EXTRACTA MATHEMATICAE Vol. 15, Núm. 2, 335 – 371 (2000)

Complemented and Uncomplemented Subspaces of Banach Spaces

Anatolij M. Plichko and David Yost

Department of Mathematics, Pedagogical University, str. Shevchenko 1, 316050 *Kirovograd, Ukraine, e-mail: aplichko@kspu.kr.ua*
Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, *Riyadh* 11451, *Saudi Arabia, e-mail: dthoyost@ksu.edu.sa*

AMS *Subject Class.* (1991): 46B26

1. Introduction

Does a given Banach space have any non-trivial complemented subspaces? Usually, the answer is: yes, quite a lot. Sometimes the answer is: no, none at all.

You all know that a closed subspace Y of a Banach space X is said to be complemented exactly when there is another closed subspace C with $X = Y + C$ and $Y \cap C = \{0\}$, or, equivalently, when it is the range of a continuous linear projection. (Of course an arbitrary linear subspace has an algebraic complement, not necessarily closed, but the associated projection may be discontinuous.) An easy consequence of the Hahn-Banach Theorem is that every finite-dimensional subspace is complemented and it follows from the definition that every closed subspace of finite codimension is complemented. These will be referred to as the trivial examples. (We do not assume that subspaces are closed.)

It follows from Auerbach's Lemma [56, Prop. 1.c.3] that if $n \in \mathbb{N}$ then every n-dimensional subspace of a given Banach space is the range of a projection with norm at most n. Let us mention a serious improvement of this due to Kadets and Snobar, [48] or [43, Chap. 5]: n can be replaced by $\sqrt{n}$. Moreover this estimate is almost the best possible [52]. At the other extreme, Pisier [75] constructed a Banach space for which there is a constant C such that every projection onto an n-dimensional subspace has norm at least $C\sqrt{n}$.

Of course, there are non-trivial complemented subspaces. In every Banach space which comes to mind quickly, it is easy to find non-trivial projections. In a Hilbert space, every closed subspace is complemented. It is interesting to recall that the proof of this is intrinsically non-linear. The projection is first defined as the closest point mapping onto the subspace, a mapping which is well-defined in a large class of Banach spaces, although often neither linear nor continuous [95]. Then particular properties of Hilbert space are used to prove linearity of the projection. Indeed linearity of these best approximation mappings, for every closed subspace, actually characterizes Hilbert spaces of dimension three or more [3, §13].

The Spectral Theorem implies that the algebra of operators on a Hilbert space, $L(H)$, is actually the closed linear span of its self-adjoint projections. Better still, Pearcy and Topping [72] showed that any operator on an infinite-dimensional Hilbert space is a sum of at most five (not necessarily self-adjoint) projections. This is obviously false in finite dimensions, where a sum of projections must have integral trace. Moreover, they also showed that any self-adjoint operator is a linear combination of at most eight self-adjoint projections, a result which is also true in finite dimensions. Amongst a number of more recent papers on this topic, it has been announced that the number eight can be reduced to four in general [32, Theorem 1.3(a)] and even further in low dimensions [67].

Which Banach spaces share the property of Hilbert spaces, that every closed subspace is complemented? This question was raised very early [8, Remarques au Chapitre XII]. It turns out that this is a characterization of spaces isomorphic to Hilbert spaces. This result was first conjectured by Sobczyk [88, p. 79] and finally proved by Lindenstrauss and Tzafriri [55].

Nevertheless it is true [83] for any Banach space, that one can put a sensible topology on the family $\mathcal{F}$ of all complemented subspaces and define a continuous map $C : \mathcal{F} \to \mathcal{F}$ in such a way that $C(Y)$ is a complement for each Y, i.e. $X = Y \oplus C(Y)$.

As in [54], we call a Banach space *decomposable* if it admits a non-trivial projection. All of the familiar Banach spaces are decomposable. This is easy to see for the $L_p(\mu)$ spaces. For $C(K)$, it is easy to prove if either K has a non-trivial convergent sequence, or if K admits a non-trivial retract.

At the other extreme, there exist Banach spaces for which the only projections are the trivial ones; such spaces are called *indecomposable*. The construction of such Banach spaces, which have many bizarre properties, is a relatively recent result of Gowers and Maurey [34]. The observation that no closed subspace of their first example is decomposable is due to W. B. Johnson [34, p. 852]; such spaces are called *hereditarily indecomposable*. This extreme lack of projections implies that such spaces have very few operators; a result of Weis [96, Corollary 2.3(a)] implies that every operator on a hereditarily indecomposable space is either upper semi-Fredholm or strictly singular; see also [1, §4]. We should also mention that there are many other senses in which a Banach space might be considered to have few operators, other than lack of projections. For a discussion of these and some interesting examples, see [35].

A longstanding question [54, p. 918] is whether all "sufficiently big" Banach spaces are decomposable. Although most known examples of hereditarily indecomposable Banach spaces are separable, a non-separable example has recently been constructed by Argyros [5]. In §2 we show that every hereditarily indecomposable Banach space has cardinality equal to the continuum. (Considering Fréchet spaces does not lead to additional examples: a hereditarily indecomposable Fréchet space must be normable [62].)

Let us make the trivial observation that, if X is not quasi-reflexive, then X^{***} has a non-trivial projection. Thus, as we move to higher duals, we get more projections. Information about the bidual also tells us about decomposability. Valdivia [94] showed that if X is not separable but not too far from reflexivity, in the sense that $\operatorname{dens}(X^{**}/X) < \operatorname{dens} X$, then X admits a non-trivial projection.

Suppose that X is quasi-reflexive but not reflexive. Must X be decomposable? Or can X be hereditarily indecomposable? The first of these questions has been floating around for many years. A positive answer would imply that every quasi-reflexive Banach space is the direct sum of order one quasi-reflexive spaces. The strongest known result in this direction is the following statement, which simply merges results of Valdivia [94] and Bellenot [10].

THEOREM 1.1. *Suppose that X is quasi-reflexive, of order n. Then there exist subspaces $R, S_1, \dots, S_n$ of X such that $X = R \oplus S_n$, R is reflexive, S_n is separable, $S_1 \subset S_2 \dots \subset S_n$ and for each j, S_j^{**}/S_j has dimension j.*

All of the examples of hereditarily indecomposable spaces in [34] and [35] are reflexive. However Gowers [33] has also shown that there is a separable hereditarily indecomposable space which contains no reflexive subspace. In fact, it contains no quasi-reflexive subspace either. More recently, Argyros and Felouzis [6] have shown that surprisingly many Banach spaces, including all separable quasi-reflexive spaces, are quotients of hereditarily indecomposable Banach spaces. Of course ℓ_1 cannot be such a quotient.

Following [30], we say that a Banach space has the *separable complementation property* (or $\mathcal{SCP}$) if every separable subspace is contained in a complemented separable subspace. Of course this property is only interesting for non-separable spaces. It is blatantly obvious that any non-separable Banach space with $\mathcal{SCP}$ is decomposable. In §3, we look at some conditions which imply $\mathcal{SCP}$: this gives a good stock of decomposable Banach spaces. In particular, we give a simple proof of the fact that weakly compactly generated Banach spaces have the $\mathcal{SCP}$.

Given two cardinal numbers $\mathfrak{k} \leq \mathfrak{m}$, we will say that a Banach space has the $(\mathfrak{k}, \mathfrak{m})$-complementation property, or $\mathcal{CP}(\mathfrak{k}, \mathfrak{m})$, if every subspace with density character at most $\mathfrak{k}$ is contained in a complemented subspace with density character at most $\mathfrak{m}$. Thus $\mathcal{SCP} = \mathcal{CP}(\aleph_0, \aleph_0)$. Johnson and Lindenstrauss [46] asked whether every Banach space has $\mathcal{CP}(\aleph_0, \mathfrak{c})$; this remains unknown. In §5, we show that many Banach spaces have $\mathcal{CP}(\mathfrak{k}, \mathfrak{m})$ for suitable $\mathfrak{k}, \mathfrak{m}$; this increases our stock of decomposable spaces. To introduce these spaces, it is necessary to define some topological properties that the dual ball of a Banach space may or may not have; this is the topic of §4.

It is possible to go down as well as up. In §6, we note that in many Banach spaces, every subspace isomorhic to c_0 contains a complemented (infinite-dimensional) subspace.

All of the techniques here lead to projections of norm one. In §7, we exhibit a renorming of $\ell_1(\aleph_1)$ which admits few norm one projections, although it obviously has many complemented subspaces.

All this summarizes the most important results about the complemented subspace problem. Now we give a historical account of the most famous uncomplemented subspaces.

2. The easiest uncomplemented subspaces

1932: The first example (as far as we know) appears in [8, Remarques au Chapitre XII], which was apparently written by Mazur [8, Préface]: ℓ_1 and hence $L_1(0,1)$ contain uncomplemented subspaces. It was known back then that every separable Banach space is a quotient of ℓ_1 and that every infinite-dimensional closed subspace of ℓ_1 contains an isomorphic copy of ℓ_1. Since ℓ_2 is reflexive, it cannot contain a copy of ℓ_1, whence it cannot be isomorphic to a subspace of ℓ_1. Alternatively, this could be deduced from Schur's Lemma. Hence if $Q : \ell_1 \to \ell_2$ is a quotient map, then $\ker Q$ must be uncomplemented in ℓ_1.

1933: The second example is due to Banach and Mazur [9, p. 107], who showed that any subspace of $C[0,1]$ isomorphic to ℓ_1 must be uncomplemented. This follows from the fact that $C[0,1]^*$ is weakly sequentially complete but ℓ_1^* is not; thus ℓ_1^* cannot be isomorphic to any subspace of $C[0,1]^*$.

1934: Fichtenholz and Kantorovitch developed a representation theory for operators on the function space $L_\infty[a,b]$, and used it to show [28, p. 92] that there is no projection from $L_\infty[0,1]$ onto $C[0,1]$.

1937: Next Murray [64], in the second of two papers, showed that for $1 < p < 2$ or $2 < p < \infty$, the spaces ℓ_p and hence also $L_p(0,1)$ contain uncomplemented closed subspaces. More precisely, he showed that for each such p, there is a sequence of constants C_n, with $C_n \to \infty$, such that for infinitely many n, $\ell_p(n)$ contains a subspace onto which any projection must have norm at least C_n. The conclusion follows immediately from this. He thus isolated the finite-dimensional nature of this problem; his papers might be considered to be the first in the local theory of Banach spaces. These were the first reflexive examples and they caused a big surprise back then [11, p. 301]. It shattered people's hopes of establishing a spectral theory for operators on L_p spaces similar to that for normal operators on Hilbert spaces. This led Murray [65] to introduce the weaker notion of *quasi-complemented* subspace, a subject we will not elaborate on here.

1940: Unaware of the work of Banach and Mazur, Komatuzaki [51] also proved that $C[0,1]$ contains uncomplemented closed subspaces. His argument was that $L_1(0,1)$ contains uncomplemented closed subspaces and is contained isometrically in $C[0,1]$. Applying the techniques of Murray, he was the first to show that c_0 contains an uncomplemented closed subspace. In fact, he essentially proved that any Banach space containing $\ell_\infty(n)$ for all $n \in \mathbb{N}$ must contain an uncomplemented closed subspace. In particular, he noted that $L_\infty(0,1)$, ℓ_∞ and the spaces of differentiable functions $C^{(k)}[0,1]$ fall into this class.

1940: Phillips [74] used his now famous lemma to show that c_0 is not complemented in ℓ_∞. He also noted that ℓ_∞ is injective, i.e. complemented in every superspace.

1941: Sobczyk [88] examined Murray's arguments carefully and presented cleaner proofs, which are also valid in the cases $p = 1, \infty$. In particular, he established the asymptotically optimal bound $|C_n - \frac{1}{2}n^{|\frac{1}{p}-\frac{1}{2}|}| \leq \frac{1}{2}$, valid for the space $\ell_p(n)$ whenever n is a power of 2. Using the same construction, he showed that many spaces with "reasonable" bases (e.g. various Orlicz spaces) contain uncomplemented closed subspaces; this led him to conjecture [88, p. 79] that only Banach spaces isomorphic to a Hilbert space have the property that every closed subspace is complemented.

1941: Although it does not belong to this class of results, it would be remiss of us not to mention Sobczyk's proof of the theorem which now bears his name [89, p. 946]: c_0 is complemented in every separable superspace. For a survey of other proofs, see [14].

