristically nilpotent if and only if* $\mathrm{Der}(\mathfrak{g})$ *is nilpotent and* $\dim \mathfrak{g} \geq 2$.

It follows from the proof that if all derivations of $\mathfrak{g}$ are nilpotent, then $\mathfrak{g}$ is also a nilpotent Lie algebra. Thus the characteristic nilpotence is a phenomena

which can only be observed in the variety of nilpotent Lie algebra laws $\mathcal{N}^n$. The theorem can be reformulated by saying that the holomorph $H(\mathfrak{g})$ of $\mathfrak{g}$ is nilpotent, where the holomorph is the split extension of $\mathrm{Der}(\mathfrak{g})$ by $\mathfrak{g}$. In connection with metabelian Lie algebras, this reformulation says that for a 2-step nilpotent Lie algebra the holomorph cannot be nilpotent. The holomorph is also useful to describe properties valid also for solvable Lie algebras, as the following:

THEOREM 6. *Let $\mathfrak{g}$ be a Lie algebra. If a Cartan subalgebra H of $\mathfrak{g}$ is characteristically nilpotent, then $\mathfrak{g}$ is a solvable Lie algebra.*

As noted by the authors, the algebra $\mathfrak{g}$ can be solvable non-nilpotent. We remark that this theorem has been generalized in 1961 by S. Tôgô [94].

It has often been asked wheter CNLA exist for any possible dimension. The answer is in the affirmative, and in fact it was enough to find examples of dimension $7 \leq n \leq 13$ to derive its existence in any dimension. The key result was the possibility of a decomposition into smaller blocks that have also the property of being characteristically nilpotent, as done in the classical theory:

THEOREM 7. *Let $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$ be a direct sum of ideals. Then $\mathfrak{g}$ is characteristically nilpotent if and only if $\mathfrak{g}_i$ is characteristically nilpotent for $1 \leq i \leq n$.*

As said, having examples from dimensions seven to thirteen, the direct sums of them give CNLA in any dimension. The nine dimensional example was given by J. Dyer in 1970, in connection with her study of nilpotent Lie groups which have expanding automorphisms. Over the basis $\{X_1, \ldots, X_9\}$ the Lie algebra is given by

$$[X_1, X_2] = X_3\,; \quad [X_1, X_3] = X_4\,; \quad [X_1, X_5] = X_7\,; \quad [X_1, X_8] = X_9\,;$$
$$[X_2, X_3] = X_5\,; \quad [X_2, X_4] = X_7\,; \quad [X_2, X_5] = X_6\,; \quad [X_2, X_7] = -X_8\,;$$
$$[X_3, X_7] = -\,[X_4, X_5] = X_9\,.$$

This was the first given CNLA with an unipotent automorphism group. Two years later G. Favre constructed a seven dimensional example with the same property. This example is one of the three filiform CNLA in dimension 7:

$$[X_1, X_i] = X_{i+1}\,, \quad 2 \leq i \leq 6\,;$$
$$[X_3, X_2] = X_6\,;$$
$$[X_4, X_2] = [X_5, X_2] = X_7\,;$$
$$[X_4, X_3] = -X_7\,.$$

To complete the construction of CNLA, there remains to find examples in dimensions $10-13$. These were given by Luks using computational methods [68]. Once the question of their existence in any possible dimension, we can ask even more: for any possible nilindex $p \geq 3$, does there exist a CNLA in any dimension? In [7] the question is answered in the affirmative for $p = 5$. This is a consequence of the classification of nilpotent Lie algebras of characteristic sequence $(5, 1, \ldots, 1)$ whose derived subalgebra is non-abelian. In fact, we prove that if a Lie algebra $\mathfrak{g}$ with this characteristic sequence is characteristically nilpotent, then it satisfies $D^2\mathfrak{g} \neq 0$.

In 1961 Tôgô published a paper reviewing most of known results about the derivation algebras of Lie algebras (over a field of charateristic zero). He also gives an example about two nonisomorphic Lie algebras whose Lie algebra of derivations is the same, proving in that manner that a Lie algebra is not entirely determined by its derivations. Among various results about classical and reductive algebras, he also generalizes the concept of CNLA to characteristically solvable Lie algebras [94]. However, here we are only concerned with results about nilpotent Lie algebras. An often asked question is the relation between a Lie algebra $\mathfrak{g}$ which is a (finite) sum of ideals and the structure of $\mathrm{Der}(\mathfrak{g})$. To this respect, in [94] the following theorem is proved:

THEOREM 8. *Let* $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$ *be a direct sum of ideals. Then* $\mathrm{Der}(\mathfrak{g}) = \bigoplus_{i=1}^{n} \mathrm{Der}(\mathfrak{g}_i)$ *if and only if* $\mathfrak{g}$ *satisfies one of the following conditions:*

(i) $Z(\mathfrak{g}) = (0)$;

(ii) $\mathfrak{g}$ *is a perfect Lie algebra* (*i.e.* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$);

(iii) *All the* $\mathfrak{g}_i$*'s except one is such that* $Z(\mathfrak{g}_i) = (0)$ *and* $\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i]$.

For a nilpotent Lie algebra $\mathfrak{g}$, this implies that the structure of its derivations is more than the sum of the derivations corresponding to its summands. The following proposition gives the precise form of $\mathrm{Der}(\mathfrak{g})$:

PROPOSITION. *Let* $\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_i$ *be a direct sum of ideals. Then*

$$\mathrm{Der}(\mathfrak{g}) = \bigoplus_{i=1}^{n} \left(\mathrm{Der}(\mathfrak{g}_i) \oplus \left(\bigoplus_{i \neq j} \mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j) \right) \right)$$

where $\mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j) = \{h \in \mathrm{End}(\mathfrak{g}) \mid h(\mathfrak{g}_k) = 0 \text{ if } k \neq i,\ h(\mathfrak{g}_i) \subset Z(\mathfrak{g}_i) \text{ and } h([\mathfrak{g}_i, \mathfrak{g}_j]) = 0\}$.

Thus if one of the conditions in theorem 8 is satisfied, then $\mathcal{D}(\mathfrak{g}_i, \mathfrak{g}_j)$ vanishes.

In the same paper Tôgô presents a list of problems of interest, specially in connection with CNLA: do there exists CNLA of derivations? From the structure of derivations for the example of Dixmier and Lister, as well as the scarceness of outer derivations, it is obvious that this algebra does not have a characteristically nilpotent algebra of derivations. As to our knowledge, nobody has answered explicitely to this question until now, though the answer is in the affirmative. In [8] we construct examples of CNLA of derivations and generalize the question to higher indexes.

EXAMPLE. Let $\mathfrak{g}$ be the Lie algebra with associated law

$$\mu_5(X_1, X_i) = X_{i+1}\,,\ i \in \{2,3,4,5\}\,; \quad \mu_5(X_5, X_2) = \mu_5(X_3, X_4) = X_6\,;$$
$$\mu_5(X_7, X_3) = X_6\,; \quad \mu_5(X_7, X_2) = X_5 + X_6\,.$$

over the basis $\{X_1, \ldots, X_7\}$. The Lie algebra of derivations $\mathrm{Der}(\mathfrak{g})$ is ten dimensional and isomorphic to

$$\begin{array}{lll}
[Z_1, Z_2] = Z_3\,, & [Z_2, Z_6] = -Z_5, & [Z_7, Z_8] = 2Z_5 - 2Z_6 + 2Z_{10}\,, \\
[Z_1, Z_3] = Z_4\,, & [Z_2, Z_8] = -Z_6\,, & [Z_7, Z_9] = Z_5 - 2Z_6 + 2Z_{10}\,, \\
[Z_1, Z_4] = Z_5\,, & [Z_2, Z_9] = -Z_4 - 2Z_6\,, & [Z_8, Z_9] = 2Z_6 - 2Z_{10}\,, \\
[Z_1, Z_7] = -Z_4\,, & [Z_2, Z_{10}] = -Z_5\,, & \\
[Z_1, Z_8] = -Z_6\,, & [Z_3, Z_8] = -Z_5\,, & \\
 & [Z_3, Z_9] = -Z_5\,. &
\end{array}$$

It is routine to verify that this algebra is a CNLA.

Among many other examples, we present the following, which is important in connection with the study of irreducible components of the variety $\mathfrak{N}^n$:

THEOREM 9. *For any $\alpha \in \mathbb{C} - \{0, 2\}$ the family of nilpotent Lie algebras given by*

$$\begin{aligned}
&[X_1, X_i] = X_{i+1}\,, \quad 2 \le i \le 5\,; \\
&[X_4, X_2] = \alpha X_6\,; \\
&[X_3, X_2] = \alpha X_5 + X_7\,; \\
&[X_7, X_3] = X_6\,; \\
&[X_7, X_2] = X_5 + X_6\,.
\end{aligned}$$

has a characteristically nilpotent Lie algebra of derivations.

This follows at once from the fact that the derivatiosn are given by:

$$
\begin{array}{lll}
[Z_1, Z_2] = Z_3\,, & [Z_2, Z_3] = -\alpha Z_5 - Z_6\,, & [Z_7, Z_9] = 2Z_5\,, \\
[Z_1, Z_3] = Z_4\,, & [Z_2, Z_6] = -Z_5\,, & [Z_7, Z_{10}] = Z_5\,, \\
[Z_1, Z_4] = Z_5\,, & [Z_2, Z_{10}] = -Z_4 - \alpha Z_5\,, & [Z_9, Z_{10}] = \frac{2}{\alpha} Z_8 + \frac{2}{\alpha} Z_4\,, \\
[Z_1, Z_7] = -Z_4\,, & [Z_3, Z_9] = -Z_5\,, & \\
[Z_1, Z_8] = -Z_5\,, & [Z_3, Z_{10}] = -Z_5\,, & \\
[Z_1, Z_9] = -Z_6\,, & & \\
[Z_1, Z_{10}] = -Z_8 - Z_4\,. & &
\end{array}
$$

This examples, as well as other considered in [8] have a common property: there always exists an outer derivation θ which belongs to the derived subalgebra of $\mathrm{Der}\,(\mathfrak{g})$. This and the method used to deduce the examples have led to the

Conjecture 1. If $\mathfrak{g}$ is a CNLA of derivations, then there exist outer derivations $\theta_1, \theta_2, \theta_3$ such that

$$[\theta_1, \theta_2] = \lambda\theta_3 \pmod{\mathbb{I}\mathrm{Der}\,(\mathfrak{g})}$$

where $\lambda \in \mathbb{C} - \{0\}$ and $\mathbb{I}\mathrm{Der}\,(\mathfrak{g})$ denotes the space of inner derivations.

We now come to the generalization announced. Let $\mathrm{Der}^{[k]}\,\mathfrak{g} = \mathrm{Der}(\mathrm{Der}(\ldots(\mathrm{Der}(\mathfrak{g}))\ldots)$ be the k-th Lie algebra of derivations. Thus we have the sequence

$$\left(\mathrm{Der}\,(\mathfrak{g})\,, \mathrm{Der}^{[2]}\,\mathfrak{g}, \ldots, \mathrm{Der}^{[k]}\,\mathfrak{g}, \ldots\right)\,.$$

Definition. A Lie algebra $\mathfrak{g}$ is called characteristically nilpotent of index k if the $(k-1)^{th}$ Lie algebra of derivations $\mathrm{Der}^{[k-1]}\,\mathfrak{g}$ is characteristically nilpotent.

Remark. It would be of great interest to know if there exist CNLA of infinite index, as this would us give the possibility to develop a theory analogue to Schenkman's one [85] for these algebras. The structure of the variety of filiform Lie algebras $\mathcal{F}^m$ for $m \geq 8$ seems to suggest the existence of such algebras, but there is no manner to prove it. Observe that the determination of such an algebra is far from being a computational problem. The question is more to find a new invariant which measures which is the gratest possible index, if any. Up to the moment, the biggest index known is 5.

5. Subspaces of CNLA

Around 1984, when some authors had already constructed infinite families of CNLA, the interest on these algebras turned to its topological and geometrical properties. R.C̃arles proved in [19] the following result:

Proposition. *The CNLA constitute a constructible set of the variety* $\mathfrak{N}^n$ *which is empty for* $n \leq 6$ *and nonempty for* $n \geq 7$.

