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The Banach Space c_0

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I. Introduction

We survey in these notes some recent progress on the understanding of the Banach space c_0 an of its subspaces. We did not try to complete the (quite ambitious) task of writing a comprehensive survey. A few topics have been selected, in order to display the variety of techniques which are required in such investigations. It is hoped that this selection can provide the reader with some intuition about the behaviour of the space. Our choice has also been influenced by the open problems which conclude these notes. It is our hope that some of these problems are not desperately hard.

The unit ball of c_0 can be (vaguely) visualized as a cubic box with no vertices; somehow, only the internal parts of the sides remain. Cubic boxes are suitable for packing, and indeed it is easy to map a space to c_0 (see § II). A cube is a polyhedron, and indeed the polyhedral nature of c_0 and its subspaces inform us of some of their isometric properties (see § III). The space c_0 contains "very few" compact sets, unlike spaces which do not contain c_0 and retain some features of reflexive spaces (see § IV). The natural norm of c_0 is flat at every point in the direction of a finite-codimensional space and c_0 is the largest space with this property. This helps to prove that the unit ball of c_0 is a sturdy box, whose shape cannot be altered by non-linear Lipschitz maps (see § V). Finally, vectors in c_0 can easily be approximated by restricting their support to given finite subsets. Subspaces of c_0 are in general not stable under this operation. However, smoothness properties of these subspaces sometimes help to build a decent approximation scheme (see § VI). A sample of open problems is displayed in the last section VII.

Notation We denote by c_0 the separable Banach space of all real sequences (u_n) such that $\lim_{n\to\infty}(u_n) = 0$. This space is equipped with its canonical norm

$$\|(u_n)\|_\infty = \sup\big\{|u_n|;\, n \geq 1\big\}.$$

Capital letters such as X, Y, ... will denote Banach spaces. By "subspace", we mean "closed subspace". We denote by e_n^* the linear form defined on c_0 by

$$e_n^*((u_k)) = u_n.$$

Of course, $e_n^* \in c_0^* = \ell^1$. We denote by $NA(X)$ the subset of the dual space X^* consisting of norm-attaining linear functionals. We otherwise follow the classical notation as it can be found e.g. in [32].

II. c_0 is an easy target

It is quite easy to give a usable representation for operators T from a Banach space X to c_0. Indeed, if we let

$$T^*(e_n^*) = x_n^* \in X^*$$

then clearly

$$T(x) = (x_n^*(x))_{n\geq 1} \tag{1}$$

and the sequence (x_n^*) is w^*-convergent to 0 in X^*. Conversely, if (x_n^*) is a w^*-null sequence in X^*, then (1) defines an operator from X to c_0.

This very simple representation provides us with a bunch of operators from X into c_0, at least when the space X is separable since in this case, bounded subsets of X^* are w^*-metrizable and thus convergent sequences are easily constructed.

This simple idea lies in the background of an old but important result on c_0: Sobczyk's theorem.

Theorem II.1. ([35]) *Let X be a separable Banach space, and Y a subspace of X. Let $T : Y \to c_0$ be a bounded operator. There exists $\widetilde{T} : X \to c_0$ such that*

(i) $\widetilde{T}_{|Y} = T$.

(ii) $\|\widetilde{T}\| \leq 2\|T\|$.

Proof. ([36]) Let $y_n^* = T^*(e_n^*)$. Clearly $\|T\| = \sup_n \|y_n^*\|$. By Hahn–Banach theorem, there exist $x_n^* \in X^*$ such that $\|x_n^*\| \leq \|T\|$ and $x^*_{n|Y} = y_n^*$ for all $n \geq 1$. We set

$$K = \{x^* \in X^*;\ \|x^*\| \leq \|T\|\}$$

and $L = K \cap Y^\perp$. When equipped with the w^*-topology, K is metrizable compact. Let d be a distance which defines the weak-star topology on K.

Since $\lim x_n^*(y) = \lim y_n^*(y) = 0$ for every $y \in Y$, every w^*-cluster point to the sequence (x_n^*) belongs to L. It follows by compactness that the distance from (x_n^*) to L in (K, d) tends to 0, hence we can pick $t_n^* \in L$ such that

$$w^*\text{-}\lim(x_n^* - t_n^*) = 0.$$

If we now define $\widetilde{T} : X \to c_0$ by

$$\widetilde{T}(x) = \big((x_n^* - t_n^*)(x)\big)_{n \geq 1}$$

it is easily seen that $\widetilde{T}$ works. Moreover

$$\|\widetilde{T}\| = \sup_n \|x_n^* - t_n^*\| \leq \sup_n \big(\|x_n^*\| + \|t_n^*\|\big) \leq 2\|T\|.$$

∎

Let us spell out two important applications.

Corollary II.2. *Let X be a separable Banach space, and let Y be a subspace which is isomorphic to c_0. Then Y is complemented in X.*

Proof. Let $T : Y \to c_0$ be an isomorphism onto c_0. In the notation of Theorem II.1, $T^{-1} \circ \widetilde{T} : X \to Y$ is a projection. ∎

Corollary II.3. *Let X be a separable Banach space which has a subspace Y such that both spaces Y and X/Y are isomorphic to subspaces of c_0. Then X is isomorphic to a subspace of c_0.*

Proof. Let $T : Y \to c_0$ and $S : X/Y \to c_0$ be isomorphisms onto subspaces of c_0. We denote again $\widetilde{T} : X \to c_0$ an operator extending T, and $Q : X \to X/Y$ the canonical map. It is easily checked that $V : X \to c_0 \oplus c_0 \simeq c_0$ defined by $V(x) = \big(\widetilde{T}(x),\, S \circ Q(x)\big)$ is an isomorphism of X into c_0. ∎

Remark II.4. Sobczyk's theorem, and the above corollaries, fail in the nonseparable case. Indeed we can take $X = \ell_\infty$ in corollary II.2 and c_0 is not complemented in ℓ_∞. There is a compact set K [9, Example VI.8.7.], such that $C(K) = X$ contains a subspace Y isomorphic to $c_0(\mathbb{N})$ such that X/Y is isomorphic to $c_0(\Gamma)$, with $|\Gamma| = c$, but X is not a subspace of $c_0(\Gamma)$ since X is not weakly compactly generated.

Although it is harder to map a nonseparable Banach space into a $c_0(\Gamma)$ space, such mappings play a fundamental role in the study of "nice" nonseparable spaces. We refer to the last three chapters of [9] for this topic.