1943: Nakamura and Kakutani [66] gave a simpler proof of Phillips's result that c_0 is not complemented in ℓ_∞. Other simple proofs of this, and a few complicated ones, have been found over the years; see [14].

For other surveys of this topic, see [47] or [61]. Amongst topics not touched upon here, the former paper mentions Auerbach bases and studies in detail a number of uncomplemented subspaces of concrete function spaces discovered in the 1950s and 1960s. The latter deals with summability domains, prime spaces and injective spaces. They both give a proof of the Lindenstrauss-Tzafriri characterization of Hilbert spaces (necessarily assuming Dvoretzky's Theorem).

3. Decomposable Banach spaces

Quite weak hypotheses guarantee the existence of decomposable subspaces. But first, take a step back and ask what it means to have a norm one projection. If $P : X \to X$ is such a projection, $E = P(X)$ and $F = P^*(X^*)$, a moment's reflection shows that F norms E (the sup is actually a max in this case) and $F \cap E^0 = \{0\}$. Conversely, if E and F are subspaces of X and X^* respectively, such that F norms E and $\overline{F}^{w^*} \cap E^0 = \{0\}$ then the natural projection from X onto $\overline{E}$ with kernel F^0 is well defined and has norm one. In other words, the only way to get a norm one projection is via a pair of norming subspaces.

Perhaps we should recall that a (not necessarily closed) subspace G of X^* is said to norm a subspace Y of X if for all $y \in Y$, $\|y\| = \sup\{g(y) : g \in G, \|g\| = 1\}$. (More precisely, we could say that G 1-norms Y.) Recall also that a vector x is said to be Birkhoff orthogonal, or simply orthogonal, to y, if $\|x\| \leq \|x + \lambda y\|$ for all scalars λ. We write $x \perp y$ in this case, and for sets A and B we write $A \perp B$ to mean that $a \perp b$ for all $a \in A, b \in B$. Easy examples show that this relation is not symmetric. Obviously $A \perp B$ implies $\overline{A} \perp \overline{B}$.

The density character of a Banach space X, denoted dens X, is the smallest cardinal $\mathfrak{k}$ for which X has a dense subset of cardinality $\mathfrak{k}$.

LEMMA 3.1. *Let X be a normed space.*

(i) *If E is any subspace of X, then there is a subspace F of X^*, with the same density character as E, which norms E.*

(ii) *If F is any subspace of X^*, then there is a subspace E of X, with the same density character as F, which norms F.*

(iii) *If $F \subset X^*$ norms $E \subset X$, then $E \perp F^0$. Likewise, if $E \subset X$ norms $F \subset X^*$, then $F \perp E^0$.*

(iv) *If A and B are closed subspaces of a Banach space X and $A \perp B$, then $A \cap B = \{0\}$, $A + B$ is closed, and the natural projection $A \oplus B \to A$ has norm one.*

Proof. (i) For each vector in some dense subset of Y of minimum cardinality, choose a support functional. Then let F be the closed linear span of all these functionals.

(ii) For each functional in some dense subset of F of minimum cardinality, choose a norming sequence of vectors. Then let Y be the closed linear span of all these vectors.

(iii) For $x \in E$ and $y \in F^0$ we have $\|x\| = \sup f(x) = \sup f(x + y) \leq \|x + y\|$.

(iv) If $x \in A \cap B$ then the inequality $\|x\| \leq \|x + (-1)x\|$ implies immediately that $x = 0$. Clearly the projection $A \oplus B \to A$ has norm one. The completeness of X and the presumed inequality imply that $A + B$ is complete also, hence closed. ∎

If E is normed by F, then E is clearly normed by any superspace of F. Thus if E_1 and F_1 are separable subspaces of X and X^* respectively, then a routine back and forth argument shows that they are contained in closed separable subspaces E and F which norm one another. This implies that the subspaces $E \oplus F^0$ and $F \oplus E^0$ are closed, and that the natural projections onto E and F are continuous with norm one. But if either F^0 or E^0 is finite dimensional, this is no great achievement. The hypotheses of the next result have been blatantly rigged to prevent those possibilities. More profound applications of this idea appear in the next section.

PROPOSITION 3.2. *If X does not admit an injective operator into ℓ_∞, in particular if its density character exceeds the continuum, then X has a decomposable subspace.*

Proof. Choose E infinite dimensional, and construct F by Lemma 3.1(i). (In this case, we only apply Lemma 3.1(i) once, without any back and forth argument.) Since F is norm separable, $\overline{F}^{w^*} = F^{00} \cong (X/F^0)^*$ will be weak* separable. This implies that X/F^0 admits an injective operator into ℓ_∞. If X itself admits no such operator, Lemma 3.1(iii) and (iv) force F^0 to be infinite dimensional. Thus $E \oplus F^0$ is a decomposable subspace of X. ∎

A similar argument establishes the following.

PROPOSITION 3.3. *If $|X| > c$, then X has a decomposable quotient.*

The same elementary reasoning shows that if X is not separable, then X^* has a decomposable subspace; however stronger results are known. In general we would prefer to prove that X and X^* are themselves decomposable, under reasonably weak hypotheses. The first result of this sort is due to Lindenstrauss [53], who showed that certain reflexive Banach spaces are decomposable. This will be discussed further in the next section. Of more interest to us now is the following result of Heinrich and Mankiewicz.

THEOREM 3.4. [41, Corollary 3.8] *If X is any non-separable Banach space, then X^* is decomposable.*

Their statement was actually that if $\operatorname{dens} X^* > c$, then X^* admits uncountably many non-trivial projections. However their proof obviously yields the preceding assertion. For a simpler proof of this, see [87, p. 55]. In other words, if we weaken the requirement "F separable" to "F isomorphic to the dual of a separable space", then E and F can always be chosen so that $X^* = F \oplus E^0$. Whether they can be chosen so that $X = E \oplus F^0$ remains open.

Here is a more general existence result. We need another definition. Given a closed subspace Y of a Banach space X, we call $T : Y^* \to X^*$ a linear extension operator (or *LEO*) if it is linear and, for each $f \in Y^*$, Tf is a norm preserving extension of f. Clearly there exists a LEO from Y^* to X^* iff Y^0 is the kernel of a norm one projection on X^*.

THEOREM 3.5. [100, Proposition 2] *Let X be any Banach space, Y a separable subspace of X, and F a separable subspace of X^*. Then X has a separable subspace M containing Y, which admits a LEO $T : M^* \to X^*$ satisfying $T(M^*) \supset F$.*

The proof of this also is available in [40, Proposition III.4.3].

Think about the last theorem: it applies to any Banach space and implies the existence of complemented subspaces in any sufficiently large dual space. Taking $F = \{0\}$ yields the result of Heinrich and Mankiewicz mentioned above. We recall the problem of Johnson and Lindenstrauss [46]: does every Banach space have the property that every separable subspace is contained in a complemented subspace with density character not exceeding the continuum? Let's call this the *Not-too-big-complementation-property*, for short property $\mathcal{N}$. The special case $Y = \{0\}$ of the last Theorem implies that every dual space has $\mathcal{N}$. In fact, no Banach space lacking $\mathcal{N}$ is known. Some years ago at a Czech winter school, S. P. Gulko exhibited an interesting $C(K)$ space and conjectured that it fails $\mathcal{N}$, but its status remains unclear. We will return to this property in §6, where we present some other positive results. First, we consider the related and more familiar problem of finding complemented separable subspaces.

4. The separable complementation property

Recall that a Banach space has the separable complementation property ($\mathcal{SCP}$, or $\mathcal{CP}(\aleph_0, \aleph_0)$) if every separable subspace is contained in a complemented separable subspace. It is easy to see that ℓ_∞ fails $\mathcal{SCP}$. For Sobczyk's Theorem and the result of Phillips obviously imply that if a separable subspace of ℓ_∞ contains c_0, then it is not complemented.

It is not too hard to verify that, for any $1 \le p < \infty$ and any measure space (S, Σ, μ), our old friend $L_p(S, \Sigma, \mu)$ has $\mathcal{SCP}$. Given a countable collection of functions in $L_p(S, \Sigma, \mu)$, we can consider the smallest σ-algebra Σ_0 with respect to which they are all measurable. The conditional expectation operator corresponding to Σ_0 will then be a norm one projection onto a separable subspace containing the countable collection. For $1 < p < \infty$, $L_p(\mu)$ is reflexive, so Theorem 4.4 below generalizes this. Without going into the details, we mention three other sufficient conditions for the separable complementation property which generalize the example of $L_1(\mu)$. As shown in [57, Proposition 1.a.9], the proof of an ancient result of Kakutani implies that any Banach lattice not containing c_0 (i.e. any order continuous Banach lattice) has $\mathcal{SCP}$.

Ghoussoub and Saab [30, p. 83] were probably the first to note this explicitly. U. Haagerup [29, pp. 111-112] showed that the predual of any von Neumann algebra has $\mathcal{SCP}$. W. B. Johnson [19, p. 38] announced without proof that every dual space with the Radon-Nikodým Property has $\mathcal{SCP}$. Theorem 3.5 generalizes this; see also §6. Diestel and Uhl [19, Problem 22] asked whether every Banach space with the Radon-Nikodým Property has $\mathcal{SCP}$; we have recently found a counterexample, details of which appear elsewhere.

As mentioned earlier, this subject really began with Lindenstrauss [53] who showed that reflexive spaces with the metric approximation property fall into this class. Much stronger results are now known, of which perhaps the most interesting is that WCG spaces (definition below) have the separable complementation property [4, Lemma 4]. We will call this the Amir-Lindenstrauss Theorem, and we will include a succinct proof.

It is curious that the properties we are about to study are all invariant under renorming, and yet the proofs automatically give us projections of norm one. We note in passing that the situation in finite dimensions is rather different. Bosznay and Garay [12] showed that most norms (in the sense of Baire category) on finite-dimensional spaces have the property that the only norm one projections are the identity and those of rank one. A new example, a renorming of $\ell_1(\aleph_1)$ for which norm one projections are not too numerous, appears in the final section.

A normed space is said to be weakly compactly generated (WCG) if it is the closed linear span of a weakly compact absolutely convex subset. For Banach spaces, the Krein-Šmulian Theorem tells us that the words "absolutely convex" can be omitted from this definition, but this is not true in general. For example, if X denotes the linear span of the basis vectors in c_0, it is not hard to show that any weakly compact absolutely convex set in X is finite dimensional. We give this slightly more complicated definition of WCG because we do need to consider incomplete spaces.

The proof of the Amir-Lindenstrauss Theorem which we now give seems to be unknown in the west. It appeared first in [76], in a journal which was not then translated. We think that the simple method of proof may be of interest. It does not require technical finite-dimensional constructions, vector spaces over the rationals, the Mackey-Arens Theorem, the Stone-Weierstraß Theorem, the Löwenheim-Skolem Theorem or any general topology. It needs only the following simple idea.

Let X be a normed space generated by a weakly compact absolutely convex set K. Assume without loss of generality that K is contained in the unit ball of X. Denote by X_1 the linear span of K and by $\|\cdot\|_1$ the gauge functional of K, and equip X_1 with the norm $\|\cdot\|_1$. Denote also by $\|\cdot\|_1$ the dual functional on X^*, i.e. $\|f\|_1 = \sup f(K)$. Obviously $\|\cdot\|$ is stronger than $\|\cdot\|_1$ on X^*; the open mapping theorem ensures that the normed space $(X^*, \|\cdot\|_1)$ is not complete except when X is reflexive. We need the following standard result, first proved by Dixmier [20].

LEMMA 4.1. *In this notation, the dual of $(X^*, \|\cdot\|_1)$ is $(X_1, \|\cdot\|_1)$, in the natural duality. In particular $(X_1, \|\cdot\|_1)$ is a Banach space.*

A brief digression: we use the preceding idea to give a simple proof of the following well known result.

THEOREM 4.2. (i) *Let K be a weakly compact subset of a Banach space X. Then the restriction map $R : X^* \to C(K)$ is weak* to weak continuous.*

(ii) *Thus a Banach space is WCG iff the unit ball of its dual, equipped with the weak* topology, is affinely homeomorphic to an Eberlein compact (i.e. a weakly compact subset of some Banach space).*

Proof. (i) We consider first the case when K is absolutely convex and X is generated by K. Obviously R is continuous when X^* is equipped with $\|\cdot\|_1$ and $C(K)$ with its natural norm. Hence it is continuous in the corresponding weak topologies. But the weak topology corresponding to $\|\cdot\|_1$ is just $\sigma(X^*, \operatorname{linsp} K)$, which is obviously weaker than the weak* topology.