This proposition is another way to prove the existence of CNLA in arbitrary dimension, and its advantage is being independent from any example. Its proof is based on the conjugacy classes of maximal tori of derivations over a nilpotent Lie algebra of dimension n ([36]), as well as the action of the general linear group $GL(n, \mathbb{C})$ on $\mathfrak{g}$ (the result is in fact true for any algebraically closed field of characteristic zero). The seven dimensional CNLA given by Favre in 1972 is generalized in the following manner: over the basis $\{X_1, \ldots, X_n, X_{n+1}\}$ the Lie algebra structure is given by:

$$\begin{aligned} &[X_1, X_i] = X_{i+1}\,, \quad 2 \leq i \leq n\,; \\ &[X_2, X_3] = X_n\,; \\ &[X_2, X_4] = [X_2, X_5] = -\,[X_3, X_4] = X_{n+1}\,. \end{aligned}$$

For $n = 6$ Favre's example is recovered. The interest of this family is that it is obtained by considering central extensions of an algebra $\mathfrak{g}'$ by $\mathbb{C}$, which proves the power of extension theory for the study of CNLA. It is also proven that any extension by the center of a CNLA is also characteristically nilpotent, where an extension by the center is a central extension of a Lie algebra $\mathfrak{g}$ by $\mathbb{C}^p$ whose center is isomorphic to $\mathbb{C}^p$. The same procedure has been used in [22] to obtain lots of CNLA in arbitrary dimension and mixed characteristic sequences. Carles also remarks that the set of CNLA is never closed, which is immediate from the preceding, and for the particular case of dimension 7 he proves that it is neither open. In [8] we have extended this result to any dimension:

Theorem 10. *For* $n \geq 8$ *the set* $\mathcal{S}_n$ *of CNLA is not open in the variety* $\mathfrak{N}^n$.

The family constructed is based on the results of the classification of 8 dimensional filiform Lie algebra due to Goze and the first author [4]. Also the deformation structure is based on this result:

Let $\mathfrak{g}_{n,17}$ $(n \geq 8)$ be the Lie algebra defined by the brackets

$$\begin{aligned}
&[X_1, X_i] = X_{i+1}\,, \quad 1 \leq i \leq n-1\,;\\
&[X_4, X_2] = X_n\,;\\
&[X_3, X_2] = X_{n-1} + X_n\,.
\end{aligned}$$

It is immediate that the algebra is filiform and characteristically nilpotent. Let $\psi \in Z^2(\mathfrak{g}_{n,17}, \mathfrak{g}_{n,17})$ be the linear expandable cocycle defined by

$$\begin{aligned}
&\psi(X_5, X_3) = X_n\,, \quad \psi(X_5, X_2) = \psi(X_4, X_3) = X_{n-1}\,,\\
&\psi(X_k, X_2) = 2X_{n-4+\left[\frac{k}{2}\right]}\,, \quad k = 3, 4\,.
\end{aligned}$$

Let $\mathfrak{g}_{n,17} + \varepsilon\psi$ be an infinitesimal deformation of $\mathfrak{g}_{n,17}$. Now we consider the change of Jordan basis $X_1' = X_1,\ X_2' = X_2 + a_3X_3 + a_4X_4 + a_5X_5$ with the relations

$$\begin{aligned}
1 + a_3^2\varepsilon - 2\varepsilon a_4 &= 0\,,\\
3a_5\varepsilon + a_3a_4 - a_3^2\varepsilon - 2\varepsilon a_4 &= 0\,.
\end{aligned}$$

Written in the new basis the algebra $\mathfrak{g}_{n,17}+\varepsilon\psi$ is isomorphic to the Lie algebra $\mathfrak{g}_{n,18}$ defined by

$$\begin{aligned}
&[X_1, X_i] = X_{i+1}\,, \quad 1 \leq i \leq n-1\,;\\
&[X_5, X_3] = \varepsilon X_n\,;\\
&[X_5, X_2] = [X_4, X_3] = \varepsilon X_{n-1}\,;\\
&[X_4, X_2] = 2\varepsilon X_{n-2}\,;\\
&[X_3, X_2] = 2\varepsilon X_{n-3}\,.
\end{aligned}$$

From the linear system (S) associated to this algebra [6] we deduce the existence of nonzero eigenvalues for diagonalizable derivations of $\mathfrak{g}_{n,18}$, so it cannot be characteristically nilpotent.

Following with the seven dimensional case, in [18] the irreducible components of the variety $\mathfrak{N}^7$ are analyzed in relation with characteristic nilpotence. It is well known that this variety has two irreducible components, one corresponding to the filiform Lie algebras, $\mathcal{F}_7$, and one consisting of non-filiform Lie algebras. The filiform components has only three CNLA, which obviously don't constitute a dense subset as none of them has an open orbit, while for the other component there exists a family of CNLA constituting a nonempty Zariski open subset. The family is precisely the one given as example above. For $n \geq 8$, the situation for $\mathcal{F}_8$ changes radically:

THEOREM 11. *Let $n \geq 8$. Then any irreducible component C of $\mathcal{F}_n$ contains a nonempty Zariski open subset A cosisting of CNLAs.*

The result is proven in [40], where even more is obtained, namely that for any open set in $\mathcal{F}_n$ a CNLA belonging to this set can be found. Other versions relative to this have been treated by H. Kraft and Ch. Riedtmann in [61]. Is it true that for dimensions $n \geq 8$ any irreducible component of the variety $\mathfrak{N}^n$ contains an open subset of CNLAs. For $n = 8$ the response is affirmative, and can be found in [8].

PROPOSITION. *For any irreducible component C of the variety $\mathfrak{N}^8$ there exists a nonempty Zariski-open subsets consisting of CNLAs.*

As commented above, the 1-parameter family that defines the second irreducible component of $\mathfrak{N}^7$ has the property of being characteristically nilpotent of index 2, according to the definition given. This leads naturally to the question wheter there exist irreducible components in $\mathfrak{N}^n$ $(n \geq 9)$ which admit nonempty open susbsets formed by CNLA od derivations. We finally remark that this problem is related with the tower problem in group theory.

5.1. CHARACTERISTICALLY NILPOTENT LIE ALGEBRAS OBTAINED FROM NILRADICALS OF BOREL SUBALGEBRAS. As commented earlier, the difficulty of constructing and characterizing CNLA led many authors to conclude that they were scarce within the variety of nilpotent Lie algebra laws, though results like those of Carles [18] pointed out their importance. The question was definitively solved by Khakimdjanov in 1988, in a series of papers ([53], [54]), where he treated with the cohomology of parabolic subalgebras of simple Lie algebras, first studied by Kostant in 1963 ([58], [59]), and applied these results to the study of deformations of the nilradicals of Borel subalgebras of simple Lie algebras. For classical topics we refer the reader to [46], [44], [23] and [58].

In [53] the author developes the cohomological tools needed, such as the fundamental cohomologies, as well as adequate filtrations for these spaces. In [54] this information is applied to prove that almost all deformations of the cited nilradicals are CNLA.

Following the notation used in [44], let L be a simple Lie algebra of rank $l > 1$, H its Cartan subalgebra, Φ the root system associated to H, Φ^+ the system of positive roots relative to a certain ordering and Δ the system of simple roots. Recall that a Borel subalgebra is a maximal solvable subalgebra of L.

We consider the subalgebra $B(\Delta) = H + \coprod_{\alpha\in\Phi^+} L_\alpha$, where L_α is the root space corresponding to the root α. This subalgebra is a Borel subalgebra of L called standard relative to the Cartan subalgebra H. Now any Borel subalgebra of L is conjugated to a standard Borel subalgebra [11], and if $\mathfrak{n}$ denotes the nilradical of an algebra $\mathfrak{g}$ we have $\mathfrak{n}(B(\Delta)) = \coprod_{\alpha\in\Phi^+} L_\alpha$. Define $\Phi(i)$ as

$$\Phi(i) = \left\{\alpha \in \Phi^+ \mid \alpha = \alpha_{j_1} + \cdots + \alpha_{j_i},\ \alpha_{j_t} \in \Delta \text{ for } 1 \le t \le i\right\}.$$

Then we can define a graduation on $\mathfrak{n}(B(\Delta D))$ by setting $F_k\mathfrak{n}(B(\Delta)) = \bigoplus_{i\ge k} \mathfrak{n}_i(B(\Delta))$, where $\mathfrak{n}_i(B(\Delta)) = \coprod_{\alpha\in\Phi(i)} L_\alpha$. The filtration in the space of cochains is given by

$$F_kC^j(\mathfrak{n},\mathfrak{n}) = \left\{c \in C^j(\mathfrak{n},\mathfrak{n}) \mid c(a_1,\ldots,a_j) \in F_{t_1+\cdots+t_j+k}\mathfrak{n}\right\}$$

whenever $a_i \in F_{t_i}\mathfrak{n}(B(\Delta))$ and where $\mathfrak{n} = \mathfrak{n}(B(\Delta))$.

This filtration extends to the cocycles and coboundaries. Until now we exclude L to be a simple algebra of the following types:

$$A_i\ (1 \le i \le 5),\ B_2,\ B_3,\ C_3,\ C_4,\ D_4,\ G_2.$$

The reason is that for these algebras certain identities among the fundamental cohomologies and the spaces $F_kH^j(\mathfrak{n},\mathfrak{n})$ for $k = 0, 1$ do not coincide [53, theorem 6]. For example, for those algebras excluded and distinct from $A_i\ (i = 1,2,3), G_2$ the cohomology space $F_0H^2(\mathfrak{n},\mathfrak{n})$ is not zero. On the other side, it is shown that the following system of cocycles suffices for a set of representatives of a basis of $F_0H^2(\mathfrak{n},\mathfrak{n})$: $\{f_{\alpha,\beta} \mid (\alpha,\beta) \in E\}$ with

$$f_{\alpha,\beta}(x_\gamma, x_\delta) = \begin{cases} x_{\sigma_\alpha\sigma_\beta(\delta)} & \text{for } (\gamma,\delta) = (\alpha, \sigma_\alpha(\beta)) \\ 0 & \text{otherwise} \end{cases}$$

where σ_α is the involution associated to the root α and E is the set of pairs of simple roots (α,β) in which (α,β) is identified with (β,α) if α is not joined to β in the Dynkin diagram.

THEOREM 12. *Let L be a simple Lie algebra and $\mathfrak{n}$ be the nilradical of a Borel subalgebra. Let $\psi = \sum_{\omega\in E} \lambda_\omega f_\omega$ an element of $F_0H^2(\mathfrak{n},\mathfrak{n})$ with $\lambda_\omega \neq 0$ for all ω. Then the Lie algebra $\mathfrak{n}(\psi)$ obtained from the linearly expandable cocycle ψ is characteristically nilpotent.*

Let $L \in \{A_4, A_5, B_3, C_3, C_4, D_4\}$. For these algebras we have $F_0H^2(\mathfrak{n}, \mathfrak{n}) \neq 0$. In [53] it is proven that the basis is composed by cocycles of the form $x_\alpha \wedge x_\beta \to x_\gamma$, where α, β and γ are enumerated in the following table:

L	α	β	γ
A_4	α_1	α_4	$\alpha_2+\alpha_3$
	α_1	$\alpha_1+\alpha_2$	$\alpha_2+\alpha_3+\alpha_4$
	α_4	$\alpha_3+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$
A_5	α_2	$\alpha_1+\alpha_2$	$\alpha_3+\alpha_4+\alpha_5$
	α_4	$\alpha_4+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$
B_3	α_1	$\alpha_1+\alpha_2$	$\alpha_2+2\alpha_3$
C_3	α_1	$\alpha_1+\alpha_2$	$2\alpha_2+\alpha_3$
C_4	α_2	$\alpha_1+\alpha_2$	$2\alpha_3+\alpha_4$
D_4	α_1	$\alpha_1+\alpha_2$	$\alpha_2+2\alpha_3$
	α_3	$\alpha_2+\alpha_3$	$\alpha_1+\alpha_2+\alpha_4$
	α_4	$\alpha_2+\alpha_4$	$\alpha_1+\alpha_2+\alpha_3$

THEOREM 13. *Let L be a simple Lie algebra of types A_4, A_5, B_3, C_3, C_4 or D_4. Let $\mathfrak{n}$ be the nilradical of the standard Borel subalgebra $B(\Delta)$ and $\varphi = \sum_{\omega \in E} \lambda_\omega f_\omega$, where $\{f_\omega \mid \omega \in E\}$ is a basis of $F_0H^2(\mathfrak{n}, \mathfrak{n})$ from the previous table, with $\lambda_\omega \neq 0$ for all $\omega \in E$. Then the nilpotent Lie algebra defined by a deformation*

$$[X, Y]_t = [X, Y] + t\varphi(X, Y) + t^2\varphi_2(X, Y), \quad t \neq 0,$$

is a CNLA.

These results are certainly of interest for the theory of CNLA. It provides not only a relation between the classical Cartan theory of Lie algebras, it moreover gives, in a certain manner, a natural interpretation of the characteristic nilpotence. On the other side, the frequency of CNLA in $\mathfrak{N}^n$ is proven in an elegant manner.

