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Smith's Counterexample about Uniform Rotundity in Every Direction

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It is an open question when a direct sum of normed spaces inherits uniform rotundity in every direction from the factor spaces. M. Smith [4] showed that, in general, the answer is negative. The purpose of this paper is carry out a complete study of Smith's counterexample.

Let X be a normed space. Its unit ball and unit sphere will be termed B and S, respectively. The space X is rotund if S has not linear segments.

The notion of normed space uniformly rotund in every direction was defined by A.L. Garkavi [3] to characterize those normed spaces in which every bounded subset has at most one Chebyshev center, that is, a point which is center of a minimum-radius ball that contains the bounded subset.

The space X is said to be uniformly rotund in a direction $z \neq 0$ (UR$\rightarrow z$ for short), if the directional rotundity modulus

$$\delta(\rightarrow z,\, \epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, x, y \in B,\, x - y = \lambda z,\, \|x-y\| \geq \epsilon\right\}$$

is strictly positive for every $0 < \epsilon \leq 2$. The spaces uniformly rotund in every non-null direction will be named URED spaces.

The space X is said to be uniformly rotund (UR for short), when the modulus of rotundity

$$\delta_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, x, y \in B,\, \|x-y\| \geq \epsilon\right\}$$

is strictly positive for $0 < \epsilon \leq 2$.

Let $(X_i, \|\cdot\|_i)$, $i \in \mathbb{N}$, be a sequence of normed spaces and let $(E, \|\cdot\|_E)$ be a normed space of real number sequences that satisfy $(\alpha_i) \in E$ and $\|(\alpha_i)\|_E \leq$

$\|(\beta_i)\|_E$, whenever $|\alpha_i| \leq |\beta_i|$ for every $i \in \mathbb{N}$, and $(\beta_i) \in E$. The direct sum space is defined by

$$E(X_i) = \{(x_i) : x_i \in X_i, (\|x_i\|_i) \in E\}$$

and it is normed by $\|(x_i)\| = \|(\|x_i\|_i)\|_E$. In Day's terminology, such a space is called a full function space [1, p. 35].

As the singleton characteristic function $\chi_{\{i\}}$ belongs to E if and only if there exists $\alpha \in E$ with $\alpha_i \neq 0$, we may and do assume that $\chi_{\{i\}} \in E$ for all $i \in I$. We note $c_i = \|\chi_{\{i\}}\|_E$. The order interval with ends $\alpha, \beta \in E$ is the set $[\alpha, \beta] = \{\gamma \in E : \alpha_i \leq \gamma_i \leq \beta_i, \ i \in I\}$.

It is easy to check that E and every X_i URED imply $E(X_i)$ URED. When E is either uniformly rotund in every direction and has compact order intervals, or weakly uniformly rotund respect to its evaluation functionals, M. Smith [4] and M.M. Day [1] have proved respectively that $E(X_i)$ is URED if and only if so are all the X_i.

However M. Smith [4] showed that, in general, E and all the X_i URED do not imply $E(X_i)$ URED. For a large family of full function spaces that include the one used by Smith, we establish an equivalent condition for $E(X_i)$ to be URED.

We use the notation summarized in the chart below.

Space	Elements	Norm	Modulus	Unit Sphere	Unit Ball
X_i	x_i, y_i, z_i	$\|\cdot\|_i$	$\delta_i(\to \cdot, \cdot)$	S_i	B_i
E	$\alpha = (\alpha_i), \beta = (\beta_i), \gamma = (\gamma_i)$	$\|\cdot\|_E$	$\delta_E(\to \cdot, \cdot)$	S_E	B_E
$E(X_i)$	$x = (x_i), y = (y_i), z = (z_i)$	$\|\cdot\|$	$\delta(\to \cdot, \cdot)$	S	B
X	x, y, z	$\|\cdot\|_X$	$\delta_X(\to \cdot, \cdot)$	S_X	B_X

1. Previous results

We will use the following results that appear in [2].