For the general case, let Y be the closed subspace generated by L, where L is the closed absolutely convex hull of K. By the previous paragraph, the composite restriction $X^* \to Y^* \to C(L) \to C(K)$ is weak* to weak continuous.

(ii) If X is a WCG space, generated say by K, then the restriction of R to $\operatorname{ball}(X^*)$ is obviously a homeomorphism onto some weakly compact subset of $C(K)$. For the converse, any such homeomorphism may be assumed to send the origin to the origin, and thus extends to a weak*-weak continuous injection $T : X^* \to Y$, for some Banach space Y. Taking the transpose finishes the proof. ∎

We have written the following so that the WCG hypothesis makes no explicit appearance in the proof. The role of weak compactness has been completely pushed into Lemma 4.1.

THEOREM 4.3. *Suppose that X is a WCG Banach space and, in the notation of the preceding paragraphs, let Y be a $\|\cdot\|$-separable subspace of X_1 and G a $\|\cdot\|_1$-separable subspace of X^*. Then there exist separable subspaces E and F of $(X_1, \|\cdot\|)$ and $(X^*, \|\cdot\|_1)$ containing Y and G respectively so that the projection from X onto $\overline{E}$ with kernel F^0 is well defined and has norm one.*

Proof. We will construct increasing sequences of separable subspaces $E_n \subset X_1$ and $F_n \subset X^*$, containing Y and G respectively, so that F_n $\|\cdot\|$-norms E_n and E_n $\|\cdot\|_1$-norms F_{n-1}.

Put $E_1 = Y$ and choose a $\|\cdot\|$-separable subspace H in X^* which $\|\cdot\|$-norms E_1. Put $F_1 = H + G$, which is evidently $\|\cdot\|_1$-separable.

Now for the inductive step: suppose that the first $n-1$ pairs of subspaces have been constructed. Since $(X_1, \|\cdot\|_1) = (X^*, \|\cdot\|_1)^*$, we may choose a $\|\cdot\|_1$-separable subspace Z_n in X_1 which $\|\cdot\|_1$-norms F_{n-1}. Since $\|\cdot\|_1$ is stronger than $\|\cdot\|$ on X_1, $E_n = Z_n + E_{n-1}$ is $\|\cdot\|$-separable and also $\|\cdot\|_1$-norms F_{n-1}. In the same way, there is a $\|\cdot\|$-separable subspace H_n of X^* which $\|\cdot\|$-norms E_n and then $F_n = H_n + F_{n-1}$ is $\|\cdot\|_1$-separable. Now put $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$. It is easily checked that E $\|\cdot\|_1$-norms F and that F $\|\cdot\|$-norms E.

Lemma 3.1(iii) tells us that the subspace E is Birkhoff orthogonal to F^0. Naturally the same conclusion holds for $\overline{E}$. It follows from Lemma 3.1(iv) that $\overline{E} + F^0$ is closed, that $\overline{E} \cap F^0 = \{0\}$ and that the projection $\overline{E} \oplus F^0 \to \overline{E}$ has norm one. To finish the proof, it is enough to show that $E + F^0$ is dense in X. For any $f \in (E + F^0)^0$, f lies in both E^0 and the weak* closure of F. But the weak* topology on X^* is simply the weak topology for the normed space $(X^*, \|\cdot\|_1)$ and so f lies in the $\|\cdot\|_1$-closure of F. But E $\|\cdot\|_1$-norms F and thus $f = 0$. Since f was arbitrary, $E + F^0$ is dense in X. In particular, $\overline{E} + F^0 = X$. ∎

We note that the assumption $Y \subset \operatorname{linsp}(K)$ is no real restriction, and so every WCG space has the Separable Complementation Property.

THEOREM 4.4. *Suppose that X is a WCG Banach space, that Y is a separable subspace of X and that G is a separable subspace of X^*. Then there is a norm one projection P from X onto a separable subspace containing Y with $P^*(X^*) \supset G$.*

Proof. Let (y_n) be a symmetric sequence dense in the unit ball of Y. Replacing Y by $Y_0 = \operatorname{linsp}\{y_1, y_2, \cdots\}$ and K by $K + \overline{\operatorname{co}}\{y_1, \frac{1}{2}y_2, \cdots\}$, we have

$Y_0 \subset \operatorname{linsp}(K)$. We define $\|\cdot\|_1$ with respect to this new K. Clearly G will be $\|\cdot\|_1$-separable. Apply the previous result and note that $\overline{Y}_0 \subset \overline{E}$ and so $Y \subset P(X)$. Finally $P^*(X^*) = F^{00} \supset G$. ∎

We would like to keep the exposition simple, by focusing just on the separable complementation problem. But finding one projection made us so happy that we are tempted to repeat this procedure again and again. Denote the projection given by this theorem as P_0. Applying the theorem again gives us another projection P_1, whose range is separable and strictly contains $P_0(X)$, and so that $P_1^*(X^*)$ strictly contains $P_0^*(X^*)$. Do it again. We get another projection P_2, whose range is separable and strictly contains $P_1(X)$, and so that $P_2^*(X^*)$ strictly contains $P_1^*(X^*)$. Get the idea? We find a nice increasing sequence of projections onto separable subspaces. So far so good, but what next? Simple, let E_n be the subspace of X_1 whose closure is the range of P_n, and let F_n be the corresponding subspace of X^*. Put $E_\omega = \bigcup_{n=0}^\infty E_n$ and $F_\omega = \bigcup_{n=0}^\infty F_n$, and let P_ω be the natural projection onto $\overline{E_\omega}$ with kernel F_ω^0. Keep going, getting projections $P_{\omega+1}, P_{\omega+2}, P_{\omega+3} \ldots$. Applying this trick at limit ordinals and the theorem at successor ordinals gives a long sequence of projections P_α, for $0 \leq \alpha \leq \omega_1$, all but the last having separable range.

But P_{ω_1} might not have separable range; can we continue to apply the theorem? Yes, once we notice that the separability hypothesis was not really used. The word "separable" can be replaced by "of density character at most $\mathfrak{k}$" for any infinite cardinal $\mathfrak{k}$. No essential changes are required for the proof. We should note that by choosing $Y \subset \operatorname{linsp}(K)$, the proof automatically gives $E_\alpha \subset \operatorname{linsp}(K)$ for all ordinals α. A transfinite induction argument then allows us to carry on, showing that any WCG space has a so-called projectional resolution of the identity. We have thus sketched the essential ideas of the proof of the following theorem. Details are largely a matter of bookkeeping.

THEOREM 4.5. [4] *Every WCG space has a PRI.*

This necessitates a definition. Let μ be the smallest ordinal of cardinality $\operatorname{dens} X$, and let ω_0 denote the smallest infinite ordinal. Then a *projectional resolution of the identity*, or PRI, is a family of projections P_α on X, $0 \leq \alpha \leq \mu$, satisfying:

1. $\|P_\alpha\| = 1$ for all α,
2. $P_\mu = \operatorname{Id}$ and $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ if $\alpha < \beta$,
3. $\operatorname{dens}(P_\alpha(X)) \leq \operatorname{card}(\alpha)$ for all $\alpha \geq \omega_0$,

4. $\overline{\bigcup_{\alpha<\beta} P_\alpha(X)} = P_\beta(X)$ if β is a limit ordinal.

Note that many authors take the first index as ω_0 rather than 0, in order to simplify condition (3).

PRIs provide a useful tool for studing WCG spaces, by transfinite induction arguments over the index set, starting from the separable case. They have a long sequence of applications [24, Chapters 6 and 8], [17, Chapter 6], about which we won't elaborate. We prefer now to concentrate on the question: what hypothesis other than WCG will guarantee a PRI? Or just the $\mathcal{SCP}$? It is well known that the existence of a PRI alone does not imply $\mathcal{SCP}$: a simple counterexample is $\ell_2(\Gamma) \oplus \ell_\infty$, where $|\Gamma| \geq c$.

Following [24], we will call a projectional generator for X any mapping $\Phi : F \to 2^X \setminus \{\emptyset\}$ such that F is a norming subspace (not necessarily closed) of X^*, each $\Phi(f)$ is a countable set, and, for every subspace V (not necessarily closed) of F, we have $\overline{V}^{w^*} \cap \Phi(V)^0 = \{0\}$. Repeating the back and forth game of the previous proofs shows that every Banach space with a projectional generator admits a PRI.

To avoid cluttering the exposition, we have simplified slightly the definition in [24, p. 106], to which we refer for all further details on this subject. Credit for the isolation of this idea and the first definition of projectional generator goes to Orihuela and Valdivia [70]. Needless to say, the concept was implicit in a number of earlier papers.

Of course every WCG space has a projectional generator: we take $F = X^*$, and for each f, choose a singleton $\Phi(f)$ so that $f(\Phi(f)) = \max f(K)$. The final step of the last proof shows that $\overline{V}^{w^*} \cap \Phi(V)^0 = \{0\}$ as required. We recall now that properties more general than WCG guarantee this.

To work in more general Banach spaces, it seems natural to exploit the weak* compactness of the dual ball, since this set always generates the dual space. Tacon [91] was the first to attempt this, and he proved (amongst other things) that if X is a sufficiently smooth Banach space, then X^* has the $\mathcal{SCP}$, and in fact a PRI. He encountered several technical difficulties, which he overcame only by imposing a smoothness hypothesis. One of these problems (essentially) was that a separable subspace of a dual space need not embed in a separable dual space. This suggests restricting attention to a smaller class of Banach spaces.

We will call a Banach space an *Asplund space* if every separable subspace has separable dual. See [73, §2] or [98] for some equivalent formulations in terms of automatic differentiability of convex functions, or [19] for the fact that Asplund spaces coincide with spaces whose duals have RNP. In particular, separable dual spaces have the Radon-Nikodým Property, and any very smooth or Fréchet smooth space is Asplund.

The following statement combines results from [25] and [70]. Details can also be found in [24, §8.2]. Note that X, when considered as a subspace of X^{**}, norms X^*.

THEOREM 4.6. *A Banach space X is an Asplund space if and only if its dual X^* admits a projectional generator defined on X, i.e. $\Phi : X \to 2^{X^*}$. In particular, the dual of any Asplund space has*

1. *a projectional generator,*
2. *a PRI*
3. *and the $\mathcal{SCP}$.*

With hindsight this all seems very natural; it is not hard to show that the existence of such a projectional generator is sufficient for a Banach space to be Asplund. However necessity is decidedly non-trivial, requiring a rather deep selection theorem of Jayne and Rogers [44], and either a result of Simons [86] (akin to James's Theorem) or a trick due to Stegall [90] which involves embedding X^* in the Lipschitz dual of X.

Of course (3) is a special case of Theorem 3.5. A previous attempt to prove something like Theorem 4.6(2) was made in [100], but the authors were unaware of the work of Jayne, Rogers, Simons and Stegall. Wallowing in ignorance, they obtained only partial results.

Before continuing, we need a bunch of definitions.

5. TOPOLOGY AND THE DUAL BALL

We have seen that the dual ball of a WCG space, equipped with its weak* topology, is not just any old compact set. There is often a strong correlation between geometric properties of a given Banach space, and topological properties of its dual ball. In this section, we consider some weaker topological properties which the dual ball might have. Here and in the next section we show that some of them lead to interesting complementation and other properties.

If Γ is any index set, let $\Sigma(\Gamma)$ denote the subset of $[0,1]^\Gamma$ of all those elements having countable support. A topological space is called a *Corson compact* if it is homeomorphic to a closed subset of $\Sigma(\Gamma)$ for some Γ. Recall that an Eberlein compact is any topological space homeomorphic to a weakly compact subset of a Banach space.

To simplify the next proof, we recall that a Markuševič basis $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ for a Banach space X is a biorthogonal system for which $(x_\gamma)_{\gamma\in\Gamma}$ generates a dense subspace of X and $(f_\gamma)_{\gamma\in\Gamma}$ generates a weak* dense subspace of X^*. As observed in [45], [54] and [84], the existence of a PRI together with a routine transfinite induction argument implies that any WCG space has a Markuševič basis. The following paraphrases the main result of [4]. Further information about Markuševič bases appears in §7.

