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Copies of ℓ_p in Tensor Products

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The problem of finding complemented copies of ℓ_p in another space is a classical problem in Functional Analysis and has been studied from different points of view in the literature. Here we pay attention to complementation of ℓ_p in an n-fold tensor product of ℓ_q spaces because we were lead to that result in the study of Grothendieck's "Probléme des topologies" as we shall comment later. It could seem strange to need results on ℓ_p spaces in order to get results on Fréchet-Montel spaces since both are different, and important, classes. Nevertheless, there are many points of relation between Banach and Fréchet-Montel spaces: if we consider a particular type of projective limit of ℓ_p spaces we can get a Köthe echelon space and, for a suitable Köthe matrix A, it can be even a Fréchet-Montel space. Another important Fréchet (not Montel) space constructed by means of projective limits of ℓ_p spaces is the so called ℓ_{p+} space, which will be used in the final section of this paper. So we shall use on Banach spaces to give results on Fréchet spaces and applications to Grothendieck's problem.

Here we shall be concerned with projective n-fold tensor products and projective symmetric n-fold tensor products of Fréchet spaces. We are not recalling here the definition of tensor product and symmetric tensor product (see [9], [11]) but we explain the notation we use.

Recall that a fundamental system of seminorms for $\hat{\otimes}_{n,\pi}E$ (the completion of the projective n-tensor product of E) is given by $\{\otimes_n p : p \in \operatorname{cs}(E)\}$, where

$$\underset{n}{\otimes} p(\theta) = \inf\Big\{ \sum_i p(x_i^1) p(x_i^n) : \theta = \sum_i x_i^1 \otimes \cdots \otimes x_i^n \Big\}.$$

In the case of a symmetric projective tensor product we can consider either the above system of seminorms nor the equivalent system $\{\otimes_n p : p \in \operatorname{cs}(E)\}$, where

$$\underset{n,s}{\otimes} p(\theta) = \inf\Big\{ \sum_i p(x_i)^n : \theta = \sum_i \varepsilon_i \underset{n}{\otimes} x_i \Big\},$$

$\otimes_n x$ is used as an abbreviation for $x \otimes \overset{n}{\overbrace{\cdots}} \otimes x$ and $\varepsilon_i \in \{-1,1\}$ for all i when E is a real space and $\varepsilon_i = 1$ for all i when E is a complex space.

Grothendieck posed in 1955 his classical "Probléme des topologies", which can be stated as follows: given two Fréchet spaces E and F and a bounded subset $B \subset E\hat{\otimes}_\pi F$, are there bounded sets $B_1 \subset E$ and $B_2 \subset F$ such that $B \subset \bar{\Gamma}(B_1\hat{\otimes}_\pi B_2)$ [13], where $\bar{\Gamma}(C)$ denotes the closed convex hull of the set C and for a pair of subsets $C_1 \subset E$, $C_2 \subset F$, $C_1 \otimes C_2$ denotes $C_1 \otimes C_2 = \{x \otimes y : x \in C_1, y \in C_2\}$. About 30 years later, in 1986, Taskinen [19] continued the study of that problem and defined the concept of (BB) property: a couple of locally convex spaces (E,F) has the (BB) property if and only if Grothendieck's "probléme des topologies" has an affirmative answer for that pair. Finally, in 1994 Dineen defined the concept of $(BB)_n$ property as an extension of Taskinen's (BB) property. In this paper we shall denote by $(BB)_{n,s}$ property the property called originally $(BB)_n$ by Dineen (since it concerns to symmetric tensor products) and reserve the term $(BB)_n$ for the analogous property in the non-symmetric case: we say a locally convex space E has $(BB)_{n,s}$ property if for every bounded set $B \subset \hat{\otimes}_{n,s,\pi}E$ there is a bounded set $B_1 \subset E$ such that $B \subset \otimes_{n,s}B_1 = \{\otimes_n x : x \in B_1\}$ whereas E has $(BB)_n$ property if for every bounded set $B \subset \hat{\otimes}_{n,\pi}E$ there is a bounded set $B_1 \subset E$ such that $B \subset \otimes_n B_1 = \{x_1 \otimes \cdots \otimes x_n : x_1, \ldots, x_n \in B_1\}$.