6. Characteristically nilpotent filiform Lie algebras

Most constructions of CNLA made are based on the deformation theory of the naturally graded filiform Lie algebra L_n. The reason is not only its simplicity; it turns out to have the most elementary law among the filiform Lie algebras. Vergne proved in [96] that any filiform Lie algebra can be obtained by a deformation of this algebra. For this reason this algebra has been the

preferred starting point for constructing families of CNLA [100], [54], though recently other authors have turned their interest into the deformations of the other naturally graded filiform Lie algebra [22].

Certain results about the cohomologies of filiform Lie algebras are contained in Vergne's paper [97]. Recall the notations introduced for the filtered cohomology:

LEMMA. *Let $\mathfrak{g}$ be a p-step nilpotent Lie algebra and $d_i = \dim F_i\mathfrak{g}$.*

(i) *If $j > d_1$ then $F_rZ^j(\mathfrak{g},\mathfrak{g}) = Z^j(\mathfrak{g},\mathfrak{g}) = 0$ for $r \in \mathbb{Z}$.*

(ii) *If $d_s < j \leq d_{s-1}$ for some $1 < s \leq p$ then $F_rZ^j(\mathfrak{g},\mathfrak{g}) = Z^j(\mathfrak{g},\mathfrak{g})$ for $r \leq q$, where $q = -\big[pd_p + (p-1)(d_{p-1} - d_p) + \cdots + s(d_s - d_{s+1}) + (s-1)(j - d_s - 1)\big]$.*

COROLLARY. *Let $\mathfrak{g}$ be an n-dimensional filiform Lie algebra. For $2r \leq (j-1)(j-2p-2)$ with $1 \leq j \leq n-1$ we have*

$$F_rZ^j(\mathfrak{g},\mathfrak{g}) = Z^j(\mathfrak{g},\mathfrak{g}).$$

Further, it can be proven (see [96] or [54]) that if $r \leq p - pj$, then $F_rZ^j(\mathfrak{g},\mathfrak{g}) = Z^j(\mathfrak{g},\mathfrak{g})$. As a consequence, any derivation of the Lie algebra $\mathfrak{g}$ will map the space $F_r\mathfrak{g}$ on $T_r\mathfrak{g}$ for any r. This leads to the equality given by Vergne, namely that for $r \leq -p$, where p is the nilindex of the algebra, we have $F_rH^2(\mathfrak{g},\mathfrak{g}) = H^2(\mathfrak{g},\mathfrak{g})$. This equality has been of importance in the study of the irreducible components of the variety of filiform laws.

Now let $\mathfrak{g} = L_n$ be the mopdel filiform Lie algebra introduced in section 1. For this algebra, it is not difficult to prove that its Lie algebra of derivations is $(2n+1)$-dimensional, where $\dim L_n = n+1$. Thus the dimension of the cohomology space $H^1(L_n, L_n)$ is also $n+1$, and from this dim $B^2(L_n, L_n) = n^2$. The description of the spaces $F_0Z^2(L_n, L_n)$ is the key to construct its characteristically nilpotent deformations. Let $\{X_0, \ldots, X_n\}$ be a basis of L_n and define the cochians $\phi(X_0, X_i) = X_j$ for $1 \leq i, j \leq n$. As they are cocycles, the determination of the space $Z^2(L_n, L_n)$ is reduced to the study of those cocycles which satisfy $\phi(X_0, X_i) = 0$ and preserve the natural graduation. In [54], the author construct the following cocycles:

$$\psi_{k,s}(X_i, X_{i+1}) = \begin{cases} X_s & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Therefore, the remaining images are given by the relation

$$\psi_{k,s}(X_i, X_j) = (-1)^{k-i} C_{k-i}^{j-k-1} (adX_0)^{i+j-1-2k} X_s .$$

Now these and the preceding cocycles describe completely the cohomology space $F_0Z^2(L_n, L_n)$:

PROPOSITION. *The cocycles $\phi_{i,j}$ and $\psi_{k,s}$ $(i < j,\ s \leq 2k+1)$ form a basis of $F_0Z^2(L_n, L_n)$.*

COROLLARY. *We have*

$$\dim F_0H^2(L_n, L_n) = \begin{cases} \frac{3n^2-4n+1}{4} & \text{for } n \equiv 1 \pmod 2, \\ \frac{n^2-2n-4}{4} & \text{for } n \equiv 0 \pmod 2. \end{cases}$$

Moreover, a basis is given by the cohomology classes of $\psi_{k,s}$ for $1 \leq k \leq n, 4 \leq s \leq n$ whenever $s \geq 2k+1$.

Using the Chevalley cohomology of the Lie algebra $\mathfrak{g}_n$ it can be shown that the elements of the space $Z^2(\mathfrak{g}_n, \mathfrak{g}_n)$ correspond to infinitesimal deformations of the algebra $\mathfrak{g}_n = (\mathbb{C}^n, \mu_n)$ (see [22], [25]). Let ψ be a cocycle and define the operation

$$[x, y]_\psi := [x, y] + \psi(x, y), \qquad x, y \in \mathbb{C}^n.$$

Then the deformation is linearly expandable if the previous operation satisfies the Jacobi condition, i.e., defines a Lie algebra structure on $\mathbb{C}^n$. Let $\psi \in \bigoplus H_i^2(L_n, L_n) = F_1H^2(L_n, L_n)$. Then the cocycle admits a decomposition $\psi = \sum_{i=1}^r \psi_i$ with $\psi_i \in H_i^2(L_n, L_n)$. The last nonzero component of this decomposition is called the sill cocycle of ψ.

The idea used in [54] is to decompose the preceding basis into layers, where a layer k_0 contains those cocycles $\psi_{k,s}$ whose entry k is k_0. Now a cocycles $\psi = \sum a_{k,s}\psi_{k,s} \in F_0H^2(L_n, L_n)$ is called degenerate in the layer k_0 if all $a_{k_0,s}$ are zero. If it is nondegenerate in this layer we choose ψ_{k_0,s_0} with $a - k_0, s_0 \neq 0$ of least class. This has been called the nondegeneracy class of ψ. Moreover, under the asumption that this last class is r, the layer k_0 is called special if $2k_0 + r + 1 < s$ for any nonzero $a_{k,s}$ for which $k > k - 0$.

DEFINITION. A nonzero cocycle $\psi \in F_1H^2(L_n, L_n)$ is called regular if it is linearly expandable and satisfies one of the following conditions:

(1) There exist two special layers in which the cocycle is nondegenerate with distincts nondegeneracy class.

(2) The cocycle belongs to $F_2H^2(L_n, L_n)$ and there exists a special layer k_0 of class r such that $a_{k_0,r+2+2k_0} \neq 0$ with $2k_0 + r + 2 < s$ for those $a_{k,s}$ with $k > k_0$.

Provided with these cocycles, Khakimdjanov shows then the following:

THEOREM 14. *Let ψ be a regular cocycle in $F_1H^2(L_n, L_n)$. Then the deformation $(L_n)_\psi$ is a CNLA.*

COROLLARY. *Let S be the set of pairs (k, s) of positive integers such that $\frac{(n-5)}{2} + 2k + 1 \leq s \leq n$ and $\psi = \sum_{(k,s)S} a_{k,s}\psi_{k,s}$. Let s_0 be the least integer such that $s_0 \geq \frac{n+1}{2}$. If one of the following conditions*

(i) *$n > 8$ and a_{1,s_0}, $a_{1,s_0+1} \neq 0$,*

(ii) *$n \geq 6$, $a_{1,s_0} = 0$ and a_{1,s_0+1}, $a_{1,s_0+2} \neq 0$,*

holds, then $(L_n)_\psi$ is a CNLA.

This and other corollaries contained in [54] allow to cosntruct large families of CNLAs. The idea is to consider subsets of the basis given above such that the elements of the linear envelope of this set gives lineraly expandable cocycles. Imposing additional conditions on the coefficients, the cocycles are made regular. It is remarked that there exist characteristically nilpotent deformations of L_n based on nonregular cocycles [54]. Moreover, the closure of the orbit corresponding to the set of CNLA of the preceding corollary is a closed irreducible set of the variety $\mathfrak{N}^{n+1}$ containing a nonempty Zariski-open subset formed by CNLAs.

Other results of the same nature due to this author are the follwoing:

LEMMA. *Let $\psi \in F_1H^2(L_n, L_n)$ be a linearly expandable nonzero cocycle. Then its sill cocycle ψ_r is also linearly expandable.*

Now let $(L_n)_\psi$ be a deformation with $\psi \in F_1H^2(L_n, L_n)$. Let ψ_r be the sill cocycle of ψ. Then the Lie algebra $(L_n)_{\psi_r}$ is called the sill algebra of $(L_n)_\psi$. The relation between these two algebras is the crucial point to construct characteristically nilpotent Lie algebras [1].

[1] It is evident that the infinitesimal deformations are filiform, for we have seen that the characteristic sequence of the deformation is greater or equal than $c(L_n)$, and this is the maximal one.

THEOREM 15. *Let $\psi \in F_1H^2(L_n, L_n)$ be a nonzero linearly expandable cocycle. Then the Lie algebra $(L_n)_\psi$ is characteristically nilpotent if and only if it is not isomorphic to its sill algebra $(L_n)_{\psi_r}$.*

From the theorem we obtain for example the following characteristic nilpotent Lie algebras with basis $\{X_0, \dots, X_{2m}\}$ and law

$$\begin{aligned} [X_0, X_i] &= X_{i+1}\,, \qquad i = 1, \dots, 2m-1\,; \\ [X_1, X_i] &= X_{i+3}\,, \qquad i = 2, \dots, 2m-3\,; \\ [X_i, X_{2m-i-1}] &= (-1)^{i+1} X_{2m}\,, \quad i = 1, \dots, m-1\,. \end{aligned}$$

For the nonfiliform Lie algebras the determination of characteristically nilpotent Lie algebras is not so well structured. In fact, for any lower characteristic sequence there will appear more naturally graded models than it was the case in the filiform algebras. This construction allowed to obtain certain results on the structure of the neighborhhods of filiform Lie algebras on the variety $\mathfrak{N}^n$ [23], so it is of interest for the determination of the irreducible components of the variety of filiform Lie algebra laws, thus for the variety $\mathfrak{N}^n$ itself. We mantain the notation for the cohomology introduced earlier.

LEMMA. *Let $s > r, s \neq 2r$. If there is a nonzero cocycle $\psi \in H_s^2(L_n, L_n)$ belonging to $H_s^2(L_n, L_n) \cap B^2((L_n)_\psi, (L_n)_\psi)$, then this cocycle is unique (up to multiples).*

The proof is based on the structure of the algebra of derivations of a sill algebra and is omitted here. It can be found in [23] and [26]. Now let $A = (L_n)_\psi$ be a filiform algebra, where $\psi \in Z^2(L_n, L_n) \cap F_1H^2(L_n, L_n)$ and ψ_r denotes the sill cocycle of ψ.

LEMMA. *Let $n \geq 8$ and V an open set of $\mathfrak{N}^n$ containing A. Then there exists a characteristically nilpotent Lie algebra in V.*

Then we obtain immediately the following:

COROLLARY. *For $\mathfrak{N}^n$ $(n \geq 7)$ there exists an open set whose elements are characteristically nilpotent Lie algebras.*

7. Lie algebras of type Q and its deformations

In this section we use the other naturally graded filiform Lie algebra, Q_n, to obtain characteristically nilpotent Lie algebras in any dimension $n \geq 9$ and

mixed characteristic sequence. This approach is perhaps not so natural, but it is based on an important property of "noncommutativity", which allows to obtain "easier" deformations. Combined with central extensions of special kind, we obtain the desired characteristically nilpotent deformations. Let us concentrate on the Lie algebra Q_n. In contrast to L_n, it can only appear in even dimension. Thus the algebra Q_n posesses a structural obstruction that forces its even-dimensionality. This obstruction is strongly related with the properties of the descending central sequence C^kQ_n.

Let $\omega_1, \dots, \omega_{2m}$ be the dual basis of the basis $X_1, \dots, X_{2m}$ of Q_n. Then the Cartan-Maurer equations of this algebra are:

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0\,, \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \le j \le 2m-1\,, \\ d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}\,. \end{aligned}$$

In particular, the nonzero exterior product $\omega_m \wedge \omega_{m+1}$ shows that the ideal $C^{p-1}Q_n$, where $\left[\frac{2m-1}{2}\right]$ and $n = 2m-1$ of the central descending sequence is not abelian, while C^pQ_n is abelian. This can be interpreted in the following manner: while L_n has abelian commutator algebra C^1L_n, the model Q_n is as far as possible from being an abelian algebra. This fact is important for deformation theory, as it can be interpreted in the sense that deforming Q_n will be easier than deforming L_n.

