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Compact Composition Operators and Carleson Measure in the Upper Half–Plane

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1. Introduction

Let $\pi^+ = \{\omega \in \mathbb{C} : \mathrm{Im}(\omega) > 0\}$, where $\mathrm{Im}(\omega)$ stands for the imaginary part of ω. For $1 \le p < \infty$, $H^p(\pi^+)$ denotes the Hardy space of functions f analytic in the upper half–plane π^+ for which

$$\|f\|_p = \sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \right)^{1/p} < \infty.$$

Then $H^p(\pi^+)$ becomes a Banach space and for $p = 2$, it is a Hilbert space with the following definition of inner product:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(x) \overline{g^*(x)} \, dx, \quad \text{for every } f, g \in H^2(\pi^+).$$

Here the boundary function f^*, defined as $f^*(x) = \lim_{y \to 0} f(x+iy)$ exists almost every where and $f^* \in L^2(m)$, where m is the Lebesgue measure on the real line. In this case [3, p. 190]

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f^*(x)|^2 \, dx.$$

Also, Lemma 2.1 of [9] makes $H^2(\pi^+)$ a functional Hilbert space and for every $a \in \pi^+$, the reproducing kernel k_a is given by

$$k_a(w) = \frac{i}{2\pi(w - \bar{a})}, \quad \text{for every } w \in \pi^+.$$

Furthermore [11, p. 381],

$$\|k_a\|_2^2 = \frac{1}{4\pi \operatorname{Im}(a)}.$$

If ϕ is an analytic self map of the upper half–plane such that $f \circ \phi \in H^2(\pi^+)$ for every $f \in H^2(\pi^+)$, then by an application of the closed graph theorem one can show that $C_\phi : H^2(\pi^+) \to H^2(\pi^+)$, defined by $C_\phi f = f \circ \phi$, is a bounded operator, called composition operator induced by ϕ. Singh [10] initiated the study of these operators on $H^2(\pi^+)$ by establishing a relation between composition operators on Hardy spaces of the unit disc and the upper half–plane. The bridge obtained between these two classes of composition operators was exploited by Singh and Sharma [11], [12] and Sharma [9], which further strengthened this theory. In [11], Singh and Sharma characterized the boundedness of these operators in terms of the behaviour of the inducing maps in the vicinity of the point at ∞. The problem of compact composition operators on $H^2(\pi^+)$ was discussed in Sharma [9] and Singh and Sharma [12].

In the Rocky Mountain Mathematics Consortium Summer Conference "Composition Operators on Spaces of Analytic Functions" organized by the University of Wyoming in July 1996, the first author raised some problems on composition operators acting on $H^2(\pi^+)$, one of them was the existence of compact composition operators on $H^p(\pi^+)$. Sharma conjectured that there do not exist compact composition operators on $H^p(\pi^+)$. Though we have not been able to solve this conjecture, we are making an attempt to provide a clue to the researchers working in this direction by supplying Carleson measure criterion for the compactness of composition operators on $H^2(\pi^+)$.

2. Carleson measure and compact composition operators

A linear operator A on a Hilbert space H is called compact if A takes bounded subsets of H into sets with compact closures. This definition is equivalent to the statement that if $f_n \to f$ weekly in H, then $Af_n \to Af$ strongly in H. We shall use this definition to give a necessary condition for the compactness of C_ϕ on $H^2(\pi^+)$ in terms of Carleson measure. As per our information Arazy et al. [1], were first to use Carleson measure in dealing with composition operators and later on Cowen and MacCluer [2], MacCluer [5] and MacCluer and Shaprio [6] made extensive use of Carleson measure in studying boundedness and compactness of composition operators on various function spaces. Matache [7] characterized boundedness of composition operators on $H^p(\pi^+)$ using Carleson measure in π^+.

For $h, t \in \mathbb{R}$, $h > 0$ the set $S_{t,h} = (t, t+h) \times (0, h)$ will be called a Carleson set. Let $\phi : \pi^+ \to \pi^+$ be holomorphic. For $y > 0$, define $\phi_y : \mathbb{R} \to \pi^+$ as $\phi_y(x) = \phi(x + iy)$ for any $x \in \mathbb{R}$ and the pullback measure μ_y by $\mu_y(B) = (m\phi_y^{-1})(B) = m(\phi_y^{-1}(B))$ for any Borel set $B \subseteq \pi^+$. Consequently, for any $y > 0$, we obtain a Borel measure μ_y on π^+. Matache [7] proved that C_ϕ is bounded on $H^p(\pi^+)$ if and only if $\mu_y(S_{t,h}) \leq Kh$, for some $K \geq 0$, *i.e.*, μ_y is a Carleson measure for any $y > 0$ [4, p. 63]. We present a stronger condition which is necessary for compactness of C_ϕ on $H^2(\pi^+)$ but not a sufficient one. At this stage we need the following result of Nordgren [8].

