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Boundary of Polyhedral Spaces: An Alternative Proof

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A Banach space space X is called *polyhedral*, if the unit ball of each of its finite-dimensional (equivalently: two-dimensional [6]) subspaces is a polytope. Polyhedral spaces were studied by various authors; most of the structural results are due to V. Fonf. We refer the reader to the surveys [1], [2] for other definitions of polyhedrality, main properties and bibliography. In this paper we present a short alternative proof of the basic result on the structure of the unit ball of a polyhedral space (Theorem 1) and a related Theorem 2.

Let us start with some definitions. Throughout the paper, X denotes an infinite-dimensional real Banach space with closed unit ball B_X, unit sphere S_X and density character dens X (i.e. the minimal cardinality of a dense subset of X).

We shall say that a set $F \subset S_X$ is a *true face* of B_X if there exists a closed hyperplane $H \subset X$ supporting B_X such that $F = H \cap B_X$ and $\mathrm{int}_H F$ (the relative topological interior of F in H) is nonempty. A set $\mathcal{B} \subset S_{X^*}$ is called *boundary* for X if for each $x \in S_X$ there exists $f \in \mathcal{B}$ such that $f(x) = 1$. (In [5], $\mathcal{B}$ is called "James boundary".)

The following theorem is a slight reformulation of Theorem 1 from [3].

Theorem 1. *Let X be a polyhedral Banach space. Then the sphere S_X is covered by the true faces of B_X. Hence the set $\mathcal{B}_0 = \{f \in S_{X^*} :\ f^{-1}(1) \cap B_X$ is a true face of $B_X\}$ is a boundary for X. In particular, $\mathcal{B}_0$ is countable whenever X is separable.*

The original proof in [3] is rather technical. About ten years later, V. Fonf considerably simplified the proof in an unpublished manuscript (see also [4]).

Our proof is quite different from those by Fonf. It is less elementary, since it uses results about generic differentiability of convex functions, but simpler than the proof in [3]. For separable X, our proof uses only the classical Mazur's theorem about generic Gâteaux differentiability of continuous convex functions. Even in view of [4], we consider our proof geometric and maybe interesting.

Let us remark the following

FACT. Since each relative interior point of a true face has a unique supporting functional of norm one, the boundary $\mathcal{B}_0$ from Theorem 1 is minimal in the sense that it is contained in each boundary of the polyhedral space.

Moreover, in separable case, B_{X^*} is the *norm*-closed convex hull of $\mathcal{B}_0$, as follows from the following result by Rodé [8]. (For a simpler proof of similar nature see [5]; a different and more geometric proof has been found recently by V. Fonf, J. Lindenstrauss and R. R. Phelps.)

THEOREM. (Rodé's Theorem [8]) *Let $\mathcal{B} \subset S_{X^*}$ be a separable boundary for X. Then $B_{X^*} = \overline{\operatorname{conv}}\,\mathcal{B}$ (the norm-closure of* $\operatorname{conv}\mathcal{B}$*).*

We shall show by a separable reduction argument that, for polyhedral spaces, the separability assumption is not necessary. We shall prove the following theorem.

THEOREM 2. *Let X be a polyhedral Banach space, and $\mathcal{B}_0$ be the boundary for X from Theorem 1. Then $B_{X^*} = \overline{\operatorname{conv}}\,\mathcal{B}_0$ and* $\operatorname{card}\mathcal{B}_0 = \operatorname{dens} X = \operatorname{dens} X^*$. *(Consequently, $B_{X^*} = \overline{\operatorname{conv}}\,\mathcal{B}$ whenever $\mathcal{B}$ is a boundary for X.)*

The *algebraic interior* of a set $A \subset X$ is the set a-int A of all points $x \in A$ such that $x \in \operatorname{int}_L(C \cap L)$ whenever $L \subset X$ is a line that contains x. Obviously, $\operatorname{int} A$ is always contained in a-int A. The following lemma about F_σ-sets is well known for closed sets. The first part of it was suggested to the author by L. Zajíček.

LEMMA 1. *Let A be an F_σ-set in X. Then* $\operatorname{int} A \neq \emptyset$ *if and only if* a-int $A \neq \emptyset$. *If, moreover, A is also convex, then* $\operatorname{int} A =$ a-int A.

Proof. Suppose $0 \in$ a-int A and $A = \bigcup A_n$ where (A_n) is a sequence of closet sets. For every $v \in S_X$ there exists $t > 0$ such that the segment $[0, tv]$ is covered by A. The Baire theorem implies that some A_n contains a nontrivial subsegment of $[0,tv]$. Consequently,

$$S_X = \bigcup\{S(n,\alpha,\beta) : \ n \in \mathbb{N}, \quad 0<\alpha<\beta, \quad \alpha,\beta \text{ rational}\},$$

where $S(n,\alpha,\beta) = \{v \in S_X : \ [\alpha v, \beta v] \subset A_n\}$.