The previous property can be expressed in terms of centralizers:

$$\begin{aligned} C_{Q_n}\left(C^pQ_n\right) &\supset C^pQ_n\,, \\ C_{Q_n}\left(C^qQ_n\right) &\not\supseteq C^qQ_n\,, \end{aligned}$$

for $n = 2m-1$, $p = \left[\frac{2m-1}{2}\right]$ and $1 \le q \le p-1$. We will say that Q_n satisfies the centralizer property.

It is rather convenient to generalize this property to any naturally graded Lie algebra:

DEFINITION. Let $\mathfrak{g}_n$ be an n-dimensional, naturally graded nilpotent Lie algebra of nilindex p. Then $\mathfrak{g}_n$ is called of type Q if

$$\begin{aligned} C_{\mathfrak{g}_n}\left(C^p\mathfrak{g}_n\right) &\supset C^p\mathfrak{g}_n\,, \\ C_{\mathfrak{g}_n}\left(C^q\mathfrak{g}_n\right) &\not\supseteq C^q\mathfrak{g}_n\,, \end{aligned}$$

for $n = 2m-1$, $p = \left[\frac{p}{2}\right]$ and $1 \le q \le p-1$.

We are principally concerned with the Lie algebras of type Q that are central extensions of the filiform Lie algebra Q_n, as well as other extensions.

Observe however that the index fixed in the previous definition is maximal, i.e, there do not exist Lie algebras which are "less abelian" with respect to the previous definition. The index, will depends only on the nilindex of the algebra, is very important and appears in apparently different contexts, such as the parabolic subalgebras [58].

THEOREM 16. *Let $\mathfrak{n}$ be the nilradical of a standard Borel subalgebra $\mathfrak{b}(\Delta)$ of a complex simple Lie algebra distinct from G_2. Then $\mathfrak{n}$ satisfies the centralizer property.*

The proof is an immediate consequence of the following result:

PROPOSITION. *Let $\mathfrak{n}$ be the nilradical of a standard Borel subalgebra $\mathfrak{b}(\Delta)$ of a complex simple Lie algebra distinct from G_2. Let $p = ht(\delta)$ be the height of the maximal root. Then there exist roots α, β whose height is $[\frac{ht(\delta)}{2}]$ such that $\alpha + \beta$ is a positive root.*

Thus we see that the classical theory provides a lot of naturally graded Lie algebras satisfying the centralizer property. However, it is usually unconvenient to manipulate these algebras, because of the great difference between its dimension and nilpotence class: the first is too high in comparison with the last.

From the definition it follows also that a central extension $\mathfrak{e}$ of Q_n by $\mathbb{C}$ of type Q cannot be filiform. This implies that the cocycle $\varphi \in H^2(Q_n, \mathbb{C})$ that defines the extension cannot be affine [17]. As a central extension of a filiform Lie algebra is filiform if and only if the cohomology class of φ is affine, we conclude that for our special case, the extension $\mathfrak{e}$ cannot be given by an affine cocycle.

Let $\mathfrak{e} \in E_{c,1}(Q_n)$ be an extension of type Q. As the nilindex is preserved, we conclude that the characteristic sequence of $\mathfrak{e}$ must be lower than $(2m, 1)$. Thus these algebras will play, in the set of Lie algebras with this characteristic sequence, the same role that Q_n plays for the filiform algebras.

Let $\widetilde{E}_{c,1}(Q_n) = \{\mathfrak{e} \in E_{c,1}(Q_n) \mid \mathfrak{e} \text{ is of type } Q\}$. If $\mathfrak{e}$ is any such element expresed over the basis $X_1, \ldots, X_{2m+1}$, it follows immediately from the definition of type Q that $\mathfrak{e}$ is naturally graded. The first $2m$ vectors are fixed in the natural graduation of the extension, thus $\mathfrak{e}$ is completely determined once we know the position of the vector X_{2m+1} in the graduation. The next lemma establishes that the positions are not arbitrary.

LEMMA. *Let $\mathfrak{e} \in E_{c,1}(Q_n)$ be an extension. If $X_{2m+1} \in \mathfrak{e}_{2t}$ $(1 \leq t \leq \left[\frac{2m-1}{2}\right])$ then $\mathfrak{e}$ is not naturally graded. In particular, $\mathfrak{e} \notin \widetilde{E}_{c,1}(Q_n)$.*

It follows that the position of the vector X_{2m+1} is only admissible if the graduation block is odd indexed. As we are not interested in split Lie algebras, we convene that $X_{2m+1} \notin \mathfrak{e}_1$. Moreover, we define the depth h of X_{2m+1} like follows:

$$h(X_{2m+1}) = t \qquad \text{if } X_{2m+1} \in \mathfrak{e}_{2t+1}, \quad 1 \leq t \leq \left[\frac{2m-1}{2}\right] - 1.$$

For convenience Lie algebras will be written usually in their contragradient representation. This will be of importance for the deformations, as linearly expandable cocycles are easier recognized when using this notation. Let $\omega_1, \ldots, \omega_{2m+1}$ be the dual basis to $X_1, \ldots, X_{2m+1}$ for the extension $\mathfrak{e} \in E_{c,1}(Q_n)$. Then its Cartan-Maurer equations are:

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0, \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}, \qquad 3 \leq j \leq 2m-1, \\ d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} (-1)^j \omega_j \wedge \omega_{2m+1-j}, \\ d\omega_{2m+1} &= \sum_{i,j} a^{ij} \omega_i \wedge \omega_j, \quad a^{ij} \in \mathbb{C}, \ i,j \geq 2, \end{aligned}$$

where $d^2\omega_{2m+1} = 0$. Then the determination of the extensions of type Q of Q_n reduces to the determination of the possible differential forms $d\omega_{2m+1}$. As known, the coefficient $a^{i,j}$ is given by a linear form over $\bigwedge^2 Q_n$ which annihilates over Ω.

Let $\varphi_{ij} \in \mathrm{Hom}(\bigwedge^2 Q_n, \mathbb{C})$, $2 \leq i,j \leq 2m$, be defined by

$$\varphi_{ij}(X_k, X_l) = \delta_{ik}\delta_{kl}, \qquad (X_k, X_l) \in \mathfrak{g}^2.$$

LEMMA. *For $m \geq 4$, $1 \leq t \leq m-2$ the cochain $\varphi_t = \sum_{j=2}^{t+1} (-1)^j \varphi_{j,3+2t-1}$ defines a cocycle of $H^2(Q_n, \mathbb{C})$. If $\mathfrak{g}_{(m,t)}$ denotes the extension defined by φ_t, then $\mathfrak{g}_{(m,t)} \in \widetilde{E}_{c,1}(Q_n)$.*

In particular, it follows from the proof [22] that the Cartan-Maurer equa-

tions of such an extension are

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0\,, \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \leq j \leq 2m-1\,, \\ d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}\,, \\ d\omega_{2m+1} &= \sum_{j=2}^{t+1} (-1)^j\, \omega_j \wedge \omega_{3-j+2t}\,, \quad 1 \leq t \leq m-2\,. \end{aligned}$$

The family of extensions (which is locally finite and depends on m) is proven to be the class of algebras we are interested in, as follows from the next:

PROPOSITION. *An extension $\mathfrak{e} \in E_{c,1}(Q_n)$ is of type Q if and only if there exists a $t \in \{1, \ldots, m-2\}$ such that $\mathfrak{e} \simeq \mathfrak{g}_{(m,t)}$.*

Let $\widehat{H}^2(Q_n, \mathbb{C}) = \left\{\varphi \in H^2(Q_n, \mathbb{C}) \mid \mathfrak{e}_\varphi \text{ is of type } Q\right\}$, where $\mathfrak{e}_\varphi$ is the extension defined by φ. The above result proves that $\dim\, \widehat{H}^2(Q_n, \mathbb{C}) = m-1$, where $n = 2m-1$. Moreover, the type of the extension $\mathfrak{g}_{(m,t)}$ satisfies

$$\begin{aligned} &p_1 = p_{2t+1} = 2\,, \\ &p_j = 1 \qquad \text{if } \; 1 \leq j \leq 2m-1\,, \;\; j \notin \{1, 2m+1\}\,. \end{aligned}$$

As we have seen, the structure of the extensions $\mathfrak{g}_{(m,t)}$ is very similar, in the sense that the differential form $d\omega_{2m+1}$ has a precise form which depends only on the depth of the (added) vector X_{2m+1} dual to X_{2m+1}.

Now a construction method for characteristically nilpotent Lie algebras is given. These deformations will be also interpretable in term of the graded cohomology spaces $H_k^2(\mathfrak{g}, \mathfrak{g})$ associated to the lie algebra $\mathfrak{g}$. The results given here are more widely covered in [22]:

NOTATION. Let $\mathfrak{g}$ be a n-dimensional Lie algebra defined over the basis $\{X_1, \ldots, X_n\}$ and let $\mathrm{Der}(\mathfrak{g})$ be its algebra of derivations. If $f \in \mathrm{Der}(\mathfrak{g})$, we will use the notation

$$f(X_i) = \sum_{j=1}^{n} f_i^j X_j\,, \qquad 1 \leq i \leq n\,.$$

We consider the following cocycle (class) for the Lie algebras $\mathfrak{g}_{(m,t)}$ and $t \geq 2$:

$$\varphi_{m,t}\left(X_2, X_{3+j}\right) = X_{2t+2+j}\,, \qquad 0 \leq j \leq 2m-2t-2\,.$$

The reason for excluding the value $t = 1$ lies in the simplicity of its last differential form. For these algebras special cocycles have to be considered [22]:

LEMMA. *For $m \geq 5,\ 1 \leq t \leq m-2$ $\varphi_{m,t}$ is linearly expandable.*

PROPOSITION. *For $m \geq 5,\ 1 \leq t \leq m-2$ the Lie algebra $\mathfrak{g}_{(m,t)} + \varphi_{m,t}$ is characteristically nilpotent.*

Note that the cocycle which defines the deformation $\mathfrak{g}_{(m,t)} + \varphi_{m,t}$ is chosen such that the incorporated brackets do not change the exterior differential of the system. The cocycle $\varphi_{m,t}$ admits the following cohomological interpretation:

PROPOSITION. *For $t \geq 3$ let $\psi \in H^2_{2t-2}\left(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)}\right)$ be a cocycle that satisfies:*

(i) *for all $X \in Z\left(\mathfrak{g}_{(m,t)}\right)$ such that $h\left(X\right) = t$, we have $\psi\left(X, \mathfrak{g}_{(m,t)}\right) = \{0\}$ and $X \notin im\left(\psi\right)$;*

(ii) *if $X \in \mathfrak{g}_{(m,t)}$ is such that there exists an $Y \in Z\left(\mathfrak{g}_{(m,t)}\right)$ with $h\left(Y\right) = t$ and $Y \notin im\ ad\left(X\right)$, then $\psi\left(X, C^1\mathfrak{g}_{(m,t)}\right) = \{0\}$.*

Then

$$\psi = \sum_{\substack{2 \leq i \leq t+2 \\ 3 \leq j \leq 2m-3}} \lambda_{ij}\psi_{ij}\,, \qquad \left(\lambda_{ij} \in \mathbb{C}\right),$$

where

$$\psi_{ij}\left(X_i, X_j\right) = X_{i+j+1}\,, \qquad i+j \leq 2m\,.$$

Writing

$$\widehat{H}^2_{2t-2}\left(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)}\right) = \{\psi \mid \psi \text{ satisfies (i) and (ii)}\}$$

we isolate the cohomology classes that give the desired deformations:

COROLLARY. *A cocycle $\psi \in \widehat{H}^2_{2t-2}(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)})$ such that $\psi(C^1\mathfrak{g}_{(m,t)}, C^1\mathfrak{g}_{(m,t)} = \{0\}$ is linearly expandable if and only if $\psi = \lambda\varphi_{m,t}$ $(\lambda \in \mathbb{C})$. Moreover, $\mathfrak{g}_{(m,t)} + \lambda\varphi_{m,t} \simeq \mathfrak{g}_{(m,t)} + \varphi_{m,t}$ for any $\lambda \neq 0$.*

From the corollary we deduce that $\varphi_{m,t}$ is fixed, up to multiples, by the restriction property to the derived subalgebra.

THEOREM 17. *Let $\psi \in \widehat{H}^2_{2t-2}\left(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)}\right)$ be a linearly expandable cocycle. Then the algebra $\mathfrak{g}_{(m,t)} + \psi$ is characteristically nilpotent.*

Any supplementary deformation to the one defined by the cocycle $\varphi_{m,t}$ changes the law $\mathfrak{g}_{(m,t)}$ in the same way as $\varphi_{m,t}$, so that it does not alter the conditions on the derivations. Further, we determine certain central extensions of the algebras $\mathfrak{g}_{(m,t)}$ obtained before. Observe that the characteristic of an extension of $\mathfrak{g}_{(m,t)}$ by $\mathbb{C}$ can be either $(2m-1,1,1,1)$ or $(2m-1,2,1)$. The first one is not interesting for our purposes, as it is linear, while the second one is mixed [2].