THEOREM 5.1. [4] *Every Eberlein compact embeds in* $c_0(\Gamma)$, *for suitable* Γ, *and hence is a Corson compact.*

Proof. Obviously any Eberlein compact is a generating set for some WCG space X. Let $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ be a Markuševič basis for X. We can suppose without loss of generality that $(f_\gamma)_{\gamma\in\Gamma}$ is bounded. (In fact, the basis can be chosen so that $(x_\gamma)_{\gamma\in\Gamma}$ is also bounded [76] or [77, Theorem 2], but the proof of this is harder.) Consideration of the map $x \mapsto (f_\gamma(x))_{\gamma\in\Gamma}$ then shows that our Eberlein compact is weakly homeomorphic to a subset of $c_0(\Gamma)$ and thus to a subset of $\Sigma(\Gamma)$. ∎

The converse is false; for some counterexamples and references, see [24, §8.4] or [50, §1]. This leads to a proper generalization of Theorem 4.4: the conclusion is valid under the weaker hypothesis that the dual ball of the Banach space under consideration is a Corson compact. One just has to show that such spaces have projectional generators: see [24, §8.3] or [17].

Now we introduce a short sequence of weaker topological properties which will be useful later. Some of them refer specifically to the convex structure of the dual ball. Others make sense for any compact (or just Hausdorff) space.

Recall that angelic spaces [22] are those regular Hausdorff spaces for which every relatively countably compact subset is relatively compact and sequentially closed. This is a bit of a mouthful; for us, it suffices to know that a compact Hausdorff space is angelic if and only if it is a Fréchet-Urysohn space, i.e. the closure of any subset coincides with its sequential closure. It is a routine exercise to show that every Corson compact is angelic. An example of an angelic space which is not a Corson compact is the unit ball of JT^{**} in its weak* topology. (See [27, §3.c] for everything you need to know but were afraid to ask about the James tree space JT.) According to [68], if X is any separable Banach space not containing ℓ_1, then the unit ball of X^{**} is weak* angelic. (Of course, the converse is also true.) Note that JT^* does not have the separable complementation property.

A topological space has countable tightness if for every subset A and for all f in the closure of A, there exists a countable $A_0 \subset A$ whose closure contains f. Obviously every compact angelic space is sequentially compact and has countable tightness.

A topological space is said to be *sequential* if, for every non-closed subset S, there is sequence in S which converges to a point outside S. Although we will not consider this property in subsequent sections, it does fit into the scheme of things at this point. It is easy to show that a compact angelic space is sequential and that a compact sequential space is sequentially compact; both converses are false. For example the ordinal interval $[0, \omega_1]$ is scattered (i.e. every non-empty subset has an isolated point), and hence it is sequentially compact, but it is obviously not sequential. Nor does it have countable tightness, and hence for two reasons it is not angelic. The following well known example shows that countable tightness does not imply angelicity.

EXAMPLE 5.2. There is a compact, sequential Hausdorff space with countable tightness which is not angelic.

Proof. An application of Zorn's Lemma shows that there is a collection $(N_\gamma)_{\gamma\in\Gamma}$ of infinite subsets of the integers, maximal with respect to the following property: The intersection of any two is finite. A standard diagonal argument shows that Γ cannot be countable.

Now we make $\mathbb{N} \cup \Gamma$ into a locally compact Hausdorff space, as in [2, Esp. A_6]. Every $n \in \mathbb{N}$ is an isolated point. For $\gamma \in \Gamma$, a set S is a neighborhood of γ if and only if $\gamma \in S$ and $N_\gamma \setminus S$ is finite.

Now let K be the one point compactification of $\mathbb{N} \cup \Gamma$. It is not hard to check that K has countable tightness. Being scattered, K is sequentially compact. However it is not angelic. The point at infinity is certainly in the closure of the countable set $\mathbb{N}$. Let A be any infinite subset of $\mathbb{N}$, considered as a sequence. By maximality, there is an index γ for $A \cap N_\gamma$ is infinite. This implies that the sequence A has a subsequence which converges to γ. Thus no sequence from $\mathbb{N}$ converges to ∞.

This argument shows that every sequence in $\mathbb{N}$ has a subsequence which converges to a point in Γ. Since every sequence in Γ converges to ∞, a moment's reflection shows that K is sequential. ∎

Moore and Mrówka [63] asked whether every compact Hausdorff spaces with countable tightness is sequential. Fedorčuk [26] and Ostaszewski [71], under some additional set theoretical axioms, independently constructed counterexamples to this. Ostaszewski's example was the one-point compactification of a locally compact, perfectly normal, countably compact space. In Fedorčuk's space, every subset is separable, yet every closed infinite subset has cardinality $2^{\mathfrak{c}}$. On the other hand, there are models of set theory in which compact and countably tight implies sequential. See [7] for a proof and some references. Thus, the Moore-Mrówka question is undecidable in ZFC.

Now we recall some convex versions of these properties.

Say that a Banach space X has property $(\mathcal{E})$ if for every bounded convex set $A \subset X^*$ and for all f in the weak* closure of A, there exists a sequence in A which coverges weak* to f. This was probably first considered in [23]. It is clear that if the unit ball of X^* is weak* angelic, then X has $(\mathcal{E})$. This will be a particularly interesting property for us in the sequel.

Following Pol [82], we say that a subspace F of X^* has the property (C*) if for every bounded convex set $A \subset F$ and for all f in the weak* closure of A, there exists a countable subset $A_0 \subset A$ whose weak* closure contains f. (More precisely, perhaps we should say that the pair (F, X^*) has the property (C*).) It is clear that X^* has the property (C*), if either the ball of X^* has weak* countable tightness, or if X has $(\mathcal{E})$.

Corson [16] was the first to consider the following property: every family of closed convex sets with empty intersection has a countable subfamily with empty intersection. Pol [82] named this property (C), and introduced the property (C*). Obviously every weakly Lindelöf Banach space has property (C). A fairly routine separation argument [82, p. 147] shows that if X has (C), then X^* has (C*). The converse of this, which is also due to Pol [82, p. 147], is more difficult.

Pol ([82, p. 145] or [15, §4.18]) showed that (C) is a 3-space property. Thus the Johnson-Lindenstrauss space $J\!L$ (see [46] or [99]) has (C), although its dual ball is not weak* angelic. (It contains the example K constructed above.) It remains unknown whether $J\!L$ has $(\mathcal{E})$, let alone whether (C) implies $(\mathcal{E})$.

Godefroy and Talagrand [31] showed that if X is a so-called representable Banach space, then the unit ball of X^* is weak* angelic if and only X has (C).

Finally, say that X has property (C) with respect to subspaces, in short (CS), if for every norm closed subspace $F \subset X^*$, any weak* limit point of ball(F) is in the weak* closure of some countable subset of ball(F). Obviously (C) implies (CS).

Although it will not be considered again in this paper, we recall that the Lindelöf property for the weak topology of a Banach space has long been of interest. Talagrand [92] (or [24, §7.1]) was the first to show that WCG spaces are weakly Lindelöf, using some descriptive set theory. We now draw the reader's attention to an alternative proof of this [69], using more functional analytic methods, in particular the separable complementation idea.

As somebody famous probably once said, functional analysis is the study of weak topologies. The point is to choose the right weak topology to work with. For studying WCG spaces, the topology of uniform convergence on weakly compact sets, i.e. the Mackey-Arens topology, would seem to be appropriate. This choice led to a simple proof of the Amir-Lindenstrauss Theorem in [93] (=[17, Lemma VI.2.4]), which was the simplest proof available in the roman alphabet. When we are interested only in one weakly compact set, then the topology of uniform convergence on that set would seem to be more appropriate; this led to the simple proof in §3. Since we are interested in the separable complementation property, the following definition seems appropriate.

If $\langle X, Y\rangle$ is a dual pairing of normed spaces, denote by $\gamma(X,Y)$ the topology (on X) of uniform convergence on bounded countable subsets of Y [69]. Of course this coincides with the topology of uniform convergence on bounded $\sigma(Y,X)$ separable subsets of Y. Since $\gamma(X,X^*)$ is stronger than the weak topology, it is obvious that any $\gamma(X,X^*)$-Lindelöf Banach space will be weakly Lindelöf.

As in [69], say that a Banach space X has the property $\mathcal{V}$ if its dual ball admits a continuous embedding $J : \mathrm{ball}(X^*) \to [0,1]^\Gamma$ in such a way that $J(\mathrm{ball}(F)) \in \Sigma(\Gamma)$ for some 1-norming subspace F. (We remark that this is equivalent to the existence of a countably 1-norming Markuševič basis. See §7 or [50] for further information.) It is obvious that a Banach space has $\mathcal{V}$ whenever the unit ball of its dual is (weak*) affinely homeomorphic to a Valdivia compact, and the homeomorphism sends 0 to 0. This includes any WCG space. The converse is false, even for Asplund spaces: $C([0,\omega_1])$ is a simple counterexample. Since an arbitrary product of Valdivia compacts is a Valdivia compact, it is easy to check that $\mathcal{V}$ is preserved by arbitrarily large ℓ_1 products. Thus any abstract L space, or any $L_1(\mu)$ space, has $\mathcal{V}$. (The same argument was used in [81] to show that $L_1(\mu)$ has a 1-norming Markuševič basis.)

THEOREM 5.3. [69, Theorem A] *If a Banach space X has the property $\mathcal{V}$, then X is $\gamma(X,F)$-Lindelöf and so is X^n for all n. (Here F of course is the 1-norming subspace of X^*.)*

We mention this result because it uses separable complemented subspaces to construct a countable subcover. We refer the reader to [69] for the full details.

We do not mean to suggest that $\mathcal{SCP}$ by itself implies that a given Banach space is Lindelöf in some relevant topology: $\ell_1(\Gamma)$ for Γ uncountable is a well known counterexample. For each countable $A \subset \Gamma$, let $C_A = \{x \in \ell_1(\Gamma) : \Sigma_{\gamma \notin A} x(\gamma) < 1\}$. Then the sets C_A form a cover of $\ell_1(\Gamma)$ by open half-spaces, which admits no countable subcover. Although $\ell_1(\Gamma)$ obviously has $\mathcal{SCP}$, its dual ball (with the weak* topology) fails most of the properties discussed above.

6. Bigger complemented subspaces

Recall that if we have two cardinal numbers $\mathfrak{k} \leq \mathfrak{m}$, then a Banach space has the property $\mathcal{CP}(\mathfrak{k}, \mathfrak{m})$, if every subspace with density character at most $\mathfrak{k}$ is contained in a complemented subspace with density character at most $\mathfrak{m}$. First we consider $\mathcal{N} = \mathcal{CP}(\aleph_0, \mathfrak{c})$. Three cases are known, of sufficient conditions for a Banach space to have $\mathcal{N}$:

1. Theorem 3.5 shows that all dual spaces have $\mathcal{N}$. We remark that the proof of this in [100] is in the spirit of the original work of Amir and Lindenstrauss. The original proof in [41] was more complicated. This result has resisted all attempts at proof by the simpler type of argument used in §4.
2. Gulko [36] showed that if K is an extremally disconnected compact Hausdorff space, then $C(K)$ has $\mathcal{N}$. We give a slight simplification of his proof later in this section.
3. If the unit ball of X^* is weak* angelic, then X has $\mathcal{N}$ [80, Proposition 5]. The proof was in the style used already in this note. The same conclusion was announced in [36] for the case when the unit ball of X^* only has weak* countable tightness. The proof was more topological, establishing a result about weak* continuous retractions in the dual ball, and then deducing the projections from this. However Gulko has kindly informed us that there is a gap in his argument, and that the proof is valid only under the assumption that the unit ball of X^* is weak* angelic. We now present two generalizations of this.

Theorem 6.1. *Let X be a Banach space with property $(\mathcal{E})$ (in particular, with weak* angelic dual ball), let Y be a subspace of X, F a subspace of X^*, $\mathfrak{m} = (\max\{\operatorname{dens} Y, \operatorname{dens} F\})^{\aleph_0}$. Then there exists a norm one projection P on X such that $P(X) \supset Y$, $P^*(X^*) \supset F$ and $\operatorname{dens} P^*(X^*) \leq \mathfrak{m}$.*

Proof. We will construct, by (relatively short) transfinite induction, sequences of closed subspaces $Z_\alpha \subset X$ and $F_\alpha \subset X^*$, for $1 \le \alpha \le \omega_1$, such that

1) $Z_1 = Y$ and $F \subset F_1$,
2) $Z_\beta \subset Z_\alpha$ and $F_\beta \subset F_\alpha$ whenever $\beta < \alpha$
3) F_α norms Z_α and $Z_{\alpha+1}$ norms F_α
4) the weak* closure of $\mathrm{ball}(F_\alpha)$ is contained in $F_{\alpha+1}$
5) $\mathrm{dens}\, F_\alpha \le \mathfrak{m}$ and $\mathrm{dens}\, Z_\alpha \le \mathfrak{m}$.