Theorem 1. *Let $z \in S$. If for every $0 < \epsilon \leq 2$,*

$$\Delta_{z,\epsilon} = \inf\Big\{\delta_E(\to \theta, \epsilon\|\theta\|_E) : \theta = (\theta_i), \frac{\|z_i\|_i}{4}\,\delta_i(\to z_i, \epsilon c_i\|z_i\|_i) \leq \theta_i \leq \|z_i\|_i\Big\} > 0,$$

then $E(X_i)$ *is* UR$\to z$.

THEOREM 2. *If* $z \in S_{\ell_\infty(X_i)}$, *then*

$$\delta\,(\to z,\, \epsilon) = \inf\{\delta_i(\to z_i,\, \epsilon\|z_i\|_i) : i \in I\}, \qquad 0 \le \epsilon \le 2.$$

If $(X_i, \|\cdot\|_i) = \mathbb{R}$ for every $i \in I$, then $\ell_\infty(X_i) = \ell_\infty$. From

$$\delta_{\mathbb{R}}(\to \zeta_i,\, \epsilon|\zeta_i|) = \delta_{\mathbb{R}}(\epsilon|\zeta_i|) = \frac{\epsilon}{2}\,|\zeta_i|,$$

it follows that

$$\delta_{\ell_\infty}(\to \zeta,\, \epsilon) = \frac{\epsilon}{2}\,\inf\{|\zeta_i| : i \in I\},$$

where $\|\zeta\|_{\ell_\infty} = 1$.

THEOREM 3. *Let* $\zeta \in S_{\ell_1}$. *Then*

$$\delta_{\ell_1}(\to \zeta,\, \epsilon) = \frac{\epsilon}{2}\,\inf\Big\{\Big|\sum_I \alpha_i\zeta_i\Big| : |\alpha_i| = 1,\ i \in I\Big\}, \qquad 0 \le \epsilon \le 2.$$

MIXED NORMS. Let $\{\|\cdot\|_i\}_{i\in I}$ be a family of norms defined on X. The mixed norm of this family, with respect to the full function space $(E, \|\cdot\|_E)$, is defined to be

$$\|x\|_X = \|(\|x\|_i)\|_E\,, \quad x \in X.$$

Let $(X_i, \|\cdot\|_i) = (X, \|\cdot\|_i)$. The application $J\colon\, x \in X \to Jx \in E(X_i)$, $(Jx)_i = x$ for every $i \in I$, enables one to identify isometrically X to JX. By means of this identification we may discuss whether $(X, \|\cdot\|_X)$ inherits uniform rotundity in a direction.

The aforementioned remark implies that

$$\delta_X(\to z,\, \epsilon) = \delta_{JX}(\to Jz,\, \epsilon) \ge \delta\,(\to Jz,\, \epsilon)\,.$$

Then one easily obtains a mixed norm version of Theorem 1.

THEOREM 4. *Let* $z \in S_X$. *If for every* $0 < \epsilon \le 2$

$$\Delta_{z,\epsilon} = \inf\Big\{\delta_E(\to \theta,\, \epsilon\|\theta\|_E) : \theta = (\theta_i), \qquad \frac{\|z\|_i}{4}\,\delta_i(\to z,\, \epsilon c_i\|z\|_i) \le \theta_i \le \|z\|_i\Big\} > 0,$$

then X *is* UR$\to z$.

As a consequence of Theorem 4 we obtain the following result.

THEOREM 5. *Let E be* UR. *If $(X, \|\cdot\|_j)$ is* URED *for some $j \in I$, then $(X, \|\cdot\|_X)$ is* URED.

2. Smith's Counterexample

For the class of full function spaces defined bellow, we establish a necessary and sufficient condition for $E(X_i)$ to be URED. A particular E in this class was used by M.A. Smith [4] to show that, in general, E and every X_i URED do not imply $E(X_i)$ URED.