Let $\mathcal{G}_2^1 = \left\{\mathfrak{g}_{(m,t)} \mid m \geq 4,\ 1 \leq t \leq m-2\right\}$. For any fixed m and t we define

$$E^1_{c,1}(\mathfrak{g}_{(m,t)}) = \{\mathfrak{e} \in E_{c,1}(\mathfrak{g}_{(m,t)}) \mid \mathfrak{e} \text{ is of type } Q \text{ and } h(X_{2m+2}) = h(X_{2m+1})+1\}$$

where $\{X_1, \ldots, X_{2m+1}\}$ is a basis of $\mathfrak{g}_{(m,t)}$, $\{X_1, \ldots, X_{2m+2}\}$ a basis of $\mathfrak{e}$ and h is the depth funtion.

PROPOSITION. *Let $t \geq 2$ and $\mathfrak{g}_{(m,t)} \in \mathcal{G}_2^1$. Then an extension $\mathfrak{e} \in E_{c,1}(\mathfrak{g}_{(m,t)})$ belongs to $E^1_{c,1}\left(\mathfrak{g}_{(m,t)}\right)$ if and only if its structural equations are*

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\,,\\
d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \leq j \leq 2m-1\,,\\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}\,,\\
d\omega_{2m+1} &= \sum_{j=2}^{t+1} (-1)^j\, \omega_j \wedge \omega_{3+2t-j}\,,\\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j\, (t+2-j)\, \omega_j \wedge \omega_{4+2t-j}\,.
\end{aligned}$$

An extension $\mathfrak{e}$ with the previous Cartan-Maurer equations will be denoted by $\mathfrak{g}^1_{(m,t)}$. Observe that the case $t = 1$ has been excluded from the proposition. The reason is that, by the simplicity of the differential form $d\omega_{2m+1}$, in this case there are two possible extensions.

[2] A characteristic sequence $c(\mathfrak{g})$ is called mixed if there are two or more Jordan blocks of dimension ≥ 2.

LEMMA. *For $m \geq 4$, $\mathfrak{e} \in E_{c,1}\left(\mathfrak{g}_{(m,1)}\right)$ belongs to $E^1_{c,1}\left(\mathfrak{g}_{(m,1)}\right)$ if the structural equations of $\mathfrak{e}$ over a basis $\{\omega_1, \ldots, \omega_{2m+2}\}$ are*

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\,,\\
d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \leq j \leq 2m-1\,,\\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}\,,\\
d\omega_{2m+1} &= \omega_2 \wedge \omega_3\,,\\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \omega_2 \wedge \omega_4 + k\omega_2 \wedge \omega_{2m+1}\,, \quad k = 0, 1\,.
\end{aligned}$$

The proof is analogous to the preceding one. The reason for the existence of the second extension is the weakness of the restrictions imposed by the differential form $d\omega_{2m+1}$. For higher depths the existence of additional exterior products in the adjoined form $d\omega_{2m+2}$ is not compatible with its closure $d^2\omega_{2m+2} = 0$.

NOTATION. For $k = 0$ the extension is denoted by $\mathfrak{g}^1_{(m,1)}$, and for $k = 1$ by $\mathfrak{g}^2_{(m,1)}$.

As known, the set of nilpotent Lie algebras $\mathfrak{g}$ of a given dimension n and characteristic sequence $c(\mathfrak{g})$ is denoted by $\delta U^n_{c(\mathfrak{g})}$ [3]. Now let $E_{c,2}(Q_n)$ be the set of central extensions of Q_n by $\mathbb{C}^2$. The following result shows that we have obtained practically all the extensions that interest us.

Let $\mathfrak{g}^{1+k}_{(m,0)}$ $(k = 0, 1)$ be the Lie algebras with structural equations

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\,,\\
d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \leq j \leq 2m-1\,,\\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{m} (-1)^j\, \omega_j \wedge \omega_{2m+1-j}\,,\\
d\omega_{2m+1} &= 0\,,\\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + k\omega_2 \wedge \omega_{2m+1}\,.
\end{aligned}$$

THEOREM 18. *For $n = 2m-1$, $m \geq 4$ the following identity holds:*

$$E_{c,2}(Q_n) \cap \delta U^{2m+2}_{(2m-1,2,1)} = \bigcup_{j=2}^{m-2} \mathcal{O}(\mathfrak{g}^1_{(m,t)}) \cup \mathcal{O}(\mathfrak{g}^2_{(m,1)}) \cup \mathcal{O}(\mathfrak{g}^{1+k}_{(m,0)})\,, \quad k = 0, 1\,,$$

where $\mathcal{O}(\mathfrak{g})$ denotes the orbit of $\mathfrak{g}$ by the action of the general linear group.

Any extension of Q_n by $\mathbb{C}^2$ must have characteristic sequence $(2m-1,1,1,1)$ or $(2m-1,2,1)$ if it preserves the nilindex. Observe however that for the first sequence, the split algebra $Q_n \oplus \mathbb{C}$ cannot generate a nonsplit central extension. Now it is convenient to introduce some notation. For $1 \leq t \leq m-2$ we can write the algebras $\mathfrak{g}^1_{(m,t)}$ formally as

$$\mathfrak{g}^1_{(m,t)} = \mathfrak{g}_{(m,t)} + d\bar{\omega}_{m,t}$$

where

$$d\bar{\omega}_{m,t} = \omega_1 \wedge \omega_{2m+1} + \sum_{j=2}^{t+1} (-1)^j \, (t+2-j) \, \omega_j \wedge \omega_{4-j+2t}$$

is called extensor of type I.

7.1. Deformations of $\mathfrak{g}^1_{(m,t)}$ $(t \geq 2)$. Let $\mathfrak{g}^1_{(m,t)}$ and consider an extensor of type I $d\bar{\omega}_{m,t}$. We know that $\mathfrak{g}^1_{(m,t)} = \mathfrak{g}_{(m,t)} + d\bar{\omega}_{m,t}$.

Consider a cocycle $\psi \in H^2(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ defined by

$$\psi(X_i, X_j) = \begin{cases} \varphi_{m,t}(X_i, X_j) & \text{if } 1 \leq i,j \leq 2m+1\,, \\ 0 & \text{if } i = 2m+2 \text{ or } j = 2m+2\,, \end{cases}$$

ψ is clearly a prolongation by zeros of the cocycle $\varphi_{m,t}$; it will be convenient to preserve the notation $\varphi_{m,t}$ to denote ψ, whenever there is no ambiguity. In the previous section we saw that the adjoined extensors have no influence on the characetristic nilpotence of the deformation $\mathfrak{g}_{(m,2)} + \varphi_{m,2}$. This property is in fact generalizable to any $t \geq 3$:

Proposition. *For any $m \geq 4$, $1 \leq t \leq m-2$ the cocycle $\varphi_{m,t} \in H^2(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ is linearly expandable.*

Corollary. *For any $m \geq 4$, $1 \leq t \leq m-2$ the following identity holds:*

$$(\mathfrak{g}_{(m,t)} + d\bar{\omega}_{m,t}) + \varphi_{m,t} = (\mathfrak{g}_{(m,t)} + \varphi_{m,t}) + d\bar{\omega}_{m,t}\,.$$

Theorem 19. *For $m \geq 4$, $1 \leq t \leq m-2$ the Lie algebra $\mathfrak{g}^1_{(m,t)} + \varphi_{m,t}$ is characteristically nilpotent.*

The cocycles $\varphi_{m,t}$ are a special case of a more wide family of cocycles of the subspace $H^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$:

LEMMA. *If $\psi \in H^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ is a prolongation by zeros of a cocycle $\varphi \in H^2_{2t-2}(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)})$, then ψ satisfies the conditions:*

(i) *for all $X \in Z(\mathfrak{g}^1_{(m,t)})$ such that $h(X) = \frac{2t+1}{2}$, we have $\psi(X, \mathfrak{g}^1_{(m,t)}) = \{0\}$ and $X \notin im(\psi)$;*

(ii) *for all $X \in Z^2(\mathfrak{g}^1_{(m,t)})$ such that $h(X) = t$, we have $\psi(X, \mathfrak{g}^1_{(m,t)}) = \{0\}$ and $X \notin im(\psi)$;*

(iii) *if $X \in \mathfrak{g}^1_{(m,t)}$ is such that there exists an $Y \in Z^2(\mathfrak{g}^1_{(m,t)})$ with $h(Y) = t$ and $Y \notin im\, ad(X)$, then $\psi(X, C^1\mathfrak{g}^{2,1}_{(m,t)}) = \{0\}$.*

PROPOSITION. *A cocycle $\psi \in H^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ is a prolongation by zeros of a cocycle $\varphi \in H^2_{2t-2}(\mathfrak{g}_{(m,t)}, \mathfrak{g}_{(m,t)})$ if and only if it satisfies conditions* (i), (ii) *and* (iii).

We note

$$\widehat{H}^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)}) = \{\psi \mid \psi \text{ satisfies (i), (ii) and (iii)}\}.$$

COROLLARY. *A cocycle $\psi \in H^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ is a prolongation by zeros of $\varphi_{m,t}$ if and only if the restriction of ψ to the derived subalgebra $C^1\mathfrak{g}^1_{(m,t)}$ is identically zero.*

THEOREM 20. *Let $\psi \in \widehat{H}^2_{2t-2}(\mathfrak{g}^1_{(m,t)}, \mathfrak{g}^1_{(m,t)})$ be linearly expandable. Then $\mathfrak{g}^1_{(m,t)} + \psi$ is characteristically nilpotent.*

These results can be resumed graphically. We introduce the following notations [22]:

$$\begin{aligned}
M^1_{m,1}(\mathfrak{g}_{(m,t)} + \varphi_{m,t}) &= \mathfrak{g}_{(m,t+1)} + \varphi_{m,t+1} \\
D^1_{1,t}(\mathfrak{g}_{(m,t)} + \varphi_{m,t}) &= \mathfrak{g}_{(m+1,t)} + \varphi_{m+1,t} \\
d\bar{\omega}_{m,t}(\mathfrak{g}_{(m,t)} + \varphi_{m,t}) &= \mathfrak{g}^1_{(m,t)} + \varphi_{m,t} \\
M^2_{m,1}(\mathfrak{g}^1_{(m,t)} + \varphi_{m,t}) &= \mathfrak{g}^1_{(m,t+1)} + \varphi_{m,t+1} \\
D^2_{1,t}(\mathfrak{g}^1_{(m,t)} + \varphi_{m,t}) &= \mathfrak{g}^1_{(m+1,t)} + \varphi_{m+1,t}
\end{aligned}$$

for $m \geq 4$ and $1 \leq t \leq m-2$.

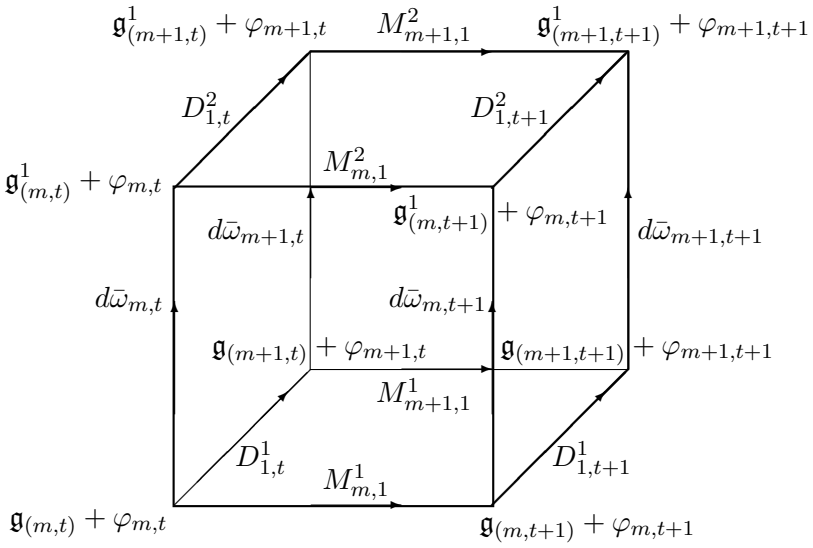

Figure 1.