III. Some isometric properties

It is easy to characterize spaces which are isometric to subspaces of c_0 by applying the formula (1) from § II to an isometric embedding.

Proposition III.1. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is isometric to a subspace of c_0.*

(ii) *There exists a sequence of (x_n^*) in X^* such that w^*-$\lim_n(x_n^*) = 0$ and for all $x \in X$,*

$$\|x\| = \sup\left\{ |x_n^*(x)|;\, n \geq 1\right\}.$$

The proof follows immediately from the consideration of

$$T(x) = \big(x_n^*(x)\big)_{n\geq 1}$$

which is an isometric embedding into c_0 if and only if (x_n^*) satisfies (ii). This simple proposition has some interesting consequences. We first need a definition.

Definition III.2. Let X be a Banach space, and $g : X \to \mathbb{R}$ a continuous function. We say that g locally depends upon finitely many coordinates if for every $x \in X$, there exist $\varepsilon > 0$, a finite subset $\{f_1, f_2, \dots, f_n\}$ of X^* and a continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $g(y) = \varphi(f_1(y), f_2(y), \dots, f_n(y))$ for every y such that $\|x - y\| < \varepsilon$.

When $g : X \to \mathbb{R}$ is a norm, we say that it locally depends upon finitely many coordinates if the above condition holds for every $x \in X \setminus \{0\}$. We note that when this happens, φ can be chosen to be a norm on $\mathbb{R}^n$. Indeed, let $T : X \to \mathbb{R}^n$ be defined by

$$T(y) = \big(f_i(y)\big)_{1 \le i \le n}$$

and pick any $y \in S_X$ with $\|x - y\| < \varepsilon$. For every $u \in \operatorname{Ker} T$, the function

$$\lambda \longrightarrow \|y + \lambda u\|$$

is convex, and ≥ 1 on some neighbourhood of 0, hence ≥ 1 on $\mathbb{R}$ and we have

$$\|y\|_{X/\operatorname{Ker} T} = 1.$$

It follows that we can take $\varphi(\cdot) = \|\cdot\|_{X/\operatorname{Ker} T}$.

Proposition III.3. *Let X be a subspace of c_0, and let $\|\cdot\|$ be the restriction of $\|\cdot\|_\infty$ to X. Then, for any $x \in X \setminus \{0\}$, there are $\delta > 0$ and a finite subset $(x_n^*)_{n \in J}$ of X^* such that if $\|x - y\| < \delta$, then*

$$\|y\| = \sup\big\{|x_n^*(y)|;\ n \in J\big\}.$$

In particular, $\|\cdot\|$ depends locally of finitely many coordinates. Moreover, X is polyhedral; that is, the unit ball of every finite dimensional subspace of X has finitely many extreme points.

Proof. Pick $x \in X \setminus \{0\}$. We have

$$\|x\| = \sup_n |e_n^*(x)|$$

and $\lim_n e_n^*(x) = 0$. Hence if

$$I = \{n \ge 1;\ |e_n^*(x)| < \|x\|\}$$

one has

$$\|x\| - \sup\big\{|e_n^*(x)|;\ n \in I\big\} = \varepsilon > 0$$

and the set $J = \mathbb{N} \setminus I$ is finite.

If $y \in X$ is such that $\|x - y\| < \varepsilon/3$, one clearly has

$$\|y\| = \sup\big\{|e_n^*(y)|;\ y \in J\big\}$$

and this show our first assertion. Note that this assertion really means that the norm is locally linear when J is a singleton.

For showing polyhedrality, we pick $E \subseteq X$ a finite dimensional subspace. Let $S_E = \{x \in E;\ \|x\| = 1\}$. For all $x \in S_E$, there is $\varepsilon(x) > 0$ and $F(x) \subseteq X^*$ a finite subset such that $\|y - x\| < \varepsilon(x)$ implies

$$\|y\| = \sup\big\{|x^*(y)|;\ x^* \in F(x)\big\}.$$

Since S_E is compact, there are $x_1, \dots, x_n$ in S_E such that

$$S_E \subseteq \bigcup_{i=1}^{n} B\big(x_i, \varepsilon(x_i)\big).$$

If $F = \cup_{i=1}^n F(x_i)$, F is a finite subset of X^* such that

$$\|x\| = \sup\big\{|x^*(x)|;\ x^* \in F\big\}$$

for all $x \in E$, hence E is isometric to a subspace of $(\mathbb{R}^{|F|}, \|\cdot\|_\infty)$ and polyhedrality follows. ∎

Subspaces of c_0 are typical examples of polyhedral spaces, but they are far from exhausting the class of polyhedral spaces. We refer to [13] for an up-to-date survey of this theory.

We shall use these notions for investigating nearest points to subspaces in the class of subspaces of c_0. We recall that a subspace Y of a Banach space X is called *proximinal* if for every $x \in X$, there exist $y_0 \in Y$ such that

$$\|x - y_0\| = \inf\big\{\|x - y\|;\ y \in Y\big\}.$$

If for instance $Y = \mathrm{Ker}(x^*)$ is an hyperplane, Y is proximinal if and only if $x^* \in NA(X)$, where $NA(X)$ denotes the subset of X^* consisting of norm-attaining functionals. That is, $x^* \in NA(X)$ when

$$\sup\big\{|x^*(x)|;\ \|x\| \le 1\big\}$$

is attained.

This observation has an easy useful generalization, namely:

PROPOSITION III.4. ([15]) *Let X be a Banach space, and Y a subspace of X of finite codimension. If Y is proximinal in X, then $Y^\perp$ is contained into $NA(X)$.*

Proof. Let $Q : X \to X/Y$ be the quotient map. Clearly Y is proximinal if and only if $Q(B_X) = B_{X/Y}$. Since X/Y is finite dimensional, every linear form on (X/Y) attains its norm. The result follows since Q^* is an isometry from $(X/Y)^*$ onto $Y^\perp$. ∎

Proposition III.4 can be reformulated as follows: if $Y \subseteq X$ is a proximinal subspace of finite codimension, every hyperplane H such that $Y \subseteq H \subseteq X$ is proximinal. The converse does not hold in general [34]. We will show however that it holds in subspaces of c_0. Indeed one has:

THEOREM III.5. ([18]) *Let X be a subspace of c_0. Let Y be a finite codimensional subspace of X. Then Y is proximinal in X if and only if the orthogonal $Y^\perp$ of Y in X^* is contained in $NA(X)$.*

Proof. We have to show that Y is proximinal if $Y^\perp$ is contained in $NA(X)$. This is obtained by combining the following facts.