The base case is easy. Lemma 3.1 gives us a subspace F_1 of X which norms $Z_1 = Y$.

For the inductive step, first suppose that α is not a limit ordinal. Again, Lemma 3.1 guarantees us a subspace Z_α of X with $\mathrm{dens}\, Z_\alpha \le \mathfrak{m}$ which norms $F_{\alpha-1}$. Since any enlargement of a norming subspace is still norming, we assume without loss of generality that Z_α contains $Z_{\alpha-1}$. Thanks to $(\mathcal{E})$, every point in $\overline{\mathrm{ball}(F_{\alpha-1})}^{w^*}$ is the limit of some weak* convergent sequence in $\mathrm{ball}(F_{\alpha-1})$, and thus $|\overline{\mathrm{ball}(F_{\alpha-1})}^{w^*}|$ is at most $(\mathrm{dens}\, F_{\alpha-1})^{\aleph_0} \le \mathfrak{m}^{\aleph_0} = \mathfrak{m}$. Thus Lemma 3.1 gives us a subspace F_α containing $\overline{\mathrm{ball}(F_{\alpha-1})}^{w^*}$ which norms Z_α and has density character at most $\mathfrak{m}$.

If α is a limit ordinal, just define $Z_\alpha = \overline{\bigcup_{\beta<\alpha} Z_\beta}$ and $F_\alpha = \overline{\bigcup_{\beta<\alpha} F_\beta}$.

Finally put $Z = \bigcup_{\alpha<\omega_1} Z_\alpha$ and $F = \bigcup_{\alpha<\omega_1} F_\alpha$. Note that Z is automatically closed, because for every convergent sequence therein, with say $x_n \in Z_{\alpha_n}$, the sequence α_n is eventually constant. Similarly F is weak* closed because its unit ball contains all the weak* cluster points of its countable subsets. Clearly F and Z norm one another and have density character at most $\mathfrak{m}$. Lemma 3.1 gives us the required projection from X onto Z. ∎

Since $\mathrm{dens}\, X^* \le 2^{\mathrm{dens}\, X}$ for every Banach space X, Theorem 3.5 shows that every dual space has $\mathcal{CP}(\mathfrak{k}, 2^{\mathfrak{k}})$ for every cardinal $\mathfrak{k}$. Theorem 6.1 improves this for certain Banach spaces. It shows that if X has $(\mathcal{E})$, then both X and X^* have $\mathcal{CP}(\mathfrak{k}, \mathfrak{k}^{\aleph_0})$ for every cardinal $\mathfrak{k}$. (Note that $2^{\mathfrak{k}} \ge \mathfrak{k}^{\aleph_0}$ always.) In particular, if X has $(\mathcal{E})$, then both X and X^* have $\mathcal{N}$ and $\mathcal{CP}(\mathfrak{c}, \mathfrak{c})$. For (CS) we have a weaker result.

THEOREM 6.2. *Let X be a Banach space with property (CS)* (*in particular, whose dual ball has weak* countable tightness*), *Y a subspace of X, F a subspace of X^*, $\mathfrak{m} = (\max\{2^{\mathfrak{c}}, \mathrm{dens}\, Y, \mathrm{dens}\, F\})^{\aleph_0}$. Then there exists* *a norm one projection P on X such that $P(X) \supset Y$, $P^*(X^*) \supset F$ and* $\operatorname{dens} P^*(X^*) \leq \mathfrak{m}$.

Proof. The only point at which the previous proof fails under this weaker hypothesis is the estimation of $|\overline{\mathrm{ball}(F_{\alpha-1})}^{w^*}|$. Since each point in $\overline{\mathrm{ball}(F_{\alpha-1})}^{w^*}$ is merely a weak* cluster point of some countable subset of $\mathrm{ball}(F_{\alpha-1})$, the previous estimate needs to be multiplied by the number of cluster points that a countable subset of a compact space might have. This maximal correction factor is clearly $|\beta\mathbb{N}| = 2^{\mathfrak{c}}$ [97, p. 149, Ex. 108]. ∎

Thus if X has property (C), in particular if X is weakly Lindelöf, then both X and X^* have $\mathcal{CP}(\mathfrak{k}, 2^{\mathfrak{c}}\mathfrak{k}^{\aleph_0})$ for every cardinal $\mathfrak{k}$. For small cardinals, Theorem 3.5 gives a better result for dual spaces, but for large cardinals, Theorem 6.2 is an improvement for the smaller class to which it applies. It is clear that replacing "countable tightness" by "tightness at most $\mathfrak{k}$" will lead to more results of this sort, but we think it is more important to decide whether every Banach space has property $\mathcal{N}$.

Since some Banach spaces fail the $\mathcal{SCP}$, the Löwenheim-Skolem Theorem suggests that for every infinite cardinal $\mathfrak{k}$, there will be a Banach space failing $\mathcal{CP}(\mathfrak{k}, \mathfrak{k})$. Let us give a simple argument which shows the existence of such a Banach space, for arbitrarily large cardinals $\mathfrak{k}$.

Example 6.3. For any cardinal $\mathfrak{k}_0$, there is a cardinal $\mathfrak{m} \geq \mathfrak{k}_0$ for which the Banach space $\ell_\infty(\mathfrak{m})$ fails $\mathcal{CP}(\mathfrak{m}, \mathfrak{m})$.

Proof. Define a sequence of cardinals by $\mathfrak{k}_{n+1} = 2^{\mathfrak{k}_n}$ and then put $\mathfrak{m} = \sup \mathfrak{k}_n$. The point of this construction is that for every cardinal $\mathfrak{k} < \mathfrak{m}$, we also have $2^{\mathfrak{k}} < \mathfrak{m}$.

Let S be a set of cardinality $\mathfrak{m}$, $X = \ell_\infty(S)$ and let Y be the closure of the subspace of functions whose supports have cardinality strictly less than $\mathfrak{m}$. It is easily checked that $\operatorname{dens} Y = \mathfrak{m}$. An ancient result of Sierpinski [85] implies that there is an index set Γ of cardinality strictly greater than $\mathfrak{m}$ and a family $\{N_\gamma : \gamma \in \Gamma\}$ of subsets of S, each of which has cardinality $\mathfrak{m}$, and for which the intersection of any two has cardinality strictly less than $\mathfrak{m}$. Considering the characteristic functions of these subsets, we check that X/Y has a subspace isometric to $c_0(\Gamma)$. If $X = A \oplus B$, where $Y \subset A$, then

$$w^*\operatorname{dens}(A/Y)^* \leq \operatorname{dens} A/Y \leq \operatorname{dens} A \text{ and } w^*\operatorname{dens} B^* \leq w^*\operatorname{dens} X^* = \mathfrak{m}.$$

Since $c_0(\Gamma)$ is isometric to a subspace of $X/Y \cong (A/Y) \oplus B$, we get

$$\mathfrak{m} < |\Gamma| = w^*\text{dens}\,(c_0(\Gamma))^* \leq w^*\text{dens}\,(A/Y)^* + w^*\text{dens}\,B^* \leq \text{dens}\,A + \mathfrak{m}.$$

This forces $\text{dens}\,A > \mathfrak{m}$ and so X fails $\mathcal{CP}(\mathfrak{m}, \mathfrak{m})$. ∎

With practically the same argument, the Generalized Continuum Hypothesis implies that for all $\mathfrak{m}$, $\ell_\infty(\mathfrak{m})$ fails $\mathcal{CP}(\mathfrak{m}, \mathfrak{m})$.

As we mentioned, another positive result is due to Gulko, who showed [36] that if K is an extremally disconnected compact Hausdorff space, then $C(K)$ has $\mathcal{N}$. His argument boils down to the topological result, Lemma 6.6 below, of which we give a short proof. Of course our arguments are completely standard, and the next two lemmas are special cases of well known results [97, Chap. 14].

Recall that a continuous surjection $f : K \to L$ from one Hausdorff space onto another is said to be *minimal* if $f(S)$ is a proper subset of L, for every proper closed subset S of K. We will say that two sets are almost disjoint if neither meets the interior of the other.

LEMMA 6.4. *Let $f : K \to L$ be a continuous surjection.*

(i) *If K is compact, then there is a closed subset $S \subset K$ for which $f|S$ is minimal.*

(ii) *If f is minimal and A, B are disjoint closed subsets of K, then $f(A)$ and $f(B)$ are almost disjoint.*

Proof. (i) Routine application of Zorn's Lemma.

(ii) Fix $x \in B$ and let N be an open neighborhood of $f(x)$. We need to show that N is not a subset of $f(A)$. Now $C = A \cup f^{-1}(N)$ is obviously closed and does not contain x. By minimality, there exists a point $y \in L \setminus f(C)$. Clearly $y \in N \setminus f(A)$. ∎

LEMMA 6.5. *Let K, L be extremally disconnected compact spaces, and let $f : K \to L$ be a minimal surjection. Then f is a homeomorphism.*

Proof. We first show that if A is any clopen subset of K, then $f(A)$ is also clopen. In this case, $B = A^c$ will also be closed, and thus $L = f(A) \cup f(B)$ is the union of two closed subsets. Then $f(B)^c$ is an open subset of $f(A)$, and extremal disconnectedness implies that $\overline{f(B)^c}$ is contained the interior of $f(A)$. In other words, $(\text{int}f(B))^c \subset \text{int}f(A)$. Lemma 6.4(ii) implies that $f(A) \subset (\text{int}f(B))^c$. Combining these two inclusions, we see that $f(A)$ is open.

It follows immediately that f sends disjoint clopen sets to disjoint clopen sets. As the topology of K has a basis of clopen sets, f is one-to-one. ∎

LEMMA 6.6. *Let K be compact and extremally disconnected, M a metric space, $f : K \to M$ continuous. Then f factors through $\beta\mathbb{N}$.*

Proof. We assume without loss of generality that f is surjective.

First, consider the special case when $K = \beta\Gamma$ is the Stone-Čech compactification of a discrete set. Since $f(K)$ is separable, there is a countable set $N \subset \Gamma$, which we identify with $\mathbb{N}$, such that $f(N)$ is dense in M. Define $g : \beta\mathbb{N} \to M$ by $g(n) = f(n)$, extending of course by continuity. Define $h : \beta\Gamma \to \beta\mathbb{N}$ by $h(n) = n$ for $n \in \mathbb{N}$, and for $\gamma \in \Gamma \setminus \mathbb{N}$, choose any $x \in \beta\mathbb{N}$ with $g(x) = f(\gamma)$ and put $h(\gamma) = x$. Clearly f agrees with gh on Γ, hence everywhere.

The more general case when K is a retract of some $\beta\Gamma$ follows easily from this.

Finally, suppose merely that K is compact and extremally disconnected. Obviously there is a continuous surjection $g : \beta\Gamma \to K$ for some discrete set Γ. Choose $L \subset K$ so that $g|L$ is minimal. By Lemma 6.5, $g|L$ is a homeomorphism and thus K is homeomorphic to a retract of $\beta\Gamma$. ∎

THEOREM 6.7. *Let K be an extremally disconnected compact Hausdorff space. Then $C(K)$ has $\mathcal{N}$.*

Proof. Fix a separable subspace Y in $C(K)$. If A is the smallest closed subalgebra containing Y and the constant functions, then A is also separable, and thus isomorphic to the space of continuous functions on some compact metric space. More precisely, there is a compact metric space M, and a continuous surjection $f : K \to M$, such that A consists precisely of those functions which factor through f. By Lemma 6.6, we may write $f = gh$, where $g : \beta\mathbb{N} \to M$ and $h : K \to \beta\mathbb{N}$ are continuous surjections. Let B be the subalgebra of $C(K)$ consisting of those functions which factor through h. Clearly B contains A which contains Y. Since $B \cong \ell_\infty$, it is complemented and has density character equal to the continuum. ∎

It is well known that a Banach space is the range of a norm one projection on every superspace if and only if it is isometric to some $C(K)$ with K extremally disconnected. Thus Gulko's result has the following curious formulation. If a Banach space is 1-complemented in every superspace, then each of its separable subspaces is contained in a 1-complemented subspace which is not too big.