There are a few known relations between (BB), $(BB)_n$ and $(BB)_{n,s}$. In particular $(BB)_{n,s}$ implies $(BB)_{m,s}$ for each $m \le n$ but it was not known any example with $(BB)_{n,s}$ property and without, for instance, $(BB)_{n+1,s}$. The first example of a Fréchet space E with $(BB)_{2,s}$ and without $(BB)_{3,s}$ was given in [2], using several results on Banach space theory. In particular, one of the steps in the construction needs to prove that ℓ_{p+} is a complemented subspace of $\ell_{2p+}\hat{\otimes}_{s,\pi}\ell_{2p+}$. The idea under the proof of that result is to prove the result for Banach spaces (i.e. proving the same statement deleting the signs "+") and there are several authors who studied that problem. We also give a different proof of it, using another theorems close to Grothendieck.

The complementation of ℓ_p in a full tensor product $\hat{\otimes}_{n,\pi}\ell_q$ or in a symmetric tensor product $\hat{\otimes}_{n,s,\pi}\ell_q$ has been studied in different times, by different authors for different reasons. Since it is an old, and interesting, problem we enunciate here the results proved and the context where they were used. The following list of theorems is not an exhaustive one, but it can help to understand the development of the subject and to give a wide view of the different techniques used.

Holub in 1970 [15] gave conditions under which ℓ_1 is complemented in $\ell_m \hat{\otimes}_\pi \ell_n$. Later he adapted this result to operator theory [16]. He proved the following

THEOREM 1. ([15], COROLLARY 5.8) *Let $1 < m, n < +\infty$ and $\frac{n}{n-1} \geq m$. Then $(e_i \otimes e_i)$ in $\ell_m \hat{\otimes}_\pi \ell_n$ is similar to (e_i) in ℓ_1.*

The first time we found complementation of ℓ_r $(r \neq 1)$ in a tensor product is in an article of Samuel [18] where he studies the problem of existence of complemented copies of ℓ_r in a 2-fold injective tensor product $\ell_p \hat{\otimes}_\varepsilon \ell_q$. It is a dual result of the result we want. He obtained the following

THEOREM 2. *Let $1 \leq p, q \leq \infty$; then ℓ_r is isomorphic to a subspace of $\ell_p \hat{\otimes}_\varepsilon \ell_q$ if and only if $r = p$, $r = q$ or*

$$r = \frac{1}{\frac{1}{q} + \frac{1}{p} - 1} \quad \text{if} \quad q' \geq p \qquad \text{or} \qquad r = \infty \quad \text{if} \quad q' \leq p,$$

being q' the conjugate exponent of q.

He also gives the following version for projective tensor products:

COROLLARY 3. *If $s \leq r'$ the space $\ell_r \hat{\otimes}_\pi \ell_s$ has a subspace isomorphic to ℓ_1.*

Oja, using Schauder decompositions and duality theory, was able to obtain the result we are interested in from Samuel's results in a paper where reflexivity of tensor products is studied. The next result is extracted from [17], Lemma 4. In the original lemma she describes the subspace isomorphic to ℓ_q complemented in $\ell_p \hat{\otimes}_\pi \ell_r$.

LEMMA 4. *Let $1 \leq p, r \leq \infty$. Then ℓ_q is isomorphic to a complemented subspace of $\ell_p \hat{\otimes}_\pi \ell_r$, where $q = 1$ if $p \leq r'$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$ (with $\frac{1}{\infty} = 0$) if $p > r'$.*

Oja also gives a "converse" result:

THEOREM 5. ([17], THEOREM 4) *Let $1 \leq p, q, r \leq \infty$. If $\ell_p \hat{\otimes}_\pi \ell_r$ contains a subspace isomorphic to ℓ_q then $q = p$ or $q = r$ or $q = s$, where $s = 1$ if $p \leq r'$ and $\frac{1}{s} = \frac{1}{p} + \frac{1}{r}$ (with $\frac{1}{\infty} = 0$) if $p > r'$.*

The jump to n-fold tensor products and symmetric tensor products has been developed during the last decade. The first of the results in this block is due to Choi and Kim. They study polynomial convergence of sequences in [6] and, as a consequence, they characterize when ℓ_1 is contained in a symmetric tensor product of spaces ℓ_q, extending Holub's result. Note that the previous results we have presented here dealt only with 2-fold full tensor products, not symmetric.