LEMMA III.6. *Let Y be a finite codimensional subspace of a Banach space X such that $Y^\perp$ is contained in $NA(X)$ and $Y^\perp$ is polyhedral. Then Y is proximinal.*

Indeed $X/Y = (Y^\perp)^*$ is polyhedral as well and thus every extreme point e of $B(Y^\perp)^*$ is in fact exposed. That is, there is $x^* \in Y^\perp$ with $\|x^*\| = 1$ and

$$\{t \in (Y^\perp)^*;\, t(x^*) = 1\} = \{e\}. \tag{2}$$

But since $x^* \in NA(X)$, there is $x \in X$ with $x^*(x) = \|x\| = 1$. If we denote by Q the quotient map from X onto $X/Y = (Y^\perp)^*$, it follows from (2) that $Q(x) = e$. Hence

$$Q(B_X) \supseteq \mathrm{Ext}(B_{X/Y})$$

and thus by the Krein–Milman theorem

$$Q(B_X) = B_{X/Y}$$

which means that Y is proximinal.

LEMMA III.7. *Let X be a subspace of c_0. For any $x^* \in S_{X^*}$, we denote*

$$J(x^*) = \{x^{**} \in X^{**};\, x^{**}(x^*) = \|x^{**}\| = 1\}.$$

If $x^* \in NA(X)$, there exists a w^*-neighbourhood V of x^* such that if $y^* \in V \cap S_{X^*}$, then $J(y^*) \subseteq J(x^*)$.

Indeed, let $j : X \to c_0$ be the canonical injection, and $j^* : \ell_1 \to X^*$ be the corresponding quotient map. We work by contradiction: if the conclusion fails, there is a sequence $(x_n^*) \subseteq S_{X^*}$ such that $w^*\text{-}\lim(x_n^*) = x^*$ and

$$J(x_n^*) \not\subset J(x^*) \tag{3}$$

for all n. Pick $y_n^* \in \ell_1$ with $\|y_n^*\| = 1$ and $j^*(y_n^*) = x_n^*$. If we let for $z^* \in S_{\ell_1}$

$$\tilde{J}(z^*) = \{t \in \ell_\infty;\ \|t\| = 1 = t(z^*)\}$$

we formally have when $\|j^*(z^*)\| = 1$

$$j^{**}\big(J(j^*(z^*))\big) = j^{**}(X^{**}) \cap \tilde{J}(z^*). \tag{4}$$

We may and do assume that the sequence (y_n^*) is w^*-convergent in B_{ℓ_1} to y^*. Clearly $j^*(y^*) = x^*$, and $y^*(x) = \|y^*\| = 1$. Hence, $y^* \in NA(c_0)$ and it follows that the set

$$\operatorname{supp}(y^*) = \{k \geq 1;\ y^*(e_k) \neq 0\}$$

is finite. It follows from (3) and (4) that one has

$$\tilde{J}(y_n^*) \not\subset \tilde{J}(y^*)$$

for all n. On the other hand, it is clear that

$$\tilde{J}(y^*) = \{t \in S_{\ell_\infty};\ t(e_k^*) = \operatorname{sign}(y^*(e_k)) \text{ for all } k \in \operatorname{supp}(y^*)\}$$

with a similar formula for $\tilde{J}(y_n^*)$. Now since $w^*\text{-}\lim(y_n^*) = y^*$, it follows that for n large enough, one has for all $k \in \operatorname{supp}(y^*)$ that

$$\operatorname{sign}(y_n^*(k)) = \operatorname{sign}(y^*(k))$$

and thus $\tilde{J}(y_n^*) \subset \tilde{J}(y^*)$, which is a contradiction.

To conclude the proof of Theorem III.5 it suffices by Lemma III.6 to show that $Y^\perp$ is polyhedral. Since it is contained in $NA(X)$, we may apply Lemma III.7 to every $x^* \in Y^\perp$. By compactness, we find $\{x_i^*;\ i \leq n\}$ in $S_{Y^\perp}$ and w^*-open neighbourhoods V_i of x_i^* such that

$$S_{Y^\perp} \subseteq \bigcup_{i=1}^{n} V_i$$

and

$$J(y^*) \subseteq J(x_i^*)$$

if $y^* \in V_i$. Therefore,

$$\bigcup \{J(y^*);\ y^* \in S_{Y^\perp}\} = \bigcup_{i=1}^{n} J(x_i^*)$$

it follows that for any $t \in (Y^\perp)^*$,

$$\|t\| = \sup \left\{ |t(x_i^*)|;\ 1 \leq i \leq n \right\}$$

hence $(Y^\perp)^*$ is polyhedral and so is $Y^\perp$. ∎

Remark III.8. By using sharper analytical tools, one can show [19] that under the assumptions of Theorem III.5, the proximinal subspace Y is in fact strongly proximinal, in the following sense. For every $x \in X$, denote $d(x, Y) = \inf\{\|x-y\|;\ y \in Y\}$ and $P_Y(x) = \{y \in Y;\ \|x-y\| = d(x,Y)\}$. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that if $y \in Y$ and $\|x - y\| < d(x, Y) + \delta$, then there is $y' \in P_Y(x)$ such that $\|y - y'\| < \varepsilon$. In other words, if y is almost a nearest point to x, then y is close to a nearest point. Another result of [19] is that under the assumption of Theorem III.5, the multivalued map $P_Y(\cdot)$ has a continuous selection.