THE FULL FUNCTION SPACE. Let E be the linear space of real bounded sequences. Let $\|\cdot\|_{**}$ be an uniformly rotund norm in E, and $|\cdot|$ a rotund norm in $\mathbb{R}^2$ such that $|(1,0)| = |(0,1)| = 1$. Define

$$\|\alpha\|_\infty = \sup_{i\in\mathbb{N}} |\alpha_i|, \quad \alpha \in E,$$

$$\|\alpha\|_* = \|(|\alpha_1| + |\alpha_2|, |\alpha_1| + |\alpha_3|, \ldots)\|_{**}, \quad \alpha \in E,$$

$$\|\alpha\|_E = |(\|\alpha\|_\infty, \|\alpha\|_*)|, \quad \alpha \in E.$$

Set $\ell_\infty = (E, \|\cdot\|_\infty)$, $\ell_{**} = (E, \|\cdot\|_{**})$, $\ell_* = (E, \|\cdot\|_*)$, and $E = (E, \|\cdot\|_E)$.

CLAIM. *The space E is* URED.

Proof. Let $\zeta \in S_E$. Since $\|\cdot\|_E$ is a mixed norm of $\|\cdot\|_\infty$ and $\|\cdot\|_*$, we use Theorem 5 to prove that E is URED. If $\inf_{i\in\mathbb{N}} |\zeta_i| > 0$, then

$$\delta_{\ell_\infty}(\rightarrow \zeta, \epsilon \|\zeta\|_\infty) = \frac{\epsilon}{2} \inf_{i\in\mathbb{N}} |\zeta_i| > 0, \quad 0 < \epsilon \le 2.$$

If $\inf_{i\in\mathbb{N}} |\zeta_i| = 0$, then

$$\delta_{\ell_*}(\rightarrow \zeta, \epsilon \|\zeta\|_*) > 0, \quad 0 < \epsilon \le 2.$$

This last implication is a consequence of the following claim, which describes the uniform rotundity directions of ℓ_*. ∎

Claim. *The space* ℓ_* *is* UR$\to \zeta$ *if and only if there exists some* $j \geq 2$ *such that* $|\zeta_1| \neq |\zeta_j|$.

Proof. Assume $|\zeta_1| = |\zeta_i|$, $i = 2, 3, \ldots$. Set $\xi_1 = -\zeta_1$, $\xi_i = 0$, $i = 2, 3, \ldots$. Then $\|\xi\|_* = \|\xi + \zeta\|_* = \|\xi + (1/2)\zeta\|_*$.

Assume the contrary. Then there exists $j \geq 2$ such that

$$(|\zeta_1| + |\zeta_j|) \min\{|\zeta_1 - \zeta_j|,\, |\zeta_1 + \zeta_j|\} > 0.$$

The space ℓ_* can be linearly isometrically identified with the linear space of sequences $J\alpha = ((\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \ldots)$, which is a subspace of $\ell_{**}(X_i)$, where X_i, $i = 2, 3, \ldots$ is the linear space $\mathbb{R}^2$ endowed with the sum norm $\|(r, s)\|_S = |r| + |s|$.

We may suppose that $\|\zeta\|_* = 1$. Theorems 1 and 3 with some manipulations yield

$$\delta_{\ell_*}(\to \zeta,\, \epsilon) \geq \delta_{\ell_{**}(X_i)}(\to J\zeta,\, \epsilon) \geq$$
$$\Delta_{J\zeta,\epsilon} = \delta_{\ell_{**}}\left(\frac{\epsilon^2}{8}\left\|\Big(a_i\big(|\zeta_1| + |\zeta_i|\big)\min\{|\zeta_1 - \zeta_i|, |\zeta_1 + \zeta_i|\}\Big)_{i \geq 2}\right\|_{**}\right) > 0, \tag{1}$$

where $a_i = \|\chi_{\{i\}}\|_{**}$, $i = 2, 3, \ldots$. ∎

Note that E has non-compact order intervals, since $\|\cdot\|_E$ is equivalent to $\|\cdot\|_\infty$.

The sum direct space. Suppose $\lim_{i\to\infty} a_i = 0$ and let $(X_i, \|\cdot\|_i)$, $i \geq 1$, be a sequence of normed spaces.