THEOREM 21. *For $m \geq 4$ and $1 \leq t \leq m-2$ the faces of the following cube are commutative diagrams.*

8. Nilpotent Lie algebras and rigidity

Let $\mathfrak{L}^n$ be the algebraic variety of complex Lie algebra laws on $\mathbb{C}^n$. Each open orbit of the natural action of $GL(n,\mathbb{C})$ on $\mathfrak{L}^n$ gives, considering its Zarisky closure, an irreducible component of $\mathfrak{L}^n$. Therefore, only a finite number of those orbits exists; or, equivalently, only a finite number of isomorphism classes of Lie algebras with open orbit. The first results about rigid Lie algebras are due to Gerstenhaber [38], Nijenhuis and Richardson [78]. The last two authors have transformed the topological problems related to rigidity into cohomological problems, proving in that an algebra is rigid if the second group in the Chevalley cohomology is trivial. This theorem allows the construction of examples or rigid Lie algebras and is used in the proof that semisimple algebras are rigid. However, the existence of rigid Lie algebras whose second cohomology group is non trivial shows that the cohomological viewpoint is not fully satisfactory in the study of rigidity.

DEFINITION. A Lie algebra $\mathfrak{g}$ is called decomposable if it can be written

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{t} \oplus \mathfrak{n}$$

where $\mathfrak{s}$ is a Levi subalgebra, $\mathfrak{n}$ the nilradical and $\mathfrak{t}$ an abelian subalgebra whose elements are *ad*-semisimple and which satisfies $[\mathfrak{s}, \mathfrak{t}] = 0$.

The abelian subalgebra T of $\operatorname{Der} \mathfrak{g}$ defined by

$$T = \{adX \mid X \in \mathfrak{t}\}$$

is called, following Malcev, an exterior torus on $\mathfrak{g}$. It is called maximal torus, if it is maximal for the inclusion. Malcev has proved that all maximal torus are pairwise conjugated, thus they have the same dimension called the *rank* of $\mathfrak{g}$ and noted $r(\mathfrak{g})$.

THEOREM 22. *Rigid Lie algebras are algebraic.*

8.1. ROOTS SYSTEM ASSOCIATED TO A RIGID SOLVABLE LIE ALGEBRA. Let μ_0 be a solvable decomposable law on $\mathbb{C}^n$. We fix a maximal exterior torus T. Let X be a non-zero vector such that $ad_{\mu_0} X$ belongs to T.

DEFINITION. We say that X is *regular* if the dimension of

$$V_0(X) = \{Y \mid \mu_0(X, Y) = 0\}$$

is minimal ; that is, $\dim V_0(X) \leq \dim V_0(Z)$ for all Z such that $ad_{\mu_0} Z \in T$.

DEFINITION. Suppose that μ_0 is not nilpotent. The *root system* of μ_0 associated to $(X, Y_1, \ldots, Y_{n-p}, X_1, \ldots, _{p-1})$ is the linear system (S) defined by the following equations:

$$\begin{aligned}
x_i + x_j &= x_k \qquad \text{if the } X_k\text{-coordinate of } \mu_0(X_i, X_j) \text{ is non-zero}, \\
y_i + y_j &= y_k \qquad \text{if the } Y_k\text{-coordinate of } \mu_0(Y_i, Y_j) \text{ is non-zero}, \\
x_i + y_j &= y_k \qquad \text{if the } Y_k\text{-coordinate of } \mu_0(X_i, X_j) \text{ is non-zero}, \\
y_i + y_j &= x_k \qquad \text{if the } X_k\text{-coordinate of } \mu_0(Y_i, Y_j) \text{ is non-zero}.
\end{aligned}$$

In these notations we state.

THEOREM 23. *If* $\operatorname{rank}(S) \neq \dim(I_0) - 1$, *the law* μ_0 *is not rigid.*

COROLLARY. *If μ_0 is rigid, the rank of a root system for μ_0 is independent of the basis $(X, Y_1, \ldots, Y_{n-p}, X_1, \ldots, X_{p-1})$ used for its definition.*

COROLLARY. *If μ_0 is rigid, there is regular vector X such that $ad_{\mu_0} X$ is diagonal and its eigenvalues are integers.*

Let $\mathfrak{R}_n$ be the variety of n-dimensional solvable Lie algebras. The principal structure theorem referring to rigid Lie algebras was proven by Carles in [18]:

THEOREM 24. *Any Lie algebra $\mathfrak{g}$ which is rigid in either $\mathfrak{L}^n$ or $\mathfrak{R}^n$ is algebraic and belongs to one of the following cases:*

(i) *The radical Rad $(\mathfrak{g})$ is not nilpotent and* $\dim \mathrm{Der}\,(\mathfrak{g}) = \dim \mathfrak{g}$ *(if moreover* $\mathrm{codim}\, C^1\mathfrak{g} > 1$, *the algebra is complete).*

(ii) *The radical is nilpotent and satisfies one of the following conditions:*

  (a) *$\mathfrak{g}$ is perfect;*

  (b) *$\mathfrak{g}$ is the direct product of $\mathbb{C}$ by a rigid perfect Liealgebra whose derivations are inner;*

  (c) *$\mathfrak{g}$ is non-perfect, has no direct abelain factor and is of rank zero; morover, for any ideal of codimension one is also of rank zero.*

COROLLARY. *Any Lie algebra $\mathfrak{g}$ rigid in $\mathfrak{R}^n$ is algebraic and satisfies one of the following conditions:*

(i) $\dim \mathrm{Der}(\mathfrak{g}) = \dim \mathfrak{g}$ *(if moreover* $\mathrm{codim}\, C^1\mathfrak{g} > 1$, *the algebra is complete);*

(ii) *$\mathfrak{g}$ is characteristically nilpotent, as well as any of its codimension one ideals.*

From the structure of the derivations for filiform Lie algebras, as found for example in [40], it follows easily that none filiform Lie algebra can be rigid in $\mathfrak{L}^n$ or $\mathfrak{R}^n$; by Carles' theorem, such an algebra would be characteristically nilpotent, and a contradiction with the dimension formulas is served. Thus the counterexamples, if any, must be searched within the nonfiliform Lie algebras. This would give an effective answer To Vergne's conjeture (1970)

CONJECTURE 2. For any $n \neq 1$ there do not exist nilpotent Lie algebras which are rigid in $\mathfrak{L}^n$ or $\mathfrak{R}^n$.

Recently we have found another curious relation between CNLA and rigid algebras. If we consider the Lie algebra $\mathfrak{g}_{(m,m-1)}$ $(m \geq 3)$ defined by the equations

$$\begin{aligned} d\omega_1 &= d\omega_2 = 0\,, \\ d\omega_j &= \omega_1 \wedge \omega_{j-1}\,, \qquad 3 \leq j < 2m\,, \\ d\omega_{2m+1} &= \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j\ \omega_j \wedge \omega_{2m+1-j}\,, \end{aligned}$$

it is immediate to see that its characteristic sequence is $(2m-1,1,1)$ and its rank is two. Then there exists deformations which are isomorphic to the nilradical of a solvable rigid law, as gives the

PROPOSITION. *The solvable Lie algebras* $\mathfrak{r}_{(m,m-1)}$ $(m \geq 3)$ *defined by the equations*

$$\begin{aligned} d\omega_1 &= \omega_{2m+2} \wedge \omega_1\,, \\ d\omega_2 &= (2m-3)\,\omega_{2m+2} \wedge \omega_2\,, \\ d\omega_j &= \omega_1 \wedge \omega_{j-1} + (2m-5+j)\,\omega_{2m+2} \wedge \omega_j\,, \qquad 3 \leq j \leq 2m-1\,, \\ d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \omega_2 \wedge \omega_3 + (4m-5)\,\omega_{2m+2} \wedge \omega_{2m}\,, \\ d\omega_{2m+1} &= \sum_{j=2}^{\left[\frac{2m+1}{2}\right]} (-1)^j\ \omega_j \wedge \omega_{2m+1-j} + (6m-9)\,\omega_{2m+2} \wedge \omega_{2m+1}\,, \\ d\omega_{2m+2} &= 0\,, \end{aligned}$$

are rigid and complete. Moreover, their nilradical has codimension one and is isomorphic to the Lie algebra $\mathfrak{g}_{(m,m-1)}+\psi$, *where* $\psi \in H^2\left(\mathfrak{g}_{(m,m-1)}, \mathfrak{g}_{(m,m-1)}\right)$ *is the linearly expandable cocycle defined by* $\psi\left(X_2, X_3\right) = X_{2m}$.

These algebras are a particular case of rigid Lie algebras whose nilradical has codimension one, characteristic sequence $(2m-1,1,1)$ and whose eigenvalues are $(1, k, k+1, \ldots, 2k+1, 3k)$. There exist classifications of rigid algebras having similar sequences of eigenvalues and filiform nilradical. However, there is nothing similar for nonfiliform Lie algebras. Now the interesting fact is that we can extend centrally the preceding nilradicals of rigid laws to obtain characteristically nilpotent Lie algebras [22]:

THEOREM 25. *The Lie algebras* $e_1\left(\mathfrak{g}_{(m,m-1)}+\psi\right)$ $(m\geq 3)$ *defined by the structural equations*

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\,,\\
d\omega_j &= \omega_1\wedge\omega_{j-1}\,, \qquad 3\leq j\leq 2m-1\,,\\
d\omega_{2m} &= \omega_1\wedge\omega_{2m-1}+\omega_2\wedge\omega_3\,,\\
d\omega_{2m+1} &= \sum_{j=2}^{\left[\frac{2m+1}{2}\right]}(-1)^j\,\omega_j\wedge\omega_{2m+1-j}\,,\\
d\omega_{2m+2} &= \omega_1\wedge\omega_{2m+1}+\sum_{j=2}^{\left[\frac{2m+1}{2}\right]}(-1)^j\,(m+1-j)\,\omega_j\wedge\omega_{2m+2-j}\,,
\end{aligned}$$

are characteristically nilpotent.

COROLLARY. *There are characteristically nilpotent Lie algebras* $\mathfrak{g}$ *with nilindex* $2m+2$ *for any* $m\geq 3$.

Observe that the previous algebras have characteristic sequence $(2m,1,1)$. This fact is directly related with the position of the vector X_{2m+1} in the graduation of $\mathfrak{g}_{(m,m-1)}$. The joined differential form involves the form $\omega_2\wedge\omega_{2m}$, so that the nilindex of the algebra increases. Moreover, observe that we have

$$e_1\left(\mathfrak{g}_{(m,m-1)}+\psi\right)\simeq e_1\left(\mathfrak{g}_{(m,m-1)}\right)+\psi$$

so that we could have constructed the algebras extending and then deforming by taking the same deformation. This gives, in a certain manner, a procedure to generate characteristically nilpotent Lie algebras by extensions and deformations of naturally graded Lie algebras (see [10]).

9. Affine structures over Lie algebras

The origin of affine structures over Lie algebras is the tudy of affine left-invariant structures over Lie groups [9]. The question wheter any solvable Lie group admits a left invariant affine structure is a problem of great interest, as it relates geometrical aspects of affine manifold theory with representation theory of Lie algebras. Translated into Lie algebra language, the question is if any solvable Lie algebra satisfies a certain condition which is called affine structure. This goes back to Milnor in the seventies, and is therefore called

the Milnor conjeture. By the time the problem was posted, all known results referred to low dimensions, where the answer is positive. The first counterexample to Milnor' s conjecture was given by Benoist [12]. He constructed a 11-dimensional filiform Lie algebra which does not admit an affine structure. Explicitely, let $\mathfrak{a}(t)$ be the filiform Lie algebra given by

$$\begin{aligned}
&[X_1, X_i] = X_{i+1}\,, \qquad 2 \leq i \leq 10\,;\\
&[X_2, X_3] = X_5\,;\\
&[X_2, X_5] = -2X_7 + X_8 + tX_9\,;
\end{aligned}$$

over the basis $\{X_1, \ldots, X_{11}\}$. The main point is to prove that this algebra does not admit a faithful representation of degree 12, which proves the nonexistence of an affine connection [12]. This example has been widely generalized in [15]:

THEOREM 26. *There exist filiform Lie algebras of dimensions* $10 \leq n \leq 12$ *which do not admit an affine structure. For* $n \leq 9$ *any filiform Lie algebra admits an affine structure.*

For this, cohomological methods are of importance, in particular the dimensions of the cohomology spaces $H^2(\mathfrak{g}, \mathbb{C})$, which are usually called Betti numbers. Let $\mathfrak{g}$ be an n-dimensional Lie algebra and G its associated Lie group. If the group posseses a left-invariant affine structure, then this induces a flat torsionfree left-invariant affine conection ∇ on G, that is

$$\begin{aligned}
\nabla_X Y - \nabla_Y X - [X, Y] &= 0\,,\\
\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z &= 0\,,
\end{aligned}$$

for all left invariant vector fields X, Y, Z on G. Now, defining

$$X.Y = \nabla_X Y$$

we obtain a bilinear product which satisfies

$$X.(Y.Z) - (X.Y).Z - Y.(X.Z) + (Y.X).Z = 0\,.$$

Observe that this implies that the product is left symmetric.