The conclusion of Lemma III.7 and the above assertions, are satisfied for a collection of spaces which is quite larger that the class of subspaces of c_0. However, it provides strong restrictions on the isometric structure. We now provide a proof of a result from [12]. If $x^* \in X^*$, we denote

$$J_X(x^*) = \{x \in S_X;\ x^*(x) = \|x^*\|\}.$$

We recall that a boundary is a subset B of S_{X^*} on which every $x \in X$ *attains* its norm. With this notation, the following holds:

Proposition III.9. *Let X be a separable Banach space such that for all $x^* \in S_{X^*} \cap NA$, there is a w^*-open neighbourhood $V_*(x^*)$ of x^* such that $y^* \in V_*(x^*) \cap S_{X^*}$ implies $J_X(y^*) \subseteq J_X(x^*)$. Then there is a boundary $B \subseteq S_{X^*}$ such that no w^*-accumulation point of B belongs to $S_{X^*} \cap NA$.*

Proof. For $x \in X$, we denote

$$J_{X^*}(x) = \{x^* \in S_{X^*};\ x^*(x) = \|x\|\}.$$

For all $x \in S_X$, the set

$$V_x = \{y \in S_X;\ J_{X^*}(y) \cap J_{X^*}(x) \neq \emptyset\}$$

is open in $(S_X, \|\cdot\|)$. Indeed if $\lim \|y_n - x\| = 0$ and $x_n^* \in J_{X^*}(y_n)$, we may and do assume that

$$w^*\text{-}\lim(x_n^*) = x^* \in J_{X^*}(x).$$

By our assumption, $J_X(x_n^*) \subseteq J_X(x^*)$ if $n \geq N$, hence $x^*(y_n) = 1$ if $n \geq N$, and thus $J_{X^*}(y_n) \cap J_{X^*}(x) \neq \emptyset$ if $n \geq N$.

Since $(S_X, \|\cdot\|)$ is a separable metric space, by paracompactness and the Lindelöf property, there is a sequence $(x_n) \subseteq S_X$ such that:

(i) $S_X = \cup_{n\geq 1} V_{x_n}$.

(ii) For all $x \in S_X$, there is $U \ni x$ open such that $\{n;\ V_{x_n} \cap U \neq \emptyset\}$ is finite.

If for $x \in S_X$, we denote

$$E_x = \{n \geq 1; J_{X^*}(x) \cap J_{X^*}(x_n) \neq \emptyset\}$$

then by (i) and (ii), E_n is a non-empty and finite set.

For all n, $J_{X^*}(x_n)$ is a w^*-compact subset of $NA \cap S_{X^*}$, hence there is a finite subset $F_n \subseteq J_{X^*}(x_n)$ such that

$$J_{X^*}(x_n) \subseteq \bigcup \{V_*(x^*);\ x^* \in F_n\}.$$

We let

$$B = \bigcup_{n\geq 1} F_n.$$

The set B is a boundary. Indeed, pick $x \in S_x$. There is n_0 such that $x \in V_{x_{n_0}}$; pick

$$x^* \in J_{X^*}(x) \cap J_{X^*}(x_{n_0}).$$

There is $b^* \in F_{n_0} \subseteq B$ such that $x^* \in V_*(b^*)$. We have $J_X(x^*) \subseteq J_X(b^*)$, hence $b^*(x) = 1$.

Finally, let $(b_k^*) \subseteq B$ and

$$w^*\text{-}\lim_{k\to\infty} b_k^* = x^*$$

with $x^* \neq b_k^*$ for all k. We have to show that $x^* \notin S_{X^*} \cap NA$.

If not, let $x \in S_X$ be such that $x^* \in J_{X^*}(x)$. For $k \geq K$, one has

$$J_X(b_k^*) \subseteq J_X(x^*).$$

Let (n_k) be such that $b_k^* \in J_{X^*}(x_{n_k})$. We note that for any $y \in S_X$

$$J_{X^*}(y) \cap B \subseteq \bigcup_{n \in E_y} F_n$$

and thus $J_{X^*}(y) \cap B$ is finite. It follows that we may and do assume that the points (x_{n_k}) are all distinct.

If $k \geq K$, we have

$$x_{n_k} \in J_X(b_k^*) \subseteq J_X(x^*)$$

hence $x^*(x_{n_k}) = x^*(x) = 1$ and thus

$$J_{X^*}(x_{n_k}) \cap J_{X^*}(x) \neq \emptyset.$$

This means that $x \in V_{n_k}$ for all $k \geq K$. But this cannot be since E_x is finite. ∎

It should be mentioned that the converse of Proposition III.9 holds true as well [12]. That is, the existence of such a boundary B implies the existence of weak* open neighbourhoods $V_*(x^*)$ satisfying the assumptions of Proposition III.9. We refer to [19, Examples III.5] for applications of this condition of proximinality results. We note that the conditions of Proposition III.9 are hereditary; every polyhedral isometric predual of ℓ^1 satisfies them.

IV. The space c_0 is minimal among non-reflexive spaces which are smooth enough

In contrast with the previous section, we now consider isomorphic properties, which are however related with the existence of special norms. We will show in particular that c_0 is minimal among the spaces whose norm locally depends upon finitely many coordinates. Let us first prove that such spaces are Asplund spaces.

Proposition IV.1. *Let X be a separable space whose norm $\|\cdot\|$ locally depends upon finitely many coordinates. Then X^* is separable.*

Proof. For every $x \in S_X$, there are $\varepsilon(x) > 0$ and a linear operator

$$T_x : X \longrightarrow (\mathbb{R}^{n(x)}, \|\cdot\|_x)$$

such that if $\|y - x\| < \varepsilon(x)$, then

$$\|y\| = \|T_x(y)\|_x$$

(see the comments after Definition III.2). By the Lindelöf property, there is a sequence $\{x_n;\, n \geq 1\}$ such that

$$S_X \subseteq \bigcup_{n \geq 1} B\big(x_n, \varepsilon(x_n)\big).$$

Let now S_n be the unit sphere of the space $(\mathbb{R}^{n(x)}, \|\cdot\|_x)^*$, and $K_n = T_x^*(S_n)$. It is clear that K_n is norm-compact, hence

$$B = \bigcup_{n \geq 1} K_n$$

is a norm-separable subset of S_{X^*}. For all $y \in S_X$, there is $x^* \in B$ such that $x^*(y) = 1$, hence B is a separable boundary. Now [16, Th. III.3] shows that X^* is separable. ∎

Remark IV.2. An alternative proof of Proposition IV.1 consists into observing that since $\|\cdot\|_x$ is strongly sub-differentiable (S.S.D.) for all $x \in S_X$ by Dini's theorem, the norm of X is S.S.D. as well. Now, every Banach space which has a S.S.D. norm is an Asplund space.

If a space does not contain $c_0(\mathbb{N})$, its supply of compact subsets allows to show a compact variational principle ([8]; see [9, Th. V.2.2]) which we state below.