7. Going down

The role of c_0 in the theory of complemented subspace is peculiar. It is complemented in every separable superspace, yet not in ℓ_∞. So far, we have increased our supply of decomposable Banach spaces by showing, for example, that separable subspaces are contained in larger complemented subspaces. Here we note that decomposability sometimes follows by looking at smaller subspaces.

We are motivated by the result of Díaz and Fernández [18, Theorem 2.2] that if a real Banach space contains an isomorphic copy of c_0 but no copy of ℓ_1, then it contains a complemented copy of c_0. More precisely, they proved the following almost isometric version:

Theorem 7.1. *Let X be a real Banach space which does not contain an isomorphic copy of ℓ_1. Suppose that X has a subspace Y isometric to c_0 and that $\varepsilon > 0$. Then Y has a subspace Z isometric to c_0, which is the range of a projection on X with norm at most $1 + \varepsilon$.*

Their proof depended on a non-trivial result of Hagler and Johnson [37, Theorem 1(a)]. We would like to point out that a variation of their result follows from the following simple argument.

Theorem 7.2. *Let X be any Banach space for which the unit ball of the dual space is weak* sequentially compact. Suppose that X has a subspace Y isometric to c. Then Y has a subspace Z isometric to c, which is the range of a norm one projection on X.*

Proof. To simplify notation, we suppose that $Y = c$. Let f_n be a sequence of norm one functionals on X whose restrictions to Y are the coordinate evaluation functionals. By hypothesis, there is a subsequence f_{k_n} which converges weak* to some $f \in X^*$. If e denotes $(1, 1, 1, \ldots) \in c$, then $f_n(e) = 1$ for all n, and so $f(e) = 1$ also. In particular, $\|f\| = 1$.

Define $T : X \to c$ by $Tx = (f_{k_n}(x))_{n=1}^{\infty}$. Clearly this is a norm one operator with $T(e_{k_n}) = e_n$ for all n and $T(e) = e$. Let Z be the closed linear span of $\{e\} \cup \{e_{k_n} : n = 1, 2, \ldots\}$ and define the spreading operator $S : c \to Z$ by $(Sy)_k = y_n$ if $k = k_n$ for some n, and $(Sy)_k = \lim_{n\to\infty} y_n$ otherwise. Then $S(e_n) = e_{k_n}$ and $P = ST$ works. ∎

We learnt recently that c can be replaced by c_0 in this result [21, Theorem 6]. Their argument is almost as short as ours.

There are many Banach spaces satisfying the hypotheses of Theorem 7.2 but not those of Theorem 7.1. For example, every WCG space satisfies the hypothesis of Theorem 7.2; in this case actually every copy of c_0 is complemented. More generally, every weak Asplund space has sequentially compact dual ball [24, 2.1.2]. Hence so does every Banach space with an equivalent smooth norm [24, Corollary 4.2.5] or [39]. Our friend ℓ_1 falls into all of these categories.

An example of a Banach space which is not weak Asplund, but whose dual ball is weak* sequentially compact, is the space of continuous functions on the "split interval" space, often denoted $D[0,1]$.

Of course a Banach space containing $\ell_1(\Gamma)$ will not satisfy Theorem 7.2, if the cardinality of Γ is equal to the continuum. However, there are models of set theory in which the unit ball of $\ell_1(\aleph_1)^*$ is weak* sequentially compact. The sequential compactness of any compact Hausdorff space of cardinality strictly less than $2^{\mathfrak{c}}$ is a consequence of Martin's axiom [60]. So this conclusion for $\ell_1(\Gamma)^*$ holds and is not vacuous for $|\Gamma| < \mathfrak{c}$, if we assume Martin's Axiom and the negation of the Continuum Hypothesis.

An easy consequence of Rosenthal's Theorem is that if X^* does not contain ℓ_1, then its unit ball is weak* sequentially compact. However this does not help us, because this hypothesis implies that X contains no copy of c_0, and the conclusion of Theorem 7.2 is then vacuous.

We must confess that there are Banach spaces to which Theorem 7.1 is applicable but Theorem 7.2 is not. Hagler and Odell [38] constructed a Banach space not containing ℓ_1, yet whose dual ball is not weak* sequentially compact. However this anomaly does not arise in Banach lattices, thanks to the following result [58]. For proofs and generalizations, see [59], [30, Theorem 7] or [13, Theorem 4.1].

THEOREM 7.3. *A Banach lattice is an Asplund space if and only if it contains no subspace isomorphic to* ℓ_1.

The title of [18] suggests that its main result is the following.

PROPOSITION 7.4. [18, Theorem 3.1] *A Banach lattice is reflexive if* (*and only if*) *it contains no subspace isomorphic to* ℓ_1 *and no complemented subspace isomorphic to* c_0.

We now see that this result can be proved without appealing to [37].

8. Big sequences of projections

Let us call a uniformly bounded increasing transfinite sequence of projections $(P_\alpha)_{1\le\alpha\le\alpha_0}$ a *weak bounded projectional resolution* (or weak BPR) if α_0 is the first ordinal with cardinality $\operatorname{dens} X$, P_{α_0} is the identity operator, and for all α,

$P_\alpha(X)$ has density character at most $|\alpha|$, for $\alpha \ge \omega_0$,

$P_\alpha(X) = \overline{\bigcup_{\beta<\alpha} P_\beta(X)}$ whenever α is a limit ordinal.

By increasing, we mean of course that $P_\alpha(X) \subset P_\beta(X)$ whenever $\alpha < \beta$. The following trivial observation will be used several times.

Lemma 8.1. *If $(P_\alpha)_{1\le\alpha\le\alpha_0}$ is a weak BPR for a Banach space X, and $\operatorname{dens} X$ is a regular cardinal, then $X = \bigcup_{\alpha<\alpha_0} P_\alpha(X)$.*

In particular, a Banach space of density character $\aleph_1$ with a weak BPR necessarily has the $\mathcal{SCP}$. For if (P_α) is a weak bounded projectional resolution for such a Banach space Y, and E is any separable subspace of Y, then for some countable ordinal α, $P_\alpha(Y)$ must contain E.

We will define the projection constant of $(P_\alpha)_{1\le\alpha\le\alpha_0}$ as $\sup_\alpha \|P_\alpha\|$. We define a weak λ-BPR as any weak bounded projectional resolution with projection constant not exceeding λ.

In case the projections also commute, we will speak of a bounded projectional resolution and a λ-BPR respectively.

Obviously a traditional PRI is simply a 1-BPR. It follows from Theorem 4.6 [25] that every dual space with RNP has a PRI. Later we will show that a Banach space with just the RNP need not have a BPR. It follows easily from [30, Theorem 5] or [57, p. 9] that any order continuous Banach lattice has a 2-BPR. Obviously admitting a BPR is a property which is invariant under renorming. However, admitting a PRI is not. Probably the simplest counterexample for this is the Banach space $C_0[0,\omega_1]$ of continuous functions vanishing at ω_1 [17, p. 289]. We do not know whether a subspace of $\ell_1(\aleph_1)$ can provide an example. However, we can show that renormings of $\ell_1(\aleph_1)$ provide further counterexamples. More precisely, we will show that for any constant $k > 1$, the space $\ell_1(\Gamma)$ (for a suitable set Γ) has an equivalent norm under which every bounded projectional resolution has projection constant at least k. This answers a question posed by M. Fabian in Jarandilla. (It is conceivable that on this space, every projection onto an infinite dimensional separable subspace has norm at least k, but so far we have been unable to prove this.)

As we have noted several times, there is an uncountable collection of infinite subsets of the integers, with the property that the intersection of any two is finite. For the construction which follows, we need a family with slightly stronger combinatorial properties. Fix a regular cardinal number $\mathfrak{m}$, not greater than $\mathfrak{c}$, and let ω denote the first ordinal number with cardinality $\mathfrak{m}$. (For simplicity, we could take $\mathfrak{m} = \aleph_1$, but the proof would be just the same.)

LEMMA 8.2. *There is a countable set D and a collection of distinct infinite subsets Θ_α of D, where $1 \leq \alpha < \omega$, such that*

(i) *if A is an uncountable collection of ordinals and $\alpha_1, \ldots, \alpha_m$ are not in A (all $< \omega$), then there is an infinite set $A_1 \subset A$ such that*

$$\bigcap_{\alpha \in A_1} \Theta_\alpha \setminus \bigcup_{k=1}^{m} \Theta_{\alpha_k} \neq \emptyset$$

(ii) *given finitely many distinct indices $\alpha_1, \ldots, \alpha_n$, all of them $\geq \omega_0$, there exist indices $\beta_1, \ldots, \beta_n$, all $< \omega_0$, such that the $2n$ sets $\Theta_{\alpha_k} \setminus \Theta_{\beta_k}$ and $\Theta_{\beta_k} \setminus \Theta_{\alpha_k}$ are pairwise disjoint.*

Proof. We define D as the dyadic tree. This is undoubtedly countable. The collection of all branches has cardinality $\mathfrak{c}$; we choose a subset of cardinality $\mathfrak{m}$, subject to the following provision, and label it as $\{\Theta_\alpha : 1 \leq \alpha < \omega\}$. The collection of branches which, from some point onwards, turn only to the left is clearly countable; the provision is that all of them belong to the subset we choose, and that they are labelled as $\{\Theta_\alpha : 1 \leq \alpha < \omega_0\}$.

(i) Suppose that every node $\nu \notin \bigcup_{k=1}^{m} \Theta_{\alpha_k}$ lies in Θ_α for only finitely many $\alpha \in A$. This implies that all but countably many of the branches Θ_α are contained in $\bigcup_{k=1}^{m} \Theta_{\alpha_k}$, which is a tall order for uncountably many branches to achieve.

(ii) Evidently, from some level m onwards, the corresponding segments of the branches Θ_{α_k} are disjoint. For each k, consider the branch which coincides with Θ_{α_k} up to level m, and thereafter turns always to the left. By construction, this branch must be Θ_{β_k} for some $\beta_k < \omega_0$. ∎

We denote by y_α the characteristic function of Θ_α, and by Y the subspace of $\ell_\infty(D)$ generated by these functions. We could consider Y as the space $C(D)$ of continuous functions on D, suitably topologized, but we do not need to.

THEOREM 8.3. *For* $\mathfrak{m}$ *as above and any integer* $n \in \mathbb{N}$, *the space* $\ell_1(\mathfrak{m})$ *has an equivalent norm under which every weak bounded projectional resolution* (P_α) *has* $\liminf \|P_\alpha\| \geq \frac{1}{3}(n-2)$.

Proof. Write e_α, $\alpha < \omega$, for the canonical basis of $\ell_1(\mathfrak{m})$. The operator $T : \ell_1(\mathfrak{m}) \to Y$, $T(e_\alpha) = y_\alpha$ is obviously well defined and bounded. We will define an equivalent norm on $\ell_1(\mathfrak{m})$ by

$$|||x||| = \max\{n^{-1}\|x\|, \|Tx\|\}.$$

Let (P_α) be any weak bounded projectional resolution for $\ell_1(\mathfrak{m})$. It follows from the argument of [78, Lemma 1] that there exist arbitrarily large indices α_0, for which the closed linear span of $\{e_\beta : \beta < \alpha_0\}$ coincides with the range of P_{α_0}. To be precise, let $\beta_0 < \omega$ be chosen arbitrarily. (In particular, we may choose $\beta_0 \geq \omega_0$.) We construct a strictly increasing sequence of ordinals β_n as follows. If n is even, choose $\beta_{n+1} > \beta_n$ so that the closed linear span of $\{e_\beta : \beta < \beta_{n+1}\}$ contains the range of P_{β_n}. If n is odd, choose $\beta_{n+1} > \beta_n$ so that $P_{\beta_{n+1}}(X)$ contains the closed linear span of $\{e_\beta : \beta < \beta_n\}$. In both cases, this is possible by Lemma 8.1. Taking $\alpha_0 = \lim \beta_n$ completes this argument.

Let U be the linear span of $\{e_\beta : \beta < \alpha_0\}$ and let $V = \ker P_{\alpha_0}$. Obviously $\ell_1(\mathfrak{m}) = \overline{U} \oplus V$. It suffices to show that $|||P_{\alpha_0}|||$ is large, and to do this we will show that the unit spheres of U and V are close.