DEFINITION. An affine structure on a Lie algebra $\mathfrak{g}$ is a bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ which is left symmetric and satisfies

$$[X, Y] = X.Y - Y.X\,, \qquad \forall\ X, Y \in \mathfrak{g}\,.$$

It is known that there exists a one-to-one correspondence between affine structures on $\mathfrak{g}$ and left invariant structures on the associated Lie group G [28]. The interesting fact is that the problem can be dealt with methods of representation theory of nilpotent Lie algebras.

PROPOSITION. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra. If $\mathfrak{g}$ admits an affine structure then $\mathfrak{g}$ possesses a faithful module M of dimension $n+1$.*

By the theorem of Ado [1], any Lie algebra admits a faithful representation. Unfortunately, the results does not say anything about the minimal degree of such a representation. Nowadays, it is accepted that the best lower bound is given in [16]. This bound, equal to $\frac{\alpha}{\sqrt{n}}2^n$ with $\alpha \sim 2,76287$, has been used to obtain other counterexamples to Milnor's conjecture [16]. In relation with the derivations structure, we have the following:

PROPOSITION. *A Lie algebra $\mathfrak{g}$ admits an affine structure if and only if there is a $\mathfrak{g}$-module M of dimension $\dim \mathfrak{g}$ such that $Z^1(\mathfrak{g}, M)$ contains a nonsingular cocycle.*

The result is a consequence of the inversibility for a nonsingular cocycle. An immediate corollary is:

COROLLARY. *If $\mathfrak{g}$ admits a nonsingular derivation, then it admits an affine structure.*

Observe in particular the importance of this for graded Lie algebras: if $\mathfrak{g}$ is naturally graded (the results remains valid for any positive indexed graded Lie algebra) then the natural operation defines a nonsingular derivation, from which we obtain that any naturally graded Lie algebra has an affine structure. As it is known that metabelian Lie algebras and those of dimensions $n \leq 6$ can be graded in such manner, all them admit an affine structure. For 3-step nilpotent Lie algebras Scheunemann [87] proved in 1974 the following:

THEOREM 27. *Any 3-step nilpotent Lie algebra $\mathfrak{g}$ admits an affine structure.*

Observe that the algebra of Dixmier and Lister is 3-step nilpotent, thus it has such a structure. Clearly all derivations are singular, which proves the existence of CNLA with affine structures. The question is which of the structural properties of CNLA allow the existence of such structures. In particular, has it any relation with the structure of the automorphism group?

For 4-step nilpotent Lie algebras the question is open, and the best result achieved can be found in [27]. However, the fundamental source (once more) for the study of affine structures is the variety $\mathcal{F}^n$ of filiform Lie algebras. In [16] the author defines the following cocycles:

DEFINITION. Let $\mathfrak{g}$ be a filiform Lie algebra. A cocycle $\omega \in Z^2(\mathfrak{g}, \mathbb{C})$ is called affine if it is nonzero over $Z(\mathfrak{g}) \wedge \mathfrak{g}$. A class $[\omega] \in H^2(\mathfrak{g}, \mathbb{C})$ is called affine if every representative is affine.

Then the next result characterizes certain extensions of filfirom Lie algebras:

PROPOSITION. *A filiform Lie algebra $\mathfrak{g}$ has a filiform extension of dimension $\dim \mathfrak{g} + 1$ if and only if there exists an affine cohomology class in $\mathfrak{g}$.*

This result has two interesting consequences:

COROLLARY. *If the filiform Lie algebra $\mathfrak{g}$ admits an affine cohomology class $[\omega]$, then it admits an affine structure.*

COROLLARY. *If $\mathfrak{g}$ is filiform of dimension $n \geq 6$ and $\dim H^2(\mathfrak{g}, \mathbb{C}) = 2$, then $\mathfrak{g}$ has no affine cohomolofgy class.*

Endowed with these methods, Burde has constructed two classes of filiform Lie algebras [17] which provide a lot of counterexamples to Milnors conjecture.

We conclude giving CNLAs which admit an affine structure but whose Lie algebra of derivations is not characteristically nilpotent: over the basis $\{X_1, \ldots, X_{11}\}$ let $\mathfrak{g}_{(a_i)}$ be the filiform Lie algebra given by

$$
\begin{aligned}
&[X_1, X_i] = X_{i+1}\,, \quad 2 \leq i \leq 10\,,\\
&[X_2, X_3] = X_5\,, && [X_3, X_6] = -\tfrac{12}{5}X_9 + a_5X_{10} + a_6X_{11}\,,\\
&[X_2, X_4] = X_6\,, && [X_3, X_7] = -\tfrac{39}{5}X_{10} + a_7X_{11}\,,\\
&[X_2, X_5] = -2X_7 + X_8\,, && [X_3, X_8] = a_8X_{11}\,,\\
&[X_2, X_6] = -5X_8 + 2X_9\,, && [X_4, X_5] = \tfrac{27}{5}X_9 + a_9X_{10} + a_{10}X_{11}\,,\\
&[X_2, X_7] = -\tfrac{13}{9}X_9 + a_1X_{10} + a_2X_{11}\,, && [X_4, X_6] = \tfrac{27}{5}X_{10} + a_9X_{10}\,,\\
&[X_2, X_8] = \tfrac{26}{5}X_{10} + a_3X_{11}\,, && [X_4, X_7] = a_{11}X_{11}\,,\\
&[X_2, X_9] = a_4X_{11}\,, && [X_5, X_6] = a_{12}X_{11}\,,\\
&[X_3, X_4] = 3X_7 - X_8\,,\\
&[X_3, X_5] = 3X_8 - X_9\,,
\end{aligned}
$$

where the following relations are satisfied:

$$\left.\begin{aligned} a_5 - 3a_1 + \tfrac{26}{5} - a_9 &= 0 \\ 3a_2 - a_3 + a_{10} - a_6 &= 0 \\ a_3 - 3a_2 - 2a_{10} &= 0 \\ a_{12} - \tfrac{27}{5}a_4 - \tfrac{54}{5} &= 0 \\ 2a_2 - a_3 + 2a_6 - a_5 - a_7 + 2 &= 0 \\ -3a_3 + a_4 + 2 + 4a_9 - \tfrac{31}{5} - a_{11} &= 0 \\ a_1 + a_5 - 2 &= 0 \\ 5a_3 - 2a_4 - 2a_8 + \tfrac{52}{2} + 5a_5 + 5a_7 - 10 &= 0 \\ a_7 + a_3 - a_1 &= 0 \\ a_4 + a_8 - \tfrac{26}{5} &= 0 \\ a_9 - 4a_5 + 6 - \tfrac{26}{5} &= 0 \end{aligned}\right\}.$$

This example, for the values

$$a_1 = \frac{51}{25}, \quad a_2 = -a_6 = a_{10} = \frac{28}{125}, \quad a_3 = \frac{28}{25}, \quad a_4 = \frac{19}{16}, \quad a_5 = -\frac{1}{25},$$
$$a_7 = \frac{23}{25}, \quad a_8 = \frac{321}{80}, \quad a_9 = -\frac{24}{25}, \quad a_{11} = -\frac{189}{16}, \quad a_{12} = \frac{1377}{80},$$

is due to Remm and Goze [83].

10. Associative characteristic nilpotent algebras

Motivated by the paper of Dixmier and Lister, in 1971 T.S. Ravisankar [81] extended the concept of being characteristically nilpotent to general algebras. This approach has been useful for the study of Malcev algebras, as for associative algebras and its deformation theory [72].

Let A be a nonassociative complex algebra (again we convene that the base field is $\mathbb{C}$, though this assumption is not generally necessary). We denote its Lie algebra of derivations by $D(A)$. Let

$$A^{[1]} = \Big\{ \sum D_i x_i \mid x_i \in A,\ D_i \in D(A) \Big\}$$

and define inductively $A^{[k+1]} = \big\{ \sum D_i y_i \mid y_i \in A^{[k]},\ D_i \in D(A) \big\}$.

Definition. An algebra A is called characteristically nilpotent (C-nilpotent) if there exists an integer n such that $A^{[n]} = 0$.

It is clear that if A is a C-nilpotent algebra, then any derivation of A is a linear nilpotent transformation on A. The converse also holds [81]. For the special case of associative algebras, in [43], let e_α be the $(r+1)^2$ matrix whose $\alpha = (i,j)$ entry is one, otherwise zero. The space generated by this vector is denoted by E_α. Let us then define

$$R = \{\alpha = (i, jt) \mid 1 \leq i, j \leq r+1\},$$
$$R^+ = \{\alpha = (i,j) \mid i < j\}.$$

It follows that $R^- = R - R^+$ and $S = \{(i, i+1) \in R \mid 1 \leq i \leq r\}$ is the set of simple roots, in analogy with the Lie algebra case [23]. Then $L = \sum_{R^+} E_\alpha$ is a nilpotent associative algebra. Consider the bilinear mappings of $L \times L \to L$ defined by

$$g_{k,m}\,(e_{\alpha_k}, e_{\alpha_m}) = e_\delta\,, \qquad \text{where } \alpha_i = (i, i+1) \text{ and } \delta = (1, r+1)\,.$$

Obviously the center of L is generated by the root δ. Let us now consider the linear combination $\psi = \sum_{1 \leq k,m \leq r} a_{k,m} g_{k,m}$ for $a_{k,m} \in \mathbb{C}$. In [43] it is proven that this is a lineraly expandable cocycle, and further that:

THEOREM 28. *Let ψ be the cocycle given by $\psi = \sum_{1 \leq k,m \leq r} a_{k,m} g_{k,m}$ with $\prod_{1 \leq i \leq r} a_{ii} \neq 0$. Then the associative algebra $L + \psi$ is characteristically nilpotent.*

Constructing families of this kind, the variety $\mathcal{N}^n$ of associative algebras can be studied as Lie algebras have been [72]. In particular, among other results the following shows the similarity between the theory of characteristically nilpotent Lie algebras and C-algebras:

THEOREM 29. *For $n \geq 2$ there exists a Zariski-open subset of $\mathcal{N}^n$ formed by characteristically nilpotent associative algebras. Moreover, its dimension is $n^2 - n$.*

REFERENCES

[1] ADO, I.D., The representation of Lie algebras by matrices, *Uspehi Mat. Nauk.* **2** (1947), 159–173.

[2] ANCOCHEA, J.M., GOZE, M., Classification des algèbres de Lie complexes de dimension 7, *Arch. Math.* **52** (1989), 175–185.

[3] ANCOCHEA, J.M., GOZE, M., On the varieties of nilpotent Lie algebras in dimension 7 and 8, *J. Pure Appl. Algebra* **77** (1992), 131–140.

[4] Ancochea, J.M., Goze, M., Classification des algèbres de Lie filiformes de dimension 8, *Arch. Math.* **50** (1988), 511 – 525.

[5] Ancochea, J.M., Goze, M., Sur la classification des algèbres de Lie nilpotentes de dimension 7, *C. R. Acad. Sci. Paris* **302** (1986), 611 – 613.

[6] Ancochea, J.M., Goze, M., Le rang du systéme lineare des racines d'une algèbre de Lie rigide resoluble complexe, *Comm. Algebra* **20** (3) (1992), 875 – 887

[7] Ancochea, J.M., Campoamor, O.R., On $(n-5)-2-$abelian Lie algebras, *Comm. Algebra*, to appear.

[8] Ancochea, J.M., Campoamor, O.R., Characteristically nilpotent Lie algebras, *Contrib. Mat.* **3** (2000), 49 – 76.

[9] Auslander, L., The structure of complete locally affine manifolds, *Topology* **3** (1964), 131 – 139.

[10] Bratzlavsky, F., Sur les algèbres admettant un tore des dérivations donné, *J. Algebra* **30** (1974), 305 – 316.

[11] Barnes, D.W., On Cartan subalgebras of Lie algebras, *Math. Z.* **101** (1967), 350 – 355.

[12] Benoist, Y., Une nilvariété non affine, *J. Diff. Geom.* **41** (1995), 21 – 52.

[13] Berman, S., On derivations of Lie algebras, *Canad. J. Math.* **27** (1976), 174 – 180.

[14] Borel, A., Mostov, G.D., On semisimple automorphisms of Lie algebras, *Ann. Math.* **61** (1955), 389 – 504.

[15] Burde, D., "Left-invariant Affine Structures on Nilpotent Lie Groups", Habilitationsschrift, Düsseldorf, 1998.

[16] Burde, D., On a refinement of Ado's theorem, *Arch. Math.* **70** (1998), 118 – 127.

[17] Burde, D., Affine structures on nilmanifolds, *Int. J. Math.* **7** (1996), 599 – 616.