Lemma 2.1. *A sequence in a functional Hilbert space is a weak null sequence if and only if it is norm bounded and pointwise null.*

We now proceed towards the main result.

Theorem 2.2. *Let ϕ be an analytic self map of the upper half–plane such that C_ϕ is a bounded operator on $H^2(\pi^+)$. Then C_ϕ is compact on $H^2(\pi^+)$ only if for any $y > 0$*

$$(*) \qquad \mu_y(S_{t,h}) = o(h) \quad \text{as } h \to 0.$$

Proof. We assume that the condition $(*)$ is false, which means that $\mu_y(S_{t,h}) \neq o(h)$ as $h \to 0$ for some $y > 0$. This implies that we can find t_n in $\mathbb{R}$, the numbers $h_n \to 0$ and $\beta > 0$ such that

$$(1) \qquad \mu_y(S_{t_n,h_n}) \geq \beta h_n.$$

Let $a_n = t_n + ih_n$ and $f_n : \pi^+ \to \mathbb{C}$ be given by $f_n = h_n^2 k_{a_n}$, where k_{a_n} is the reproducing kernel defined by a_n. Then

$$\|f_n\|_2^2 = \frac{h_n^4}{4\pi\,\mathrm{Im}(a_n)} = \frac{h_n^3}{4\pi}, \quad \text{for } n = 1,\, 2,\, 3,\, \ldots\,.$$

Let $g_n = f_n/\|f_n\|_2$. Then $\{g_n\}$ is a norm bounded sequence in $H^2(\pi^+)$ and for $n = 1, 2, 3 \ldots$

$$\begin{aligned} g_n(w) &= h_n^{1/2}\, 2\sqrt{\pi} k_{a_n}(w) \\ &= \frac{ih_n^{1/2}}{\sqrt{\pi}(w - \bar{a}_n)} \to 0 \quad \text{as } n \to \infty \quad \text{pointwise in } \pi^+. \end{aligned}$$

Hence, by Lemma 2.1, $g_n \to 0$ weakly in $H^2(\pi^+)$. But for this y, by change of variables formula,

$$\int_{-\infty}^{\infty} |(g_n \circ \phi)(x+iy)|^2 \, dx = \frac{1}{\|f\|_2^2} \int_{-\infty}^{\infty} |f_n(\phi_y(x))|^2 \, dx$$
$$= \frac{1}{\|f_n\|_2^2} \int_{\pi^+} |f_n(w)|^2 \, dm\phi_y^{-1}(w) \,,$$

or,

$$\int_{-\infty}^{\infty} |(g_n \circ \phi)(x+iy)|^2 \, dx \geq \frac{1}{\|f_n\|_2^2} \int_{S_{t_n,h_n}} |f_n(w)|^2 \, d\mu_y(w). \tag{2}$$

However, if $\omega = x+iy \in S_{t_n,h_n}$, then $t_n < x < t_n + h_n$, $0 < y < h_n$ and

$$f_n(w) = \frac{ih_n^2}{2\pi\big((x-t_n)+(y+h_n)i\big)} \,.$$

Therefore,

$$|f_n(w)|^2 = \frac{h_n^4}{4\pi^2\big(\sigma_1^2 + (1+\sigma_2)^2\big)h_n^2}$$
$$\geq \frac{h_n^2}{20p^2},$$

for $0 < \sigma_1, \sigma_2 < 1$ and all $w \in S_{t_n,h_n}$. Hence from the inequality (2), we have

$$\|C_\phi g_n\|_2^2 \geq \int_{-\infty}^{\infty} |(g_n \circ \phi)(x+iy)|^2 \, dx$$
$$\geq \frac{4\pi}{h_n^3} \frac{h_n^2}{20\pi^2} \mu_y(S_{t_n,h_n}).$$

Using inequality (1), we obtain

$$\|C_\phi g_n\|_2^2 \geq \frac{\beta}{5\pi},$$

which shows that $\{\|C_\phi g_n\|_2\}$ remains bounded away from zero. Hence $\{C_\phi g_n\}$ does not converge to the zero function strongly. This implies that C_ϕ is not compact. Thus the condition $(*)$ is a necessary condition for the compactness of C_ϕ on $H^2(\pi^+)$. ∎

The following example shows that the condition $(*)$ is not a sufficient one for the compactness of C_ϕ on $H^2(\pi^+)$.