Theorem IV.3. *Let X be a Banach space not containing $c_0(\mathbb{N})$. let $U \ni 0$ be a bounded open symmetric subset of X. let $f : \overline{U} \to \mathbb{R}$ be a continuous function such that:*

(i) *$f(x) > 0$ for all $x \in \partial U$,*

(ii) *$f(-x) = f(x)$,*

(iii) *$f(0) \leq 0$,*

then there is a compact subset K of U, and $V \ni 0$ open such that $K+V \subseteq U$, and such that if we let

$$f_K(x) = \sup\{f(x+h);\ h \in K\}$$

then $f_K(0) = 0$ and $f_K(x) > 0$ if $x \in V \setminus \{0\}$.

Proof. By a result of Bessaga–Pelczynski [4], if $X \not\supset c_0(\mathbb{N})$ and $(x_i)_{i\in\mathbb{N}}$ is such that

$$\sup\left\{\|\sum \varepsilon_i x_i\|;\ (\varepsilon_i) \in \{-1,1\}^{[\mathbb{N}]}\right\} < \infty$$

then the series $(\sum x_i)$ is unconditionally convergent, and thus the set

$$E = \left\{\sum \varepsilon_i x_i;\ (\varepsilon_i) \in \{-1,1\}^{[\mathbb{N}]}\right\}$$

is $\|\cdot\|$-relatively compact.

Let $x_0 = 0 \in U$. If $x_0, \cdots, x_n$ have been chosen, let

$$K_n = \left\{\sum_{i=0}^{n} \varepsilon_i x_i;\ \varepsilon_i \in \{-1,1\}\right\}$$

and

$$E_n = \left\{x \in X;\ \text{ for all } k \in K_n,\ (x+k) \in U \text{ and } f(x+k) \leq 0\right\}.$$

Denote

$$\alpha_n = \sup\left\{\|x\|;\ x \in E_n\right\}$$

and choose $x_{n+1} \in E_n$ such that

$$\|x_{n+1}\| \geq \frac{\alpha_n}{2}.$$

Note that the construction can be continued since $0 \in E_n$ for all n. Let now

$$K = \overline{\cup K_n}.$$

Since $K \subseteq \overline{U}$, it is bounded, and Bessaga–Pelczynski's result shows that K is compact and that $\lim(\alpha_n) = 0$.

We observe now that $f \leq 0$ on K, hence $K \cap \partial U = \emptyset$ and $f_K(0) \leq 0$. Let V be a neighbourhood of 0 such that $K + V \subseteq U$. If $x \in V \setminus \{0\}$, there is $n \geq 1$ such that $\|x\| > \alpha_n$, and thus $f_K(x) > 0$; it also follows that $f_K(0) = 0$. ∎

The unit ball of $X \not\supset c_0(\mathbb{N})$ may fail to contain extreme points; however, it contains "extreme compact sets". Indeed:

Corollary IV.4. *If X does not contain c_0, there is a compact subset K of B_X such that $(K+h) \not\subset B_X$ for all $h \neq 0$.*

Proof. Let $U = \{x \in X;\ \|x\| < 2\}$. We denote

$$f(x) = \operatorname{dist}(x, B_X) = \|x\| - 1.$$

Theorem IV.3 provides us with a compact symmetric set K such that $f_K(0) = 0$ (hence $K \subseteq B_X$) and $f_K(h) > 0$ (hence $(K+h) \not\subset B_X$) if $0 < \|h\| < \delta$ for some $\delta > 0$. Clearly, this set K works. ∎

Note that if K is a compact subset of c_0, one has

$$\lim_{n\to\infty} \big(\sup \big\{|x(i)|;\ x \in K,\ i \geq n\big\}\big) = 0.$$

Hence, c_0 equipped with its natural norm does not satisfy the conclusion of Corollary IV.4, which says that c_0 is minimal for this property. It follows that it is also minimal for local dependence upon finitely many coordinates.

Corollary IV.5. *If the norm of X locally depends upon finitely many coordinates and $\dim X = \infty$, then X contains c_0.*

Proof. If not, pick $K \subseteq B_K$ satisfying the conclusion of Corollary IV.4. We may and do assume that $0 \notin K$. For all $x \in K$, there is $\varepsilon(x) > 0$ such that if $\|x - y\| < \varepsilon(x)$, then

$$\|y\| = \varphi_x(T_x(y))$$

where T_x is a finite rank map. By compactness, we have

$$K \subseteq \bigcup_{i=1}^{n} B\big(x_i, \varepsilon(x_i)\big).$$

Since $\dim X = \infty$, the space $H = \cap_{i=1}^{n} \operatorname{Ker}(T_{x_i})$ is not reduced to $\{0\}$. Hence we can pick $h \in H$ with $0 < \|h\| < \varepsilon = \inf_{1 \leq i \leq n} \varepsilon(x_i)$. Clearly, $(K+h) \subseteq B_X$, contradicting our choice of K. ∎

The compact variational principle (Theorem IV.3) can be used to provide some uniform continuity in spaces which do not contain c_0. For instance, if $X \not\supset c_0$, and if there is a C^k-smooth bump function $b_0 : X \to \mathbb{R}$ ($k \geq 1$), then there is a C^k-smooth bump function b such that $b^{(k-1)}$ is *uniformly* continuous (see [9, Th. V.3.2]). In particular, if $X \not\supset c_0$ and X has a C^2-smooth bump, then X is superreflexive of type 2 (see [9, Cor. V.3.3]).

Note that c_0 has a C^∞-smooth bump which locally depends upon finitely many coordinates (see [9, Chapter V]). Indeed, pick $f : \mathbb{R} \to \mathbb{R}$ a C^∞-smooth function such that $f(t) = 1$ if $|t| \leq 1$ and $f(t) = 0$ if $|t| \geq 2$. If $x = (x(i)) \in c_0$, we let

$$b(x) = \prod_{i=1}^{\infty} f(x(i)).$$

Note that for this bump function b, like for any differentiable bump function on c_0, the derivable b' is not uniformly continuous.

V. The space c_0 is maximal among asymptotically flat spaces

The following "modulus of asymptotic smoothness" has been defined in [33]. If $x \in S_X$, $\tau > 0$ and Y is a linear subspace of X, define

$$\rho(x, \tau, Y) = \sup\left\{\|x + \tau y\| - 1;\ y \in S_Y\right\}$$

and then

$$\rho(x, \tau) = \inf\left\{\rho(x, \tau, Y);\ \dim X/Y < \infty\right\}$$

and finally

$$\rho_X(\tau) = \sup\left\{\rho(x, \tau);\ x \in S_X\right\}.$$

In other words, $\rho_X(\tau)$ measures the uniform smoothness of the norm of X, when one is allowed to "neglect" at every point $x \in S_X$ a well-chosen finite dimensional space.

