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Cyclic Behavior of Linear Fractional Composition Operators

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1. Cyclic linear fractional composition operators

For each real number ν the weighted Dirichlet space $\mathcal{S}_\nu$ is the space of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that the following norm

$$\|f\|_\nu^2 = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2\nu}$$

is finite. Observe that $\|\cdot\|_\nu$ comes from an inner product and, therefore, the spaces $\mathcal{S}_\nu$ are Hilbert spaces. Also, notice that the above condition on the Taylor coefficients of the function f implies that $\limsup_{n\to\infty} |a_n|^{1/n} \le 1$ and, thus, each $f \in \mathcal{S}_\nu$ is analytic, at least, on the unit disk $\mathbb{D}$ of the complex plane. For some values of ν the spaces $\mathcal{S}_\nu$ are very well known classical analytic function spaces: for $\nu = 1/2$ it is the Dirichlet space $\mathcal{D}$; for $\nu = 0$ it is the Hardy space $\mathcal{H}^2$ and for $\nu = -1/2$ it is the Bergman space $\mathcal{A}^2$.

The easiest composition operators are those induced by linear fractional maps

$$\varphi(z) = \frac{az+b}{cz+d} \qquad ad - bc \neq 0$$

and $\varphi(\mathbb{D}) \subset \mathbb{D}$. In this case, the linear fractional composition operator defined by

$$C_\varphi f = f \circ \varphi \qquad (f \in \mathcal{S}_\nu)$$

is always bounded on any of the $\mathcal{S}_\nu$ spaces (see [8], for instance). In some of the $\mathcal{S}_\nu$ spaces, this fact fails to be true for certain holomorphic selfmaps of $\mathbb{D}$ (see [4]). As weighted shifts, composition operators are a class of concrete operators that, in general, do not fall into a class of operators satisfying a prescribed property invariant under similarity. This along several nice features makes them worth studying.

A bounded linear operator T acting on a separable Hilbert space $\mathcal{H}$ is said to be cyclic if there is an $f \in \mathcal{H}$ such that the linear span of the orbit $\{T^n f\}_{n\geq 0}$ is dense in $\mathcal{H}$. There have been much interest for a long time in cyclic operators because their relation to invariant subspaces. More recently, stronger forms of cyclicity are being investigated. If there is $f \in \mathcal{H}$ such that $\{\lambda T^n f : \lambda \in \mathbb{C} \text{ and } n = 0, 1, \dots\}$ is dense in $\mathcal{H}$, then the operator T as well as the vector f are called supercyclic. If the orbit itself is dense in $\mathcal{H}$, then T and f are called hypercyclic.

In [5] the authors have completely characterized the cyclicity and hypercyclicity of scalar multiples of linear fractional composition operators in all the $\mathcal{S}_\nu$ spaces (see Table I and [2] for the classification of linear fractional maps). Previous results for $\nu = 0$ and $\nu > 1/2$ (or $\nu > 3/2$, depending on which type of φ) were already known (see [3] and the page 150). Cyclic composition operators for $\nu = 0$ were first studied in [13]. Observe the central role played by the Dirichlet space $\mathcal{D}$ in the cut-off of any of the cyclic properties. Since supercyclicity is an intermediate property between cyclicity and hypercyclicity, many of the results for the supercyclicity column follow immediately. If φ has a fixed point in $\mathbb{D}$ (as in the elliptic case), it is easy to show that there is no supercyclic operators (see [5, Sect. 8]). For the Dirichlet space $\mathcal{D}$ ($\nu = 1/2$), the supercyclicity follows because λC_φ is hypercyclic whenever $|\lambda| > 1$ (see [5, Sect. 4]). Therefore, to complete the characterization it is just sufficient to know the supercyclic behavior of composition operators induced by parabolic non automorphisms. In this sense, Shapiro [11], using that C_φ is similar to λC_φ acting on certain subspace, had proved that λC_φ is not hypercyclic on $\mathcal{H}^2$. But this is far to mean that C_φ is supercyclic on $\mathcal{H}^2$. We stress here that even an invertible operator can be supercyclic and not hypercyclic for any scalar multiple of it. This can be seen as a consequence of Theorem 5.2 in [7]. Better still, Salas [10] has provided examples of this behavior in which even the set of normal eigenvalues is the empty set.

The following result that completes the supercyclic behavior of linear fractional composition operators in the Hardy space $\mathcal{H}^2$ was firstly proved in [6].

Theorem 1. *Let φ be a parabolic non automorphism that takes the unit disk into itself. Then C_φ acting on the Hardy space $\mathcal{H}^2$ is not supercyclic.*

Sketch of the proof. Parabolic non automorphisms that take the unit disk into itself have just one fixed point on the boundary $\partial\mathbb{D}$ and because any cyclic property is invariant under similarity it may be assumed that $\varphi(1) = 1$ and, thus, the following formula holds

$$\varphi(z) = \frac{(2-a)z + a}{-az + 2 + a}$$

where $\Re a > 0$ because φ is not an automorphism of $\mathbb{D}$.