Theorem 6. ([6], Corollary 3.2) *Let $1 < p < \infty$.*

a) *For each positive integer m, $1 \leq m < p$, $\hat{\otimes}_{m,s,\pi}\ell_p$ contains no copy of ℓ_1.*

b) *For each positive integer m, $p \leq m$, $\hat{\otimes}_{m,s,\pi}\ell_p$ contains a copy of ℓ_1.*

Dineen also got the same result, using a different technique:

Theorem 7. ([8]) *Let $1 \leq p < \infty$ and $n \geq p$. Then ℓ_1 is a complemented subspace of $\hat{\otimes}_{n,s,\pi}\ell_p$.*

He also gives the polynomial version (i.e. the dual version):

Corollary 8. *Let $1 < p < \infty$ and $n \geq p$. Then ℓ_∞ is a (complemented) subspace of $\mathcal{P}(^n\ell_p)$.*

In 1996 Dineen and Lindström [10] obtained other results in a more general framework: they give conditions on a Fréchet space E (related to ℓ_p) under which ℓ_∞ is a complemented subspace of $\mathcal{P}(^nE)$. In 1998 the author gave another generalization of the result we are studying, this time for Köthe echelon spaces [4]. But it was also in 1996 when Arias and Farmer [3] gave the theorem concerning complementability of ℓ_r in a n-fold projective tensor product of ℓ_p spaces. They apply that result to several aspects of tensor product theory. In particular, they study the primarity of projective tensor products of ℓ_p spaces. Since they prove that the main diagonal in a tensor product is complemented and isomorphic to a ℓ_r space, the result is also true for symmetric tensor products.

Theorem 9. ([3], Theorem 1.3) *Let $X = \ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_N}$. Then the main diagonal $\mathcal{D} = [e_n \otimes \cdots \otimes e_n : n \in \mathbb{N}]$ is 1-complemented and satisfies $\mathcal{D} \equiv \ell_r$, where $\frac{1}{r} = \min\left\{1, \sum_{i=1}^N \frac{1}{p_i}\right\}$.*

Now we shall prove that ℓ_p is a complemented subspace of $\hat{\otimes}_{n,s,\pi}\ell_{np}$ (and, consequently of using a different technique than Arias and Farmer. One of the keys in the proof is a theorem of Grothendieck concerning weakly p-summable sequences, a concept of Banach space theory. The proof of the complementation result follow the ideas under Proposition 3.2 in [2].

Let us begin with the well-known definition of weakly p-summable sequence and an easy lemma.

DEFINITION 10. A sequence $(x_k)_k$ in a Banach space X is called weakly p-summable if $(x^*x_k)_k \in \ell_p$ for every $x^* \in X^*$.

LEMMA 11. *Let $1 \le p < \infty$ and $(e_k)_k$ be the canonical basis of ℓ_{np}. Then $(\otimes_n e_k)_k \subset \hat{\otimes}_{n,s,\pi}\ell_{np}$ is weakly p'-summable (where p' is such that $\frac{1}{p}+\frac{1}{p'}=1$).*

Proof. Let $\varphi \in (\hat{\otimes}_{n,s,\pi}\ell_{np})^*$. We know ([9], for instance) there is $P \in \mathcal{P}(^n\ell_{np})$ such that $\varphi(\otimes_n x) = P(x)$ for every $x \in \ell_{np}$.

We have, for each $n \in \mathbb{N}$, that

$$\sum_{k=1}^{N} \left|\varphi\Big(\underset{n}{\otimes}\, e_k\Big)\right|^{p'} = \sum_{k=1}^{N} |P(e_k)|^{p'} \le \left\|\big(P(e_k)\big)_k\right\|_{p'},$$

where the last inequality is obtained using that under the hypotheses in the lemma $\big(P(e_k)\big)_k \in \ell_{\frac{np}{np-n}}$ (see [20], Corollary 2). ∎

Although there are more direct ways to obtain the complementation of ℓ_p in a tensor product, since we are interested in one problem posed by Grothendieck, we shall use a result on weakly p-summable sequences due to him, stated in the next theorem. Its proof can be read in the original Grothendieck's paper [14] or, in more recent papers of Castillo and Sánchez [5] or Gonzalo and Jaramillo [12].