Following [29], one says that X is asymptotically uniformly smooth if

$$\lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} = 0.$$

Let us say that X is asymptotically uniformly flat if there is $\tau_0 > 0$ such that $\rho_X(\tau_0) = 0$ (or equivalently, $\rho_X(\tau) = 0$ for all $\tau \in [0, \tau_0]$).

It is easily seen that when $(X = c_0, \|\cdot\|_\infty)$, then $\rho_X(1) = 0$. But it turns out that c_0 is the largest space with this property.

Theorem V.1. *Let X be an asymptotically uniformly flat Banach space. Then X is isomorphic to a subspace of c_0.*

Proof. Let $x \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Let (u_n^*) be a sequence in X^* with w^*-$\lim(u_n^*) = 0$ and $\lim \|x^* + u_n^*\| = \alpha > 0$. Pick any $\varepsilon > 0$. Since X is asymptotically uniformly flat, there is $\tau_0 > 0$ independent of x such that we can find $Y \subset \mathrm{Ker}(x^*)$ finite codimensional, such that for all $y \in S_Y$,

$$\|x + \tau_0 y\| \leq 1 + \varepsilon\tau_0.$$

Since w^*-$\lim(u_n^*) = 0$, there exists by [20, Lemma 2.5] a sequence $(y_n) \subset S_Y$ with w-$\lim(y_n) = 0$ such that

$$\underline{\lim}\langle u_n^*, y_n\rangle \geq \frac{\alpha}{2}$$

and thus we have

$$\underline{\lim}\langle x^* + u_n^*, x + \tau_0 y_n\rangle \geq 1 + \tau_0\frac{\alpha}{2}.$$

It follows that

$$\underline{\lim}\, \|x^* + u_n^*\| \geq \frac{1 + \tau_0\frac{\alpha}{2}}{1 + \tau_0\varepsilon},$$

hence since $\varepsilon > 0$ is arbitrary

$$\underline{\lim}\, \|x^* + u_n^*\| \geq 1 + \tau_0\frac{\alpha}{2}. \tag{5}$$

Summarizing, we have shown (5) for all w^*-null sequences (u_n^*) with $\lim \|u_n^*\| = \alpha$, provided that x^* is norm-attaining. Since such functionals are norm-dense by the Bishop–Phelps theorem, (5) holds for every $x^* \in S_{X^*}$.

In the terminology of [21], (5) means that the norm of X is “Lipschitz w^*-Kadec–Klee”. Now [21, Th. 2.4] shows that X is isomorphic to a subspace of c_0. ∎

It should be noted that the proof of [21, Th. 2.4], which is quite technical, can be made simpler if one assumes that X has a shrinking F.D.D., in which case it boils down to a skipped blocking argument. Now if E is an arbitrary space with E^* separable, there is $Y \subseteq E$ such that Y and (E/Y) both have a shrinking F.D.D. To conclude the proof, it suffices to show that “being a subspace of c_0” is a 3-space property (see Cor II.3 above). This approach is followed in [29] (see also [17, Th. V.7]).

A motivation for Theorem V.1 is the following non-linear result: if $(X, \|\cdot\|_X)$ is asymptotically uniformly flat and $U : X \to Y$ is a bijective map such that U and U^{-1} are both Lipschitz maps, then the formula

$$|||y^*|||_{Y^*} = \sup\left\{\frac{|y^*(Ux - Ux')|}{\|x - x'\|};\ (x, x') \in X^2,\ x \neq x'\right\}$$

defines an equivalent dual norm on Y^* whose predual norm is asymptotically uniformly flat. It follows from this (non trivial) fact and Theorem V.1 that the class of subspaces of c_0 is stable under Lipschitz isomorphisms, and in particular that a space which is Lipschitz isomorphic to c_0 is linearly isomorphic to c_0 [21].

It should be noted that "asymptotic uniform flatness" is in fact equivalent to "Lipschitz w^*-uniform Kadec Klee"; more generally, asymptotic uniform smoothness is equivalent to w^*-uniform Kadec Klee. This is a non-reflexive version of the well-known duality between uniform convexity and uniform smoothness.

We recall that a Banach space Y is said to be M-embedded if for every $y^* \in Y^*$ and $t \in Y^\perp \subset Y^{***}$, one has $\|y^* + t\| = \|y^*\| + \|t\|$. We refer to [28] for numerous examples and applications. Note that any non reflexive M-embedded space contains an isomorphic copy of c_0 [2] (see [23, Th. 3.5] for a more general result).

It is clear that $(c_0, \|\cdot\|_\infty)$ is M-embedded. We prove now that this property characterizes c_0 among the isomorphic preduals of ℓ^1.

Corollary V.2. *Let X be a separable $\mathscr{L}^\infty$ space. If X is* M-embedded*, then X is isomorphic to c_0.*

Proof. Since X is M-embedded, the weak* and weak topologies coincide on S_{X^*} (see [28, Cor. III.2.15]) and thus X^* is separable. Since X is $\mathscr{L}^\infty$, X^* is a separable $\mathscr{L}^1$ dual space, and thus [31] it is isomorphic to ℓ^1. Let us show that $\|\cdot\|_X$ is asymptotically uniformly flat.

To show this, take a sequence (u_n^*) with w^*-$\lim(u_n^*) = 0$ and $\lim \|u_n^*\| = \varepsilon$, such that $\overline{\lim}\, \|x^* + u_n^*\| \leq 1$. We have to show that $\|x^*\| \leq 1 - K\varepsilon$ for some fixed constant $K > 0$. Since X^* is isomorphic to ℓ^1, it has the strong Schur property (see [5]). Hence there is a subsequence $(u_{n_k}^*)$ of (u_n^*) such that $\|u_{n_k}^* - u_{n_l}^*\| \geq \varepsilon/2$ for all $k \neq l$, and the strong Schur property provides a further subsequence which we denote (v_n^*) which is $(K_0\varepsilon)$-equivalent to the canonical basis of ℓ^1. It follows that (v_n^*) has a w^*-cluster point G in X^{***}

such that $\mathrm{dist}(G, X^*) \geq K\varepsilon$ for some $K > 0$. Note that $G \in X^\perp$ since w^*-$\lim(v_n^*) = 0$ and thus $\mathrm{dist}(G, X^*) = \|G\| \geq K\varepsilon$. Now since $\overline{\lim}\,\|x^* + v_n^*\| \leq 1$, we have $\|x^* + G\| \leq 1$ and thus by the M-ideal property,

$$\|x^*\| \leq 1 - \|G\| \leq 1 - K\varepsilon.$$

By Theorem V.1, X is isomorphic to a subspace of c_0. The conclusion follows since any $\mathscr{L}^\infty$ subspace of c_0 is isomorphic to c_0. ∎

Corollary V.2 has been originally shown in [25] through an application of Zippin's converse to Sobczyk's theorem [37]. We refer to [21] for non separable versions of Corollary V.2.