[18] Carles, R., Sur le structure des algèbres de Lie rigides, *Ann. Inst. Fourier* **34** (3) (1984), 65 – 82.

[19] Carles, R., "Sur les Algèbres Caractéristiquement Nilpotentes", Publ. Univ. Poitiers, 1984.

[20] Carles, R., "Introduction aux Déformations d'Algèbres de Lie de Dimension Finie", Publ. Univ. Poitiers, 1986.

[21] Carles, R., Diakité, E.Y., Sur les variétés d'algèbres de Lie de dimension ≤ 7, *J. Algebra* **91** (1984), 53 – 63.

[22] Campoamor, R., "Álgebras de Lie Característicamente Nilpotente", Tesis Univ. Complutense, Madrid, 2000.

[23] Carter, R.W., Simple Lie groups and simple Lie algebras, *Bull. London Math. Soc.* **40** (1965), 193 – 240.

[24] Castro-Jiménez, F.J., Nuñez-Valdés, J., Gröbner basis in the classification of characteristically nilpotent filiform Lie algebras of dimension 10, *Progress in Math.* **143** (1996), 115 – 133.

[25] Chao, C.Y., Uncountably many non isomorphic nilpotent Lie algebras, *Proc. Amer. Math. Soc.* **13** (1962), 903 – 906.

[26] Chao, C.Y., Some characterisations of nilpotent Lie algebras, *Math. Z.* **103** (1968), 40 – 42.

[27] Dekimpe, K., Hartl, M., Affine structures on 4-step nilpotent Lie algebras, *J. Pure Appl. Algebra* **120** (1997), 123 – 134.

[28] Dekimpe, K., Malfait, W., Affine structures on a class of virtually nilpotent groups, *Topology Appl.* **73** (1996), 97 – 119.

[29] Dixmier, J., Lister, W.G., Derivations of nilpotent Lie algebras, *Proc. Amer. Math. Soc.* **8** (1957), 155 – 157.

[30] Dixmier, J., Cohomologie des algèbres de Lie nilpotentes, *Acta Sci. Math. Szeged* **16** (1955), 246 – 250.

[31] Dixmier, J., Sur les représentations unitaires des groupes de Lie nilpotentes III, *Canad. J. Math.* **16** (1955), 246 – 250.

[32] Dozias, J. Sur les dérivations des algèbres de Lie, *C. R. Acad. Sci. Paris* **259** (1964), 2748 – 2750.

[33] Dyer, J.L., A nilpotent Lie algebra with nilpotent automorphism group, *Bull. Amer. Math. Soc.* **76** (1970), 52 – 56.

[34] Eilenberg, S., Extensions of general algebras, *Ann. Soc. Polon. Math.* **21** (1948), 125 – 134.

[35] Favre, G., Un algèbre de Lie caractéristiquement nilpotente en dim 7, *C. R. Acad. Sci. Paris* **274** (1972), 1338 – 1339.

[36] Faure, G., Systéme des poids sur une algèbre de Lie nilpotente, *Manuscripta Math.* **9** (1973), 53 – 90.

[37] Galitski, L.Y., Timashev, D.A., On classifications of metabelian Lie algebras, *J. Lie Theory* **9** (1999), 125 – 156.

[38] Gerstenhaber, M., On the deformations of rings and algebras, *Ann. Math.* **79** (1964), 59 – 103.

[39] Gerstenhaber, M., On nilalgebras and linear varieties of nilpotent matrices III, *Duke Math. J.* **27** (1960), 21 – 32.

[40] Goze, M., Khakimdjanov, Y.B., Sur les algèbres de Lie admettant un tore des dérivations, *Manuscripta Math.* **84** (1994), 115 – 124.

[41] Goze, M., Perturbations of Lie algebras, *ASI Nato*, Serie C247 (1986), 265 – 356.

[42] Goze, M., Critéres cohomologiques pour la rigidité de lois algebriques, *Bull. Soc. Math. Belgique* **43** (1991), 33 – 42.

[43] Goze, M., Khakimdjanov, Y.B., Makhlouf, A., Sur les algèbres associatives caractéristiquement nilpotentes, *Comm. Algebra* **22** (8) (1994), 2961 – 2968.

[44] Humphreys, J.E., "Introduction to Lie Algebras and Representation Theory", GTM 9, Springer-Verlag, 1972.

[45] Ito, K., The rigidity of universal solvable Lie algebras of Iwasawa subalgebras, *Su. Kokyuroku* **875** (1994), 72 – 90.

[46] Jacobson, N., "Lie Algebras", Academic Press, 1962.

[47] Jacobson, N., Cayley numbers and normal simple Lie algebras of type G, *Duke Math. J.* **5** (1939), 775 – 783.

[48] Jacobson, N., A note on automorphisms and derivations of Lie algebras, *Proc.Amer. Math. Soc.* **6** (1955), 281 – 283.

[49] Jacobson, N., Abstract derivations and Lie algebras, *Trans. Amer. Math. Soc.* **42** (1937), 206 – 224.

[50] Johnson, R.W., Homogeneous Lie algebras and expanding automorphisms, preprint, 1974.

[51] Kantor, I.L., Graded Lie algebras, *Trudy Sem. Vect. Anal.* **15** (1970), 227 – 266.

[52] Khakimdjanov, Y.B., Variété des lois d'algèbres de Lie nilpotentes, *Geom. Dedicata* **40** (1991), 269 – 295.

[53] Khakimdjanov, Y.B., Cohomologies et deformations des certaines algèbres de Lie nilpotentes, *Comm. Algebra* **16**(10), 1988, 2149 – 2192.

[54] Khakimdjanov, Y.B., Characteristically nilpotent Lie algebras, *Math. USSR Sbornik* **70** (1) (1990), 65 – 78.

[55] Khakimdjanov, Y.B., On characteristically nilpotent Lie algebras, *Soviet Math. Dokl.* **41** (1) (1991), 105 – 107.

[56] Kirillov, A.A., Unitary representations of nilpotent Lie groups, *Russ. Math. Surv.* **17** (4) (1962), 53 – 104.

[57] Kirillov, A.A., Neretin, Y.A., The variety A_n of n-dimensional Lie algebra structures, *Amer. Math. Soc. Transl.* **137** (1987), 21 – 30.

[58] Kostant, B., Generalized Borel-Weil theorem, *Ann. Math.* **74** (1961), 329 – 387.

[59] Kostant, B., Lie algebra cohomology and generalized Schubert cells, *Ann. Math.* **77** (1963), 72 – 144.

[60] Koszul, J.L., Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France* **78** (1950), 41 – 64.

[61] Kraft, H., Riedtmann, C., "Geometry of Representations of Quivers. In Representations of Algebras", London Math. Soc. Lecture Notes **116**, Cambridge, 1986.

[62] Leger, G.F., A note on the derivations of Lie algebras. *Proc. Amer. Math. Soc.* **4** (1953), 511 – 514.

[63] Leger. G.F., Derivations of Lie algebras III, *Duke Math. J.* **30** (1963), 637 – 645.

[64] Leger, G.F., Luks, E., On a duality for metabelian Lie algebras, *J. Algebra*, **21** (1972), 266 – 270.

[65] Leger, G.F., Luks, E., On derivations and holomorphs of nilpotent Lie algebras, *Nagoya Math. J.* **44** (1971), 39 – 50.

[66] LEGER, G.F., LUKS, E., On nilpotent groups of algebra automorphisms, *Nagoya Math. J.* **46** (1972), 87 – 95.

[67] LEGER, G.F., TÔGÔ, S., Characteristically nilpotent Lie algebras, *Duke Math. J.* **26** (1959), 623 – 628.

[68] LUKS, E., "What is the Typical Nilpotent Lie Algebra, Computers in Nonassociative Rings and Algebras", Acad. Press, 1977.

[69] LUKS, E., A characteristically nilpotent Lie algebra can be a derived algebra, *Proc. Amer. Math. Soc.* **56** (1976), 42 – 44.

[70] LUTZ, R., GOZE, M., "Non Standard Analysis: a Practical Guide with Applications", LNM 881, Springer-Verlag, 1981.

[71] MAGNIN, L., Sur les algèbres de Lie nilpotentes de dimension ≤ 7 *J. Geom. Phys.* **39** (1), 119 – 144 (1986).

[72] MAKHLOUF, A., "Sur les Algébres Associatives Rigides", Thése, 1990.

[73] MALCEV, A., On solvable Lie algebras, *Izvest. Akad. Nauk. SSR, Ser Mat.* **9** (1945), 329 – 356.

[74] MILNOR, J., On fundamental groups of complete affine flat manifolds, *Advances Math.* **25** (1977), 178 – 187.

[75] MOROZOV, V.V., Classification des algèbres de Lie nilpotentes de dimension 6, *Izv. Vyssh. Ucheb. Zar.* **4** (1958), 161 – 171.

[76] MOSTOV, G.D., Fully reductible subgroups of algebraic groups, *Amer. J. Math.* **68** (1956), 200 – 221.

[77] MURRAY, F.J., Perturbation theory and Lie algebras, *J. Math. Phys.* **3** (1962), 89 – 105.

[78] NIJENHUIS, A., RICHARDSON, R.W., Cohomology and deformations in graded Lie algebras, *Bull. Amer. Math. Soc.* **72** (1966), 1 – 29.

[79] NIJENHUIS, A., RICHARDSON, R.W., Deformations of Lie algebra structures, *J. Math. Mech.* **17** (1967), 89 – 105.

[80] RAUCH, G., Effacement et déformation, *Ann. Inst. Fourier* **22** (1972), 239 – 259.

[81] RAVISANKAR, T.S., Characteristically nilpotent algebras, *Canad. J. Math.* **23** (1971), 222 – 235.

[82] REED, B., Representation of solvable Lie algebras, *Michigan Math. J.* **16** (1969), 227 – 233.

[83] REMM. E., GOZE, M., Non complete affine connections on filiform Lie algebras, available at xxx.lanl.gov/math.RA/0007067.

[84] SATO, T., The derivations of a Lie algebra, *Tohuku Math. J.* **23** (1971), 21 – 36.

[85] SCHENKMAN, E., A theory of subinvariant Lie algebras, *Amer. J. Math.* **73** (1951), 453 – 474.

[86] SCHENKMAN, E., On the derivations algebra and holomorphs of a nilpotent Lie algebra, *Mem. Amer. Math. Soc.* **14** (1955), 15 – 22.

[87] Scheuneman, J., Affine structures on three-step nilpotent Lie algebras, *Proc. Amer. Math. Soc.* **46** (1974), 451 – 454.

[88] Seeley, C., Some nilpotent Lie algebras of even dimension, *Bull. Austral. Math. Soc.* **45** (1992), 71 – 77.

[89] Segal, D., On the automorphism groups of certain Lie algebras, *Math. Proc. Camb. Phil. Soc.* **106** (1989), 67 – 76.

[90] Skjelbred, T., Sund, T., Sur la classification des algèbres de Lie nilpotentes, *C. R. Acad. Sci. Paris* **286** (1978), 241 – 242.

[91] Sund, T., Classification of filiform solvable Lie algebra, *Comm. Algebra* **22** (11) (1994), 4303 – 4359.

[92] Tits, J., "Sur les Constantes de Structure et le Theorem d'Existence des Algèbres de Lie Semi-simples", Publ. Math. I.H.E.S 31, (1966).

[93] TÔGÔ, S., Outer derivations of Lie algebras, *Trans. Amer. Math. Soc.* **128** (1967), 264 – 276.

[94] TÔGÔ, S., On the derivation algebras of Lie algebras, *Canad. J. Math.* **13** (2) (1961), 201 – 216.

[95] Umlauf, K., "Ueber die Zusammensetzung der Endlichen Continuirlichen Transformationsgruppen, Insbesondere der Gruppen vom Range Null", Doctorat, Leipzig, 1891.

[96] Vergne, M., "Variété des Algèbres de Lie Nilpotentes", These 3^{eme} cycle, Paris, 1966.

[97] Vergne, M., Cohomologie des algèbres de Lie nilpotentes. Applications a l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. Math. France* **98** (1970), 81 – 116.

[98] Vranceaunu, G., "Leçons de Géométrie Différentielle, Vol 4", Bucarest, 1975.

[99] Weisfeiler, B.Ju., Infinite dimensional filtered Lie algebras and their connection with graded Lie algebras, *Funct. Anal. Appl.* **2** (1968), 88 – 89.

[100] Yamaguchi, S., Derivations and affine structures of some nilpotent Lie algebras, *Mem. Fac. Sci. Kyushu Univ.*, Ser. A, **34** (1980), 151 – 170.

[101] Yamaguchi, S., On some classes of nilpotent Lie algebras and their automorphism group, *Mem. Fac. Sci. Kyushu Univ.*, Ser. A, **34** (1981), 241 – 251.