EXAMPLE 2.3. Let $\phi : \pi^+ \to \pi^+$ be defined as $\phi(w) = w + w_0$, $w_0 = x_0 + iy_0 \in \pi^+$. Then by Theorem 2.1 of Singh and Sharma [11] C_ϕ is a bounded operator on $H^2(\pi^+)$. Since ϕ is invertible, by Theorem 3.1 of [11], C_ϕ is also invertible and so it can not be a compact operator. Nevertheless we shall show that ϕ satisfies the condition $(*)$. For $y > 0$, $\phi_y(x) = x + x_0 + i(y + y_0)$ for any $x \in \mathbb{R}$. Since for $t \in \mathbb{R}$, $h > 0$ and any $y > 0$,

$$\phi_y^{-1}\big(\{(x, y + y_0) : \, t < x < t + h\}\big) = (-x_0 + t, \, -x_0 + t + h),$$

we have

$$\phi_y^{-1}(S_{t,h}) = \begin{cases} (-x_0 + t, \, -x_0 + t + h) & \text{if } \; h > y + y_0, \\ \emptyset & \text{if } \; h \leq y + y_0. \end{cases}$$

This further implies that for any $y > 0$

$$\mu_y(S_{t,h}) = \begin{cases} h & \text{if } \; h > y + y_0, \\ 0 & \text{if } \; h \leq y + y_0. \end{cases}$$

Thus, if h is sufficiently small, then

$$\lim_{h \to 0} \frac{\mu_y(S_{t,h})}{h} = 0,$$

i.e., $\mu_y(S_{t,h}) = o(h)$ as $h \to 0$ for any $y > 0$. Hence ϕ satisfies the condition $(*)$.

ADDED IN PROOF

After this paper was accepted by the editor, a paper of V. Matache [Composition operators on Hardy spaces of a half–plane, *Proc. Amer. Math. Soc.* **127** (5) (1999), 1483–1491] in which it has been exhibited that no composition operator is compact on $H^p(\pi^+)$, was brought to the attention of the authors. However, our statements of results, ideas and methods of proofs are different from those used in the above paper.

References

[1] Arazy, J., Fisher, S.D., Peetre, J., Mobius invariant function spaces, *J. Reine Angew. Math.,* **363** (1985), 110 – 145.

[2] Cowen, C.C., MacCluer, B.D., "Composition Operators on Spaces of Analytic Functions", CRC Press, BocaRaton, New York,, 1995.

[3] Duren, P.L., "Theory of H^p Spaces", Academic Press, New York, 1970.

[4] Garnett, J., "Bounded Analytic Functions", Academic Press, New York, 1981.

[5] MacCluer, B.D., Composition operators on S^p, *Houston J. Math.,* **13** (1987), 245 – 254.

[6] MacCluer, B.D., Shapiro, J.H., Angular derivative and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.,* **38** (1986), 878 – 906.

[7] Matache, V., Composition operators on H^p of the upper half–plane, *An. Univ. Timisoara Ser. Stiint. Mat.,* **119** (1989), 63 – 66.

[8] Nordgren, E.A., Composition operators on Hilbert spaces, Lecture Notes in Math. 693, Springer–Verlag, Berlin, 1978, 37 – 63.

[9] Sharma, S.D. Compact and Hilbert–Schmidt composition operators on Hardy spaces of the upper hall–plane, *Acta Sci. Math. (Szeged)* **46** (1983), 197 – 202.

[10] Singh, R.K., A relation between composition operators on $H^2(D)$ and $H^2(\pi^+)$, *Pure and Appl. Math. Sci.,* **1** (1974), 1 – 5.

[11] Singh, R.K., Sharma, S.D., Composition operators on a functional Hilbert space, *Bull. Austral. Math. Soc.,* **20** (1979), 377 – 384.

[12] Singh, R.K., Sharma, S.D., Non-compact composition operators, *Bull. Austral. Math. Soc.,* **21** (1980), 125 – 130.