VI. Approximation properties in subspaces of c_0

We recall that a Banach space X has the approximation property (in short, A.P.) if for any compact set K and any $\varepsilon > 0$, there is a finite rank operator R such that $\|x - Rx\| < \varepsilon$ for all $x \in K$. If there is $M < \infty$ such that R can be chosen with $\|R\| \leq M$, we say that X has the bounded approximation property (B.A.P.); if $M = 1$, X has the metric approximation property (M.A.P.). It is easily seen that a separable space X has the B.A.P. if and only if there is a sequence (R_n) of finite rank operators such that $\|R_n\| \leq M$ for all n and $\lim \|x - R_n(x)\| = 0$ (with $M = 1$ if and only if X has M.A.P.). A remarkable result from [6] asserts that if X is any separable Banach space with M.A.P., there is a sequence (R_n) of finite rank operators such that

(i) $\|R_n\| \leq 1$ for all n.

(ii) $\lim \|x - R_n(x)\| = 0$ for all $x \in X$.

(iii) $R_n R_k = R_k R_n$ for all n, k.

We now investigate some aspects of the approximation properties in subspaces of c_0.

Theorem VI.1. *Let X be an* M-*embedded separable Banach space, such that there exists a sequence (R_n) of finite rank operators satisfying* (ii), (iii) *and such that $\|R_n\| < 2 - \varepsilon$ for some $\varepsilon > 0$. Then there is a sequence (R'_n) of finite rank operators satisfying* (i), (ii) *and* (iii).

Note that being M-embedded is a hereditary property; hence Theorem VI.1 applies in particular to every subspace of c_0.

Proof. We consider the conjugate operators (R_n^*) on X^*. It follows from (ii) that

$$w^*\text{-}\lim R_n^*(x^*) = x^*$$

for all $x^* \in X^*$. We claim that

$$\overline{\bigcup_{n\geq 1} R_n^*(x^*)} = X^*. \tag{6}$$

If not, there is $F \in X^{**}$ with $\|F\| = 1$ such that $F(R_n^*(x^*)) = 0$ for all $x^* \in X^*$ and for all n. We pick $x^* \in X^*$ such that $\|x^*\| \leq (2-\varepsilon)^{-1}$ and $F(x^*) > 1/2$.

We have

$$\|R_n^*(x^*)\| \leq \|R_n^*\|\|x^*\| \leq 1.$$

Let $t \in X^{***}$ be a w^*-cluster point in X^{***} to the sequence $(R_n^*(x^*))$. It is clear that $F(t) = 0$ and $t|_X = x^*$. Since X is M-embedded, we have

$$\|x^* - t\| = \|t\| - \|x^*\| \leq 1 - F(x^*) < \frac{1}{2}.$$

On the other hand, $F(x^* - t) > 1/2$, a contradiction which shows (6).

We now claim that in fact

$$\lim \|x^* - R_k^*(x^*)\| = 0 \tag{7}$$

for all $x^* \in X^*$. Indeed by (6), for any $\varepsilon > 0$, there is $y^* \in X^*$ and $n \geq 1$ such that

$$\|x^* - R_n^*(y^*)\| < \varepsilon.$$

Using (iii), we have for all $k \geq 1$

$$\|R_k^*(x^*) - R_n^* R_k^*(y^*)\| = \|R_k^*(x^*) - R_k^* R_n^*(y^*)\| \leq 2\varepsilon.$$

Now since R_n^* is a conjugate finite rank operator, it is $(w^*\text{-}\|\cdot\|)$ continuous and thus

$$\lim_k \|R_n^*(y^*) - R_n^* R_k^*(y^*)\| = 0.$$

It follows that there is $A(\varepsilon)$ such that if $k \geq A(\varepsilon)$ then

$$\|x^* - R_k^*(x^*)\| < 4\varepsilon$$

and this shows (7).

To conclude the proof, we note that (7) implies that X^* is a separable dual with the approximation property. By a theorem of Grothendieck (see [32, Th. I.e.15]) it follows that X^* has the M.A.P., and thus X has M.A.P. as well (see [32, Th. I.e.7]) and we may apply [6]. ∎

We refer to [24] for an alternative construction of commuting approximating sequences in spaces not containing ℓ^1, which applies in particular to M-embedded spaces.

It follows, immediately from Theorem VI.1 and [6] that we have

Corollary VI.2. *Let X be a separable* M*-embedded space. If there exists a Banach space Y with the metric approximation property such that $d_{BM}(X,Y) < 2$ then X has the metric approximation property.*

It is natural to wonder whether 2 can be replaced by an arbitrary constant in Theorem VI.1 and Corollary VI.2. It is not so, as shown by the following example due to W.B. Johnson and G. Schechtman (see [30]).

Theorem VI.3. *There exists a subspace X of c_0, such that there is a sequence (R_n) of finite rank operators satisfying* (ii), (iii) *and* $\sup \|R_n\| \leq 8$, *but X fails the metric approximation property.*

Proof. It follows from Enflo's theorem [10] that there is a subspace Y of c_0 failing A.P. (see [32, p. 37]). Let $(Y_n)_{n\geq 1}$ be an increasing sequence of finite dimensional subspaces of Y such that $Y = \overline{\cup_n Y_n}$.

We consider the space

$$c((Y_n)) = \{y_n \in Y_n : (y_n) \text{ is norm-convergent}\}.$$

When equipped with the supremum norm, $c((Y_n))$ is clearly a Banach space. The map

$$\begin{array}{rccc} L: & c((Y_n)) & \longrightarrow & Y \\ & (y_n) & \mapsto & \lim(y_n) \end{array}$$

is a quotient map, whose kernel is the space $c_0((Y_n))$.