The method of the proof is to get estimates from above and from below of C_φ acting on certain subspaces. These subspaces are built up from the eigenfunctions of C_φ. For each $t \geq 0$ an elementary computation shows that the inner function

$$e_t(z) = \exp\left[t\frac{z+1}{z-1}\right]$$

is an eigenfunction of C_φ in $\mathcal{H}^2$ corresponding to the eigenvalue e^{-at} (see [4] for more about the spectrum of C_φ). The proof also uses the following result

$$\overline{\operatorname{span}}\,\{e_t : t \geq 0\} = \mathcal{H}^2$$

that is already contained in Ahern and Clark's work [1]. By taking quotients by the C_φ-invariant subspace $F = \operatorname{span}\,\{e_\tau, e_\sigma\}$, where $\tau > \sigma > 0$, the space $\widehat{\mathcal{H}} = \mathcal{H}^2/F$ can be decomposed in an orthogonal sum with some useful properties

$$\widehat{\mathcal{H}}^2 = \widehat{X}_\tau \oplus \widehat{Z}_{\tau\sigma} \oplus \widehat{Y}_\sigma. \tag{1}$$

First, if σ is chosen large enough, then there are constants $0 < c < C$ such that the operator $\widehat{C}_\varphi$ defined by $\widehat{C}_\varphi \hat{f} = \widehat{C_\varphi f}$ satisfies $\|\widehat{C}_\varphi^n\big|_{\widehat{Y}_\sigma} \hat{f}\| \leq c^n \|\hat{f}\|$ for $\hat{f} \in \widehat{Y}_\sigma$ and $\|\widehat{C}_\varphi^n\big|_{\widehat{X}_\tau} \hat{f}\| \geq C^n \|\hat{f}\|$ for $\hat{f} \in \widehat{X}_\tau$. Second, the spaces in (1) are $\widehat{C}_\varphi$-invariant. The estimates above can be obtained by using Gerschgorin's Theorem (see [12]) about localization of eigenvalues. This theorem is applied to $n \times n$ matrices that represent the restrictions of $\widehat{C}_\varphi$ to certain n-dimensional $\widehat{C}_\varphi$-invariant subspaces. The matrices are obtained thank to very nice orthogonality properties that possess the eigenfunctions $e_t(z)$ in the Hardy space $\mathcal{H}^2$.

Finally, suppose that C_φ is supercyclic, then so is $\widehat{C}_\varphi$. Now, choose a supercyclic vector $\widehat{f} = \hat{f}_\tau \oplus \hat{f}_{\tau\sigma} \oplus \widehat{f}_\sigma$. Then $\hat{f}_\tau$ must be different from zero; otherwise $\widehat{C}_\varphi$ is not supercyclic. Finally, for $\hat{g} \neq 0$ orthogonal to $\widehat{X}_\tau$, we have

$$\frac{|\langle \widehat{C}_\varphi^n \hat{f}, \hat{g}\rangle|}{\|\widehat{C}_\varphi^n \hat{f}\|\|\hat{g}\|} \leq \frac{c^n \|f_\sigma\|}{C^n \|f_\tau\|}$$

that goes to zero. Therefore, C_φ is not supercyclic, a contradiction. ∎

The last idea in the sketch of the proof has occurred first in [9] in relation to infinite dimensional subspaces of supercyclic vectors (see [6] for more details).

Type of φ	Cyclic	Supercyclic	Hypercyclic	Example
Hyperbolic Automorphism	$\nu < 1/2$	$\nu < 1/2$	$\nu < 1/2$	$\frac{3z+1}{z+3}$
Parabolic Automorphism	$\nu < 1/2$	$\nu < 1/2$	$\nu < 1/2$	$\frac{(1+i)z-1}{z+i-1}$
Hyperbolic Non-Automorphism	Always	$\nu \leq 1/2$	$\nu < 1/2$	$\frac{1+z}{2}$
Parabolic Non-Automorphism	$\nu \leq 3/2$	Never	Never	$\frac{1}{2-z}$
Interior & Exterior	Always	Never	Never	$\frac{-z}{2+z}$
Interior & Boundary	Never	Never	Never	$\frac{z}{2-z}$
Elliptic Irrational Rotation	Always	Never	Never	$e^{2i/3}z$
Elliptic Rational Rotation	Never	Never	Never	$e^{2\pi i/3}z$

Table 1

The comparison principle for supercyclic operators (first noticed in [10]) only shows that C_φ is not supercyclic for any $\mathcal{S}_\nu$ for $\nu \geq 0$. However, a careful analysis shows that the methods we have just described can also be used to prove that C_φ is not supercyclic in any of the $\mathcal{S}_\nu$ spaces. Although there is no good orthogonality properties of the functions $e_t(z)$ in $\mathcal{S}_\nu$ for $\nu > 0$, one can still obtain rather manageable C_φ-invariant subspaces. In fact, we can prove the following result that completes the cyclic behavior of linear fractional composition operators in all the $\mathcal{S}_\nu$ spaces.

THEOREM 2. *Let φ be a parabolic non automorphism that takes the unit disk into itself. Then C_φ is not supercyclic in any of the weighted Dirichlet spaces $\mathcal{S}_\nu$.*

Actually, our methods are still valid for spaces of analytic functions that are not Hilbert spaces, and can be used to get non supercyclicity of C_φ, for instance, in the Bergman spaces $\mathcal{A}^p$, $1 \leq p < \infty$ (and by the comparison principle in any space that is densely contained in some of these spaces and has a stronger topology). The comparison principle is one of the motivations for the work discussed in this note (see [5] for more details).

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