Let S be a dense subset of U with cardinality $|\alpha_0|$. Then for any $\alpha > \alpha_0$, it is trivial that $e_\alpha \in \overline{U} + V$, so there exists an element $w = w_\alpha \in S$ such that $d(e_\alpha + w, V) < n^{-1}$. Of course this distance is calculated with respect to the new norm. Since the cardinality of $[\alpha_0, \omega)$ equals $\mathfrak{m}$ and the cardinality of S is strictly less than $\mathfrak{m}$, there must be an uncountable set $A \subset [\alpha_0, \omega)$ and an element $w \in S$ such that

$$d(e_\alpha + w, V) < n^{-1} \text{ for } \alpha \in A.$$

The element w must have the form $w = \sum_{k=1}^m \lambda_k e_{\gamma_k}$ for some ordinals $\gamma_k < \alpha_0$ and real numbers λ_k.

Let $A_1 \subset A$ and $\nu \in \bigcap_{\alpha \in A_1} \Theta_\alpha \setminus \bigcup_{k=1}^m \Theta_{\gamma_k}$ be given by Lemma 8.2(i). Since A_1 is infinite, it contains at least n elements $\{\alpha_1, \ldots \alpha_n\}$. Let $\{\beta_1, \ldots, \beta_n\}$ be the corresponding set of ordinals given by Lemma 8.2(ii). It is easily checked that

$$\|\sum_{k=1}^n y_{\alpha_k} - \sum_{k=1}^n y_{\beta_k}\| \leq 1.$$

Put $x = \sum_{k=1}^{n} e_{\alpha_k} + nw$. Then

$$\begin{aligned} d(x,U) &\leq |||x - nw - \sum_{k=1}^{n} e_{\beta_k}||| \\ &= \max\left\{\frac{1}{n}\|x - w - \sum_{k=1}^{n} e_{\beta_k}\|, \|T(x - w - \sum_{k=1}^{n} e_{\beta_k})\|\right\} \\ &= \max\left\{\frac{1}{n}\|\sum_{k=1}^{n} e_{\alpha_k} - \sum_{k=1}^{n} e_{\beta_k}\|, \|\sum_{k=1}^{n} y_{\alpha_k} - \sum_{k=1}^{n} y_{\beta_k}\|\right\} \\ &\leq \max\{2,1\} = 2 \end{aligned}$$

and

$$\begin{aligned} |||x||| \geq \|Tx\| &= \|\sum_{k=1}^{n} y_{\alpha_k} + n\sum_{k=1}^{m} \lambda_k y_{\gamma_k}\| \\ &\geq (\sum_{k=1}^{n} y_{\alpha_k} + n\sum_{k=1}^{m} \lambda_k y_{\gamma_k})(\nu) = \sum_{k=1}^{n} y_{\alpha_k}(\nu) + 0 = n. \end{aligned}$$

Also, $d(x,V) \leq \sum_{k=1}^{n} d(e_{\alpha_k} + w, V) < 1$.

Finally if $u \in U$ and $v \in V$, then $|||u||| \leq |||P_{\alpha_0}||| |||u - v|||$, which gives the required estimate. ∎

Kalenda [49, Theorem 2] has recently shown that if $|\Gamma|$ is a regular cardinal, then $\ell_1(\Gamma)$ has an equivalent norm under which there is no projectional resolution of the identity. His proof does not yield the uniform bound for the projection constant given above. He uses a different argument, obtaining the result as a corollary of deeper work concerning Valdivia compact dual balls.

Replacing D in the preceding argument by a tree of uncountable height, it is clear that we will find arbitrarily large cardinals $\mathfrak{m}$ for which the conclusion of the theorem remains valid. Whether the proof will work for any cardinal number whatsoever is not clear, and we have not bothered to check the transfinite arithmetic. The corollary below is probably more interesting.

LEMMA 8.4. *Let X be a Banach space of density character $\aleph_1$ which has a λ-BPR. If Q is any projection on X, then its range $Q(X)$ has a weak $\lambda\|Q\|$-BPR.*

Proof. Write $Y = Q(X)$, and let $(P_\alpha)_{\alpha<\omega_1}$ be the given BPR on X. The important thing is that there are arbitrarily large indices α for which $QP_\alpha|_Y$

is a projection on Y. To see this, fix α_1, and inductively define a strictly increasing sequence α_n so that $QP_{\alpha_n}(X)$ is contained in $P_{\alpha_{n+1}}(X)$. This is always possible, by Lemma 8.1 and the separability of $QP_{\alpha_n}(X)$. If $\alpha = \lim \alpha_n$, then it is clear for each n that $QP_{\alpha_n}(X) \subset P_\alpha(X)$, whence $P_\alpha QP_{\alpha_n} = QP_{\alpha_n}$. Taking the pointwise limit, we see that $P_\alpha QP_\alpha = QP_\alpha$ and so $QP_\alpha|_Y$ is a projection on Y.

The condition $P_\alpha QP_{\alpha_1} = QP_{\alpha_1}$ above makes it easy to construct a weak BPR on Y by transfinite induction. Clearly $\|QP_\alpha|_Y\| \leq \lambda\|Q\|$. ∎

COROLLARY 8.5. *There exists a Banach space X with the Radon-Nikodým Property but without any bounded projectional resolution.*

Proof. We know that for each integer n, there is a Banach space X_n, isomorphic to $\ell_1(\aleph_1)$, without any weak n-BPR. If X is the ℓ_1 sum of these spaces, it is clear from the preceding Lemma that X will have no BPR. It is well known that each X_n will have the Radon-Nikodým Property [19], and easily proved that X does too. ∎

Finally, a few more words about Markuševič bases. As noted in §4, the existence of a PRI in every WCG space guarantees the existence of a Markuševič basis. This can be generalized easily. In [17, p. 286] the property $\mathcal{P}$ is defined inductively as follows:

(1) Separable spaces have $\mathcal{P}$.

(2) If X has a PRI (P_α) and every $(P_{\alpha+1} - P_\alpha)(X)$ has $\mathcal{P}$, then X has $\mathcal{P}$.

Then, by the same argument, every X with $\mathcal{P}$ has a Markuševič basis. Further consequences of $\mathcal{P}$ can be found in [17] and [42]. (For example [17] any X with $\mathcal{P}$ has an equivalent locally uniformly convex norm.)

Conversely, one might ask what conclusion can be drawn from the existence of a Markuševič basis. In general, not much: there is a Banach space with a Markuševič basis but neither the $\mathcal{SCP}$, nor any BPR. Let $X = JL$ be the Johnson-Lindenstrauss space [46, Example 1], [99]. A minor modification to its definition allows us to assume that dens $JL = \aleph_1$, whilst preserving all other essential properties. Since JL^* is weak* separable, we may apply [79, Theorem 2] to find a Banach space Y with a Markuševič basis of cardinality $\aleph_1$ which contains X. As c_0 is not complemented in X, it cannot be complemented in Y either. It follows from Sobczyk's Theorem that Y fails the $\mathcal{SCP}$. As noted earlier, a Banach space of density character $\aleph_1$ with a BPR necessarily has the $\mathcal{SCP}$.

But with stronger hypotheses, a positive result holds. Recall from [77] that a Markuševič basis $(x_\gamma, f_\gamma)_{\gamma\in\Gamma}$ is said to be countably λ-norming if the collection of functionals $f \in X^*$ for which $\{\gamma : f(x_\gamma) \neq 0\}$ is countable forms a λ-norming subspace. (As noted, for $\lambda = 1$ this is equivalent to the property $\mathcal{V}$ defined in §4.) It was shown in [77, Theorem 1] that the existence of a countably λ-norming Markuševič basis implies the existence of a λ-BPR.

Acknowledgements

Part of this work was done at the Conference on Banach Spaces in Jarandilla. The first author would like to thank all of his Spanish colleagues for their hospitality during his visit to Spain. The second author's contribution to this work was supported by the Junta de Extremadura.

References

[1] Aiena, P., González, M., On inessential and improjective operators, *Studia Math.,* **131** 56(1998), 271–287.
[2] Alexandrov, P., Urysohn, P., Mémoire sur les espaces topologiques compacts, *Vehr. Kon. Akad. Wetensch. Amsterdam Afd. Natuurk.,* **14** (1929), 1–96.
[3] Amir, D., "Characterizations of Inner Product Spaces", Birkhäuser, Basel, 1986.
[4] Amir, D., Lindenstrauss, J., The structure of weakly compact sets in Banach spaces, *Ann. of Math.,* **88** (1968), 35–46.
[5] Argyros, S., *(Personal communication),* (1999).
[6] Argyros, S., Felouzis, V., Interpolating hereditarily indecomposable Banach spaces, *J. Amer. Math. Soc.,***13** (2000), 243–294.
[7] Balogh, Z., On compact Hausdorff spaces of countable tightness, *Proc. Amer. Math. Soc.,* **105** (1988), 755–764.
[8] Banach, S., "Théorie des Opérations Linéaires", Monografje Matematyczne, vol. 1, Warsaw, 1932.
[9] Banach, S., Mazur, S., Zur Theorie der linearen Dimension, *Studia Math.,* **4** (1933), 100–112.
[10] Bellenot, S., Somewhat quasireflexive Banach spaces, *Ark. Mat.,* **22** (1984), 175–183.
[11] Bohnenblust, F., Convex regions and projections in Minkowski space, *Ann. of Math.,* **39** (1938), 301–308.
[12] Bosznay, Á.P., Garay, B.M., On norms of projections, *Acta. Sci. Math. (Szeged),* **50** (1986), 87–92.
[13] Bukhvalov, A.V., Veksler, A.I., Lozanskii, G.Ya., Banach lattices: some Banach aspects of their theory, *Uspekhi Mat. Nauk,* **34.2** (1979), 137–183, English transl.: *Russian Math. Surveys,* **34.2** (1979), 159–212.
[14] Cabello Sánchez, F., Castillo, J.M.F., Yost, D., Sobczyk's Theorem from A to Z, *Extracta Math.,* (to appear).

[15] Castillo, J.M.F., González, M., "Three-space Problems in Banach Space Theory", Lecture Notes in Math. 1667, Springer Verlag, Berlin, 1997.
[16] Corson, H.H., The weak topology of a Banach space, *Trans. Amer. Math. Soc.,* **101** (1961), 1 – 15.
[17] Deville, R., Godefroy, G., Zizler, V., "Smoothness and Renormings in Banach Spaces", Pitman Monographs and Surveys in Pure and Applied Math. 64, Longman, Essex, 1993.
[18] Díaz, S., Fernández, A., Reflexivity in Banach lattices, *Archiv Math.,* **63** (1994), 549 – 552.
[19] Diestel, J., Uhl Jr., J.J., The Radon-Nikodým theorem for Banach space valued measures, *Rocky Mountain J. Math.,* **6** (1976), 1 – 46.
[20] Dixmier, J., Sur un théorème de Banach, *Duke Math. J.,* **15** (1948), 1057 – 1071.
[21] Dowling, P.N., Randrianantoanina, N., Turett, B., Remarks on James's distortion theorem II, *Bull. Austral. Math. Soc.,* **59** (1999), 515 – 522.
[22] Edgar, G.A., Measurability in a Banach space II, *Indiana Univ. Math. J.,* **28** (1979), 559 – 579.
[23] Efremov, N.M., A condition for a Banach space to be a dual (Russian), *Izv. Vyssh. Uchebn Zaved. Mat.,* **34.4** (1984), 11 – 13, English transl.: *Soviet Math. (Iz. VUZ),* **28:4** (1984), 12 – 14, MR 86e:46007.
[24] Fabian, M.J., "Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces", Wiley-Interscience, New York, 1997.
[25] Fabian, M., Godefroy, G., The dual of every Asplund space admits a projectional resolution of the identity, *Studia Math.,* **91** (1988), 141 – 151.
[26] Fedorčuk, V.V., On the cardinality of hereditarily separable compact Hausdorff spaces (Russian), *Dokl. Akad. Nauk SSSR,* **222** (1975), 302 – 305, English transl.: *Soviet Math. Dokl.,* **16** (1975), 651 – 655.
[27] Fetter, H., Gamboa de Buen, B., "The James Forest", London Math. Soc. Lecture Note Series 236, Cambridge Univ. Press, Cambridge, 1997.
[28] Fichtenholz, G., Kantorovitch, L., Sur les opérations linéaires dans l'espace des fonctions bornées, *Studia Math.,* **5** (1934), 69 – 98.
[29] Ghoussoub, N., Godefroy, G., Maurey, B., Schachermayer, W., "Some Topological and Geometrical Structures in Banach Spaces", Mem. Amer. Math. Soc. 378, Providence, 1987.
[30] Ghoussoub, N., Saab, E., On the weak Radon-Nikodým property, *Proc. Amer. Math. Soc.,* **81** (1981), 81 – 84.
[31] Godefroy, G., Talagrand, M., Espaces de Banach représentables, *Israel J. Math.,* **41** (1982), 321 – 330.
[32] Goldstein, S., Paszkiewicz, A., Linear combinations of projections in von Neumann algebras, *Proc. Amer. Math. Soc.,* **116** (1992), 175 – 183.
[33] Gowers, W.T., A Banach space not containing c_0, ℓ_1 or a reflexive subspace, *Trans. Amer. Math. Soc.,* **344** (1994), 407 – 420.
[34] Gowers, W.T., Maurey, B., The unconditional basic sequence problem, *J. Amer. Math. Soc.,* **6** (1993), 851 – 874.
[35] Gowers, W.T., Maurey, B., Banach spaces with small spaces of operators, *Math. Ann.,* **307** (1997), 543 – 568.