We observe that by Corollary II.3, the space $c((Y_n))$ is isomorphic to a subspace of c_0, since its subspace $c_0((Y_n))$ and the corresponding quotient space Y are both isomorphic (in fact, isometric) to subspaces of c_0. An examination of the proof of Corollary II.3 shows in the notation used there that $\|V\| \leq 2$ and $\|V^{-1}|_{\mathrm{Im}(V)}\| \leq 4$. It follows that there is a subspace X of c_0 such that

$$d_{BM}\big(c_0((Y_n)), X\big) \leq 8.$$

Since $c_0((Y_n))$ trivially has the M.A.P., the existence of the sequence (R_n) with (ii), (iii) and $\sup \|R_n\| \leq 8$ follows.

The proof of Theorem VI.1 shows in particular that if a subspace of c_0 has the M.A.P. then its dual has the M.A.P. as well. Hence it suffices to show that the space X^* fails A.P.; this follows from the

FACT VI.4. The space Y^* is isomorphic to a complemented subspace of $c((Y_n))^*$.

Indeed, $L^* : Y^* \to c((Y_n))^*$ is an isomorphic embedding, and it suffices to find $P : c((Y_n))^* \to Y^*$ such that $PL^* = \mathrm{Id}_{Y^*}$.

For all $n \geq 1$, let $j_n : Y_n \to c((Y_n))$ be defined as follows: if $j_n(y) = z$ then

$$z_k = \begin{cases} 0 & \text{if } \; k < n \\ y & \text{if } \; k \geq n. \end{cases}$$

Pick any $f \in c((Y_n))^*$, and let $f_n = j_n^*(f)$. Clearly, $\|f_n\| \leq \|f\|$. Let $\overline{f_n} \in Y^*$ be an extension of f_n to Y with $\|\overline{f_n}\| = \|f_n\|$. Fix a free ultrafilter $\mathscr{U}$ on $\mathbb{N}$. It is easily seen that $w^*\text{-}\lim_{n\to\mathscr{U}}(\overline{f_n})$ does not depend upon the choice of the extension $\overline{f_n}$; we can therefore set

$$P(f) = w^*\text{-}\lim_{n\to\mathscr{U}} \overline{f_n}.$$

Finally, $PL^*(y^*) = y^*$ for every $y^* \in Y^*$. Indeed, if $(g_n) \subseteq Y^*$ is any sequence such that $\|g_n\| \leq \|y^*\|$ and $g_n = y^*$ on Y_n, then (g_n) is w^*-convergent to y^*.

To conclude the proof of Theorem VI.3, we observe that since Y fails A.P., the space Y^* fails A.P. as well (see [32, Th. I.e.7]). Hence $c((Y_n))^*$ fails A.P. since it contains a complemented subspace which fails it. ∎

Note that it follows from Theorem VI.1 that the space X, which has the B.A.P., is such that $d_{BM}(X, Y) \geq 2$ for all spaces Y which have the M.A.P. It is a well-known and important open problem [32, Pb. I.e.21] whether every Banach space which has the B.A.P. is isomorphic to a Banach space with the M.A.P.; or equivalently by [6], if every Banach space with the B.A.P. has this property with commuting operators. The space X of Theorem VI.3 is, by its construction, isomorphic to a space which has M.A.P.

VII. Open problems

§ II. As indicated in Remark II.4, Sobczyk's theorem fails in the non-separable case. However, the separable complementation property shows that c_0 is complemented in every W.C.G. space X containing it. This extend to

$c_0(\Gamma)$ spaces if $\mathrm{dens}(X) < \aleph_{\omega_0}$, but not further [1]. We refer to [9, Th. II.8.3] for a link between extensions of G-smooth norms and the existence of linear projections. Let us mention that the relevant Problem II.2 from [9, p. 90] on extensions of Fréchet smooth norms is still open.

§ III. Proximality of finite codimensional subspaces can be understood as an n-dimensional version of Bishop–Phelps theorem (see Prop. III.4). It is not known whether every Banach space contains a proximinal subspace of codimension 2. It is also an open problem to know if every dual space contains a 2-dimensional subspace consisting of norm-attaining functionals.

§ IV. The gist of this section is that c_0 is minimal among non superreflexive C^k smooth spaces, with $k > 1$. A related open problem is whether every continuous convex function f on c_0 has points of "Lipschitz smoothness" (see [11]); that is, let $f : c_0 \to \mathbb{R}$ be a convex continuous function. Does there exists a point x_0 of Fréchet smoothness of f such that

$$f(x_0 + h) = f(x_0) + f'(x_0)h + O(\|h\|^2)\,?$$

Note that Fréchet smoothness means that the remainder is $o(\|h\|)$.

It follows from the existence of a C^∞ smooth bump function on c_0 that every continuous function $g : c_0 \to \mathbb{R}$ is uniformly approximable by C^∞ smooth functions (see [9, Th. VIII.3.2]). If g is assumed to be *uniformly* continuous, it is uniformly approximable by real-analytic functions ([7], [14]). It is not known if this conclusion still holds for continuous functions. It would be interesting to link this topic with Gowers' stability theorem ([27]; see [3, Th. 13.18]).

§ V. The results from [20] (see [3, Th. 10.17]) show that c_0 is somehow a "rigid" space. We refer again to [27] (see [3, Th. 13.18]) for an independent and deep result of non-linear rigidity of c_0. It is not known whether a Banach space which is uniformly homeomorphic to c_0 is isomorphic to c_0, although it can be shown that such a space is an isomorphic predual of ℓ^1 which shares many features of c_0 [21]. Corollary V.2 is one of the characterizations of c_0 among preduals of ℓ^1. It is not known whether c_0 is the only isomorphic predual of ℓ^1 which has Pelczynski's property (u) (see [23, Th. 7.7] for a partial result).

§ VI. No effort was made in the proof of Theorem VI.3 to tighten the constant and it is unlikely that 8 is the critical value. A natural guess is 2. No example is known of a subspace of c_0 having A.P. but failing B.A.P. It is not known whether the assumption of commutativity (iii) can be removed in

Theorem VI.1; in other words, if a subspace of c_0 with λ-B.A.P. for $\lambda < 2$ has the M.A.P. Note that [26, Ex. 4.7] shows that the proof of Theorem VI.1 fails to provide a positive answer. By [22, Prop. 3.2], a subspace X of c_0 with the metric approximation property whose dual embeds into L^1 is isomorphic to a quotient of c_0; it is not known whether one can dispense with assuming the metric approximation property.

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