Claim. $E(X_i)$ *is* URED *if and only if so are all the* X_i *and*

$$z \in S,\ \inf_{i\in\mathbb{N}}\{\|z_i\|_i\} > 0 \quad \Rightarrow \quad \inf_{i\in\mathbb{N}}\{\delta_i(\to z_i,\, \epsilon)\} > 0, \quad 0 < \epsilon \leq 2. \tag{2}$$

Proof. The trick of the proof is to look at the norm in $E(X_i)$ as the mixed norm $\|x\| = |(\|x\|_{\ell_\infty(X_i)}, \|x\|_{\ell_*(X_i)})|$. Suppose X_i URED for every $i \geq 1$ and (2). Let $z \in S$. From Theorem 2, if $\inf_{i\in\mathbb{N}} \|z_i\|_i > 0$, then

$$\delta_{\ell_\infty(X_i)}\Big(\to z,\, \epsilon\,\|z\|_{\ell_\infty(X_i)}\Big) = \inf_{i\in I}\Big\{\delta_i(\to z_i,\, \epsilon\|z_i\|_i)\Big\} > 0, \quad 0 < \epsilon \leq 2.$$

Using Theorem 5 yields $E(X_i)$ UR$\to z$. If $\inf_{i\in\mathbb{N}} \|z_i\|_i = 0$, then we must prove $\delta_{\ell_*(X_i)}\Big(\to z,\, \epsilon \|z\|_{\ell_*(X_i)}\Big) > 0$, $0 < \epsilon \le 2$. From inequality (1) we obtain

$$\begin{aligned}
&\delta_{\ell_*(X_i)}\Big(\to z,\, \epsilon \|z\|_{\ell_*(X_i)}\Big) \ge \Delta_{z,\hat{\epsilon}} \\
&= \inf\Big\{\delta_{\ell_*}(\to \theta,\, \hat{\epsilon}\,\|\theta\|_*) :\, \gamma_i \le \theta_i \|z\|_{\ell_*(X_i)} \le \|z_i\|_i\,,\ i \ge 1\Big\} \\
&\ge \inf\Big\{\delta_{\ell_{**}}\Big(\Big\|\Big(\frac{\hat{\epsilon}^2}{8}a_i(\theta_1+\theta_i)\min(\theta_1+\theta_i, |\theta_1-\theta_i|)\Big)_{i\ge 2}\Big\|_{**}\Big) \\
&\qquad\qquad :\, \gamma_i \le \theta_i \|z\|_{\ell_*(X_i)} \le \|z_i\|_i\,,\ i \ge 1\Big\},
\end{aligned}$$

where $\hat{\epsilon} = \epsilon \|z\|_{\ell_*(X_i)}$, $\gamma_i = (1/4)\,\|z_i\|_i\,\delta_i(\to z_i,\, \epsilon a_i \|z_i\|_i)$ and $a_1 = \|\chi_{\{1\}}\|_*$.

Let $\gamma_i \le \theta_i \|z\|_{\ell_*(X_i)} \le \|z_i\|_i$, $i \ge 1$. If $z_1 = 0$, then

$$\|z\|^2_{\ell_*(X_i)}(\theta_1+\theta_i)\min\{\theta_1+\theta_i, |\theta_1-\theta_i|\} = \|z\|^2_{\ell_*(X_i)}{\theta_i}^2 \ge {\gamma_i}^2, \quad i \ge 2.$$

There exists $j \ge 2$ such that $\gamma_j > 0$. Therefore

$$\begin{aligned}
\delta_{\ell_*(X_i)}(\to z, \epsilon\|z\|_{\ell_*(X_i)}) &\ge \delta_{**}\left(\frac{\epsilon^2\|z\|^2_{\ell_*(X_i)}}{8}\left\|\left(\frac{a_i\gamma_i^2}{\|z\|^2_{\ell_*(X_i)}}\right)_{i\ge 2}\right\|_{**}\right) \\
&= \delta_{**}\left(\frac{\epsilon^2}{8}\left\|\left(a_i\gamma_i^2\right)_{i\ge 2}\right\|_{**}\right) \\
&\ge \delta_{**}\left(\frac{\epsilon^2}{8}\left\|\left(a_j\gamma_j^2\chi_{\{j\}}(i)\right)_{i\ge 2}\right\|_{**}\right) \\
&= \delta_{**}\left(\frac{\epsilon^2}{8}a_j^2\gamma_j^2\right) > 0.
\end{aligned}$$