[36] GULKO, S.P., On complemented subspaces of Banach spaces of the weight continuum (Russian), *Ekhstremal'nye Zadachi Teor. Funkts.,* **8** (1990), 34 – 41.

[37] HAGLER, J., JOHNSON, W.B., On Banach spaces whose dual balls are not weak* sequentially compact, *Israel J. Math.,* **28** (1977), 325 – 330.

[38] HAGLER, J., ODELL, E., A Banach space not containing ℓ_1 whose dual ball is not weak* sequentially compact, *Illinois J. Math.,* **22** (1978), 290 – 294.

[39] HAGLER, J., SULLIVAN, F., Smoothness and weak* sequential compactness, *Proc. Amer. Math. Soc.,* **78** (1980) 497 – 503.

[40] HARMAND, P., WERNER, D., WERNER, W., "M-ideals in Banach Spaces and Banach Algebras", Lecture Notes in Math. 1547, Springer-Verlag, New York, 1993.

[41] HEINRICH, S., MANKIEWICZ, P., Applications of ultrapowers to the uniform and Lipschitz classification of Banach spaces, *Studia Math.,* **73** (1982), 225 – 251.

[42] JAIN, P.K., ARORA, K.K., SINHA, D.P., Weak* sequential compactness of dual unit balls of Banach spaces with PRI, *J. Indian Math. Soc.,* **63** (1997), 123 – 132.

[43] JAMESON, G.J.O., "Summing and Nuclear Norms in Banach Space Theory", London Math. Soc. Student Texts 8, Cambridge University Press, Cambridge - New York, 1987.

[44] JAYNE, J., ROGERS, C., Borel selectors for upper semicontinuous mappings, *Acta Math.,* **155** (1985), 41 – 79.

[45] JOHNSON, W.B., Existence theorems for Markushevich bases in Banach spaces (Abstract 677-46-1), *Notices Amer. Math. Soc.,* **17** (1970), 782.

[46] JOHNSON, W.B., LINDENSTRAUSS, J., Some remarks on weakly compactly generated Banach spaces, *Israel J. Math.,* **17** (1974), 219 – 230 and **32** (1979), 382 – 383.

[47] KADETS, M.I., MITYAGIN, B.S., Complemented subspaces in Banach spaces (Russian), *Uspekhi Math. Nauk,* **28:6** (1973), 77 – 94, English Transl.: *Russian Math. Surveys,* **28:6** (1973), 77 – 95.

[48] KADETS, M.I., SNOBAR, M.G., Certain functionals on the Minkowski compactum, *Math. Zametki,* **10** (1971), 453 – 457, English Transl.: *Math. Notes,* **10** (1971), 694 – 696.

[49] KALENDA, O., Valdivia compacta and equivalent norms, *Studia Math.,* **138** (2000), 179 – 191.

[50] KALENDA, O., Valdivia compacts spaces in topology and Banach space theory, (to appear).

[51] KOMATUZAKI, H., Sur les projections dans certains espaces du type (B), *Proc. Imp. Acad. Japan,* **16** (1940), 274 – 279.

[52] KÖNIG, H., TOMCZAK-JAEGERMANN, N., Norms of minimal projections, *J. Funct. Anal.,* **119** (1994), 253 – 280.

[53] LINDENSTRAUSS, J., On nonseparable reflexive Banach spaces, *Bull. Amer. Math. Soc.,* **72** (1966), 967 – 970.

[54] LINDENSTRAUSS, J., Decomposition of Banach spaces, *Indiana Univ. Math. J.,* **20** (1971), 917 – 919.

[55] LINDENSTRAUSS, J., TZAFRIRI, L., On the complemented subspaces pro-

blem, *Israel J. Math.,* **9** (1971), 263 – 269.
[56] Lindenstrauss, J., Tzafriri, L., "Classical Banach Spaces I. Sequence Spaces", Springer-Verlag, Berlin, 1977.
[57] Lindenstrauss, J., Tzafriri, L., "Classical Banach Spaces II. Function Spaces", Springer-Verlag, Berlin, 1979.
[58] Lotz, H.P., The Radon-Nikodým Property in dual Banach lattices, Abstract 726-46-20, *Notices Amer. Math. Soc.,* **22** (1975), A-571.
[59] Lotz, H.P., Rosenthal, H.P., Embeddings of $C(\Delta)$ and $L^1[0,1]$ in Banach lattices, *Israel J. Math.,* **31** (1978), 169 – 179.
[60] Malykhin, V.I., Sapirovskii, B.E., Martin's axiom and properties of topological spaces, *Dokl. Akad. Nauk. SSSR,* **213** (1973), 532 – 535, English transl.: *Soviet Math. Dokl.,* **14** (1973), 1746 – 1751.
[61] Mascioni, V., Topics in the theory of complemented subspaces in Banach spaces, *Exposition. Math.,* **7** (1989), 3 – 47.
[62] Miñarro, M.A., Every non-normable Fréchet space contains a non-trivial direct sum, *Turkish J. Mathematics,* **18** (1994), 165 – 167.
[63] Moore, R.C., Mrówka, S.G., Topologies determined by countable objects, Abstract no. 614-88, *Notices Amer. Math. Soc.,* **11** (1964), 554.
[64] Murray, F.J., On complementary manifolds and projections in spaces L_p and ℓ_p, *Trans. Amer. Math. Soc.,* **41** (1937), 138 – 152.
[65] Murray, F.J., Quasi-complements and closed projections in reflexive Banach spaces, *Trans. Amer. Math. Soc.,* **58** (1945), 77 – 95.
[66] Nakamura, M., Kakutani, S., Banach limits and the Čech compactification of an uncountable discrete set, *Proc. Imp. Acad. Japan,* **19** (1943), 224 – 229.
[67] Nakamura, Y., Any Hermitian matrix is a linear combination of four projections, *Lin. Alg. Appl.,* **61** (1984), 133 – 139.
[68] Odell, E., Rosenthal, H.P., A double dual characterization of separable Banach spaces containing ℓ_1, *Israel J. Math.,* **20** (1975), 375 – 384.
[69] Orihuela, J., On weakly Lindelöf Banach spaces, *North Holland Math. Studies,* **170** (1992), 279 – 291.
[70] Orihuela, J., Valdivia, M., Projective generators and resolutions of identity in Banach spaces, *Rev. Mat. Univ. Compl. Madrid,* **2** (núm. suppl.) (1989), 179 – 199.
[71] Ostaszewski, A., On countably compact, perfectly normal spaces, *J. London Math. Soc.,* **14** (1976), 505 – 516.
[72] Pearcy, C., Topping, D., Sums of small numbers of idempotents, *Michigan Math. J.,* **14** (1967), 453 – 463.
[73] Phelps, R.R., "Convex Functions, Monotone Operators and Differentiability", Lecture Notes in Math. 1364, Springer-Verlag, Berlin, 1989.
[74] Phillips, R.S., On linear transformations, *Trans. Amer. Math. Soc.,* **48** (1940), 516 – 541.
[75] Pisier, G., "Factorization of Linear Operators and Geometry of Banach Spaces", CBMS Regional Conference Series 60, Amer. Math. Soc., Providence, 1986.
[76] Plichko, A.M., Existence of a bounded Markuševič basis in every WCG space (Russian), *Teor. Funktsiĭ Funktsional Anal. Priložen (Kharkov),* **32:4**

(1979), 61–69, MR 80k:46020, Zbl 421.46011.
[77] PLICHKO, A.M., On projective resolutions of the identity operator and Markuševič bases (Russian), *Dokl. Akad. Nauk SSSR,* **263:3** (1982), 543–546, English transl.: *Soviet Math. Dokl.,* **25** (1982), 386–389.
[78] PLICHKO, A.M., Projective resolutions, Markuševič bases and equivalent norms (Russian), *Mat. Zametki,* **34:2** (1983), 719–726, English transl.: *Math. Notes,* **34:2** (1983), 851–855, MR85f:46028, Zbl 545.46010.
[79] PLICHKO, A.M., On bases and complements in nonseparable Banach spaces (Russian), *Sibirsk. Mat. J.,* **25:4** (1984), 155–162, English transl.: *Siber. Math. J.,* **25:4** (1984), 636–266.
[80] PLICHKO, A.M., On bases and complements in nonseparable Banach spaces. II, (Russian) *Sibirsk. Mat. J.,* **27:2** (1986), 149–153, English transl.: *Siber. Math. J.,* **27:2** (1986), 263–266, MR88g:46025, Zbl 604.46016.
[81] PLICHKO, A.M., POPOV, M.M., "Symmetric Function Spaces on Atomless Probability Spaces", Dissertationes Math., **306** 1990.
[82] POL, R., On a question of H. H. Corson and some related problems, *Fund. Math.,* **109** (1980), 143–154.
[83] PORTA, H., RECHT, L., Continuous selections of complemented subspaces, *Contemp. Math.,* **52** (1986), 121–125.
[84] REIF, J., A note on Markuševič bases in weakly compactly generated Banach spaces, *Comment. Math. Univ. Carolinae,* **15** (1974), 335–340.
[85] SIERPINSKI, W., Sur une décomposition d'ensembles, *Monatsh. Math. Physik,* **35** (1928), 239–242.
[86] SIMONS, S., A convegence theorem with boundary, *Pacific J. Math.,* **40** (1972), 703–708.
[87] SIMS, B., YOST, D., Linear Hahn-Banach extension operators, *Proc. Edinburgh Math. Soc.,* **32** (1989), 53–57.
[88] SOBCZYK, A., Projections in Minkowski and Banach spaces, *Duke Math. J.,* **8** (1941), 78–106.
[89] SOBCZYK, A., Projection of the space (m) on its subspace (c_0), *Bull. Amer. Math. Soc.,* **47** (1941), 938–947.
[90] STEGALL, C., Spaces of Lipschitz functions on Banach spaces, in "Functional Analysis (Essen, 1991)", Lecture Notes in Pure and Appl. Math. 150, Dekker, New York, 1994, 265–278, MR94k:46074.
[91] TACON, D.G., The conjugate of a smooth Banach space, *Bull. Austral Math. Soc.,* **2** (1970), 415–425.
[92] TALAGRAND, M., Sur une conjecture de H. H. Corson, *Bull. Sci. Math.,* **99** (1975), 211–212.
[93] VALDIVIA, M., Espacios de Fréchet de generación débilmente compacta, *Collect. Math.,* **38** (1987), 17–25.
[94] VALDIVIA, M., Topological direct sum decompositions of Banach spaces, *Israel J. Math.,* **71** (1990), 289–296.
[95] VESELÝ, L., Metric projections after renorming, *J. Approx. Theory,* **66** (1991), 72–82.
[96] WEIS, L., Perturbation classes of semi-Fredholm operators, *Math. Zeit.,* **178** (1981), 429–442.

[97] Wilansky, A., "Topology for Analysis", Xerox, Massachussetts, 1970.
[98] Yost, D., Asplund spaces for beginners, *Acta Univ. Carolinae Math. Phys.,* **34** (1993), 159 – 177.
[99] Yost, D., The Johnson-Lindenstrauss space, *Extracta Math.,* **12** (1997), 185 – 192.
[100] Yost, D., Sims, B., Banach spaces with many projections, *Proc. Centre Math. Anal. Austral. Nat. Univ.,* **14** (1986), 335 – 342.