THEOREM 12. ([14]) *A sequence $(x_k) \subset X$ is weakly p-summable if and only if there is a bounded linear operator $T : \ell_{p+} \to X$ such that $T(e_k) = x_k$.*

With this result in mind we can enunciate, and give a new proof, the following theorem:

THEOREM 13. *Let $1 \le p < \infty$. Then ℓ_p is a complemented subspace of $\hat{\otimes}_{n,s,\pi}\ell_{np}$.*

Proof. By Theorem 12 we know there is a bounded linear operator $J_p : \ell_p \to \ell_{np}$ such that $J_p(u_k) = \otimes_n e_k$, where $(u_k)_k$ and $(e_k)_k$ are the canonical basis of ℓ_p and ℓ_{np} respectively. This is the injection we were looking for.

Next define the projection $\Pi_p : \otimes_{n,s,\pi}\ell_{np} \to \ell_p$ by $\Pi_p\big(\otimes_n (x_k)_k\big) = (x_k^n)_k$ (and extending it to the whole $\otimes_{n,s,\pi}\ell_{np}$ by linearity). The mapping is well defined and surjective since $(x_k)_k \in \ell_{np}$ if and only if $(x_k^n)_k \in \ell_p$. Moreover, $\|(x_k^n)_k\|_p \le \|(x_k)\|_{pn}^n = \big(\otimes_{n,s} \|\cdot\|_{pn}^n\big)\big(\otimes_n (x_k)_k\big)$, which gives continuity of this mapping. It only remains to extend it by continuity to the completion $\hat{\otimes}_{n,s,\pi}\ell_{np}$ of $\otimes_{n,s,\pi}\ell_{np}$.

Finally, it is trivial that $\Pi_p \circ J_p(\theta) = \theta$ for every $\theta \in \ell_p$. ∎

Remark 14. In the complex case, the operator J_p used in Theorem 13 is an isometry between ℓ_p and the main diagonal in $\hat{\otimes}_{n,s,\pi}\ell_{np}$, when we use the symmetric projective norm in $\hat{\otimes}_{n,s,\pi}\ell_{np}$. Indeed:

$$\underset{n,s}{\otimes}\|\cdot\|_{np}\left(\sum_{j=1}^{m} x_j \underset{n}{\otimes} e_j\right) = \sup\left\{\left|\sum_{j=1}^{m} x_j P(e_j)\right| : P \in \mathcal{P}(^n\ell_{np}),\ \|P\| = 1\right\}$$

but, for every $P \in \mathcal{P}(\ell_{np})$ with $\|P\| = 1$ we have that

$$\left|\sum_{j=1}^{m} x_j P(e_j)\right| \le \left\|\sum_{j=1}^{m} x_j e_j\right\|_p \left\|\big(P(e_j)\big)_j\right\|_{p'} \le \|(x_j)_j\|_p.$$

In [3] it is proved that there is an isometry between ℓ_p and the main diagonal in $\hat{\otimes}_{n,s,\pi}\ell_{np}$, using the projective norm (not the symmetric one), so the main diagonal of $\hat{\otimes}_{n,s,\pi}\ell_{np}$ is a subspace where both projective norms (symmetric and not symmetric) coincide.

The nature of the continuos seminorms of the space ℓ_{p+} allow to extend easily the above theorem for Banach spaces to a particular case of Fréchet spaces. It has a direct proof in [1].

THEOREM 15. ([1], PROPOSITION 3.2) *Suppose $1 \le p < \infty$ and $n \in \mathbb{N}$. Then the space ℓ_{p+} is isomorphic to a complemented subspace of $\hat{\otimes}_{n,s,\pi}\ell_{np+}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} \ell_q & \leftrightarrows & \underset{n,s,\pi}{\hat{\otimes}}\ \ell_{nq} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \ell_{p+} & \leftrightarrows & \underset{n,s,\pi}{\hat{\otimes}}\ \ell_{np+} \end{array}.$$

Having in mind how are defined the continuous seminorms in the above spaces and Theorem 13 it is easy to finish the proof. ∎

As we commented in the introduction, Theorem 15 played an important role when constructing the first example of a Fréchet space with $(BB)_{2,s}$ property but without $(BB)_{3,s}$ property, using several results on ℓ_{p+} developed by Defant and Peris [7]. That example is plenty of connections between Banach Space Theory and Fréchet Space Theory: in addition to complementation of ℓ_p in $\hat{\otimes}_{n,s,\pi}\ell_{np}$, results on type and cotype are used.

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