If $z_1 \ne 0$, take $j \ge 2$ such that $\|z_j\|_j < (\gamma_1/2)$. Hence

$$\begin{aligned}
&\|z\|^2_{\ell_*(X_i)}(\theta_1+\theta_j)\min(\theta_1+\theta_j, |\theta_1-\theta_j|) \\
&\qquad \ge \theta_1(\theta_1-\theta_j)\|z\|^2_{\ell_*(X_i)} \ge (\theta_1^2/2)\|z\|^2_{\ell_*(X_i)} \ge (\gamma_1^2/2) > 0.
\end{aligned}$$

Again

$$\begin{aligned}
\delta_{\ell_*(X_i)}\Big(\to z,\, \epsilon\|z\|_{\ell_*(X_i)}\Big) &\ge \delta_{\ell_{**}}\left(\frac{\epsilon^2}{8}\left\|a_j\frac{\gamma_1^2}{2}\left(\chi_{\{j\}}(i)\right)_{i\ge 2}\right\|_{**}\right) \\
&= \delta_{\ell_{**}}\left(\frac{\epsilon^2}{8}a_j^2\frac{\gamma_1^2}{2}\right) > 0.
\end{aligned}$$

To prove the reverse, suppose that $E(X_i)$ is URED and that there exists $z \in S$ such that $\inf_{i\in\mathbb{N}} \|z_i\|_i > 0$ and $\inf_{i\in\mathbb{N}} \delta_i(\to z_i,\, \epsilon) = 0$ for some $0 < \epsilon \leq 1$. Taking a sub-sequence if necessary, we may suppose $\lim_{i\to\infty} \delta_i(\to z_i,\, \epsilon) = 0$. Thus there exists $(v_i)_{i\geq 1}$ such that $v_i,\, v_i + \epsilon z_i / \|z_i\|_i \in S_i$ and

$$\lim_{i\to\infty} \left\| v_i + \frac{\epsilon z_i}{2\|z_i\|_i} \right\|_i = 1.$$

Define

$$x_i^n = \begin{cases} -\dfrac{\epsilon z_1}{\|z_1\|_1}, & \text{if } i = 1, \\ v_n, & \text{if } i = n, \\ 0, & \text{if } i \neq 1,\, n. \end{cases}$$

From $\lim_{n\to\infty} a_n = 0$, some manipulations yield

$$\begin{aligned} \lim_{n\to\infty} \|x^n\| &= \lim_{n\to\infty} \left\| \left(x^n + \epsilon \left(\frac{z_i}{\|z_i\|_i} \right) \right) \right\| \\ &= \lim_{n\to\infty} \left\| \left(x^n + \frac{\epsilon}{2} \left(\frac{z_i}{\|z_i\|_i} \right) \right) \right\| = \left| \left(1, \|(\epsilon, \epsilon, \epsilon, \ldots)\|_{**} \right) \right|. \end{aligned}$$

from which it follows that $E(X_i)$ is non-URED.

When $\|\alpha\|_{**} = \left(\sum_{i\geq 2} a_i^2 \alpha_i^2\right)^{1/2}$, $\alpha = (\alpha_2, \alpha_3, \ldots)$, $|(r,s)| = (r^2 + s^2)^{1/2}$, $(r,s) \in \mathbb{R}^2$ and $(X_i, \|\cdot\|_i)$ is $\mathbb{R}^2$ endowed with $\|(r,s)\|_{i+1} = (|r|^{i+1} + |s|^{i+1})^{1/(i+1)}$, $i = 1, 2, 3, \ldots$, we have Smith's counterexample.

References

[1] Day, M.M., "Normed Linear Spaces", Third Edition, Springer-Verlag, Berlin–Heidelberg–New York, 1973.

[2] Fernández, M., Palacios, I., Directions of uniform rotundity in direct sums of normed spaces, *Arch. Math.*, **73** (1999), 366 – 372.

[3] Garkavi, A.L., The best possible net and the best possible cross-section of a set in a normed space, *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 87 – 106; *Amer. Math. Soc. Transl.*, Ser. 2, **39** (1964), 111 – 132.

[4] Smith, M.A., Products of Banach spaces that are uniformly rotund in every direction, *Pacific J. Math.* **73** (1) (1977), 215 – 219.