Since the sets $S(n,\alpha,\beta)$ are easily seen to be closed and they are countably many, another application of the Baire category theorem implies that some $S(\overline{n},\overline{\alpha},\overline{\beta})$ has nonempty interior in S_X. Thus $A_{\overline{n}}$ (and hence A) contains the nonempty open set

$$\bigcup\{(\overline{\alpha}v,\overline{\beta}v) : \ v \in \operatorname{int}_{S_X} S(\overline{n},\overline{\alpha},\overline{\beta})\}.$$

The assertion concerning convex sets follows from the Hahn-Banach theorem (indeed, if A is convex and $\operatorname{int} A$ is nonempty, no boundary point of A can belong to a-int A because it is a support point). ∎

If $A \subset Y$ and Y is an affine set in X, we denote by a-int$_Y A$ the relative algebraic interior of A in Y:

$$\text{a-int}_Y A = \{x \in A : \ x \in \operatorname{int}_L(A \cap L) \text{ whenever } L \text{ is a line and } x \in L \subset Y\}.$$

Remark. (a) Lemma 1 clearly implies: if A is a set of the first category in a Banach space, then a-int A is empty. (Indeed, A is contained in an F_σ-set with empty interior.)

(b) The equality $\operatorname{int} A =$ a-int A does not hold in general. Consider the origin in $X = \mathbb{R}^2$ and the set $A = \{(x,y) : \ y \geq x^2\} \cup \{(x,y) : \ y \leq 0\}$.

(c) Lemma 1 remains valid if we replace X by a closed affine subspace of a Banach space (and consider relative interior and relative algebraic interior).

LEMMA 2. *Let X be polyhedral, $x_0 \in S_X$. Then the following assertions are equivalent.*

(i) *x_0 is interior point of a true face of B_X;*

(ii) *B_X is Fréchet smooth in x_0;*

(iii) *B_X is Gâteaux smooth in x_0.*

Proof. The implications (i)⇒(ii)⇒(iii) are obvious. Suppose (iii) holds. Then B_X has a unique supporting hyperplane Y at x_0. For any two-dimensional subspace $Z \subset X$ that contains x_0, the line $Y \cap Z$ is the unique supporting line of the polygon $B_X \cap Z$ at x_0, hence the line intersects the polygon in a nontrivial line segment that contains x_0 as its (relative) interior point. Consequently, $x_0 \in \operatorname{int}_Y(Y \cap B_X)$. Then Lemma 1 implies that $Y \cap B_X$ is a true face and (i) holds. ∎

Proof of Theorem 1. Let Q be the set of the points from S_X that are not contained in the union of all true faces.

Fix a point $u \in Q$ and a functional $f \in S_{X^*}$ with $f(u) = 1$. Let $Z = f^{-1}(0)$ and let $\pi\colon X \to Z$ be the linear projection along u, i.e. $\pi(z+tu) = z$ whenever $z \in Z$, $t \in \mathbf{R}$. It is easy to see that π is a homeomorphism of an open neighborhood G of u in S_X onto $G_0 := Z \cap \operatorname{int}\left(\frac{1}{2}B_X\right)$. Define $p\colon G \to G_0$ by $p(x) = \pi(x)$. Then for each $z \in G_0$ we have

$$p^{-1}(z) = z + \varphi(z)u$$

where $\varphi\colon G_0 \to \mathbb{R}$ is continuous and concave. Let $Q_0 = p(Q \cap G)$.

Claim: the point $u_0 = p(u)$ belongs to a-$\operatorname{int}_Z Q_0$.

Let z be an arbitrary nonzero vector from Z. Since the unit ball of $\operatorname{span}\{u, z\}$ is a polygon that contains u as a boundary point, the boundary of this polygon contains two non-overlapping nondegenerate segments $[v_1, u]$ and $[u, v_2]$ with $v_1, v_2 \in G$. It is easy to see that the segment $p([v_1, u] \cup [u, v_2]) = [p(v_1), p(v_2)]$ is parallel to z and contains u_0 as an interior point. Now it is not difficult to see that $(v_1, u] \cup [u, v_2) \subset Q$. Indeed, if some point $y \in (v_1, u)$ belonged to a true face, the hyperplane that defines this face would support B_X at y and hence also at each point of $[v_1, u]$. But this is impossible since $u \in Q$. (Similarly for $y \in (u, v_2)$.) This implies that $(p(v_1), p(v_2)) \subset Q_0$. The claim is proved.

Lemma 2 implies that no point of Q is a point of Gâteaux differentiability of B_X; hence Q_0 contains only points of Gâteaux nondifferentiability of φ.

(α) If X is separable, φ is generically Gâteaux differentiable on G_0 by Mazur's theorem ([7], [5]). By Remark (a), we must have a-$\operatorname{int}_Z Q_0 = \emptyset$. But this contradicts our Claim. Thus Theorem 1 holds for separable spaces.

(β) If X is not separable, then each separable subspace of X has a countable boundary by (α), and hence, by Rodé's theorem, a separable dual. Thus φ is generically Fréchet differentiable on G_0 (cf. [7]). By Remark (a), we get again a-$\operatorname{int}_Z Q_0 = \emptyset$, a contradiction with our Claim. ∎

Proof of Theorem 2. Suppose that $\operatorname{dist}(f, \overline{\operatorname{conv}}\, \mathcal{B}_0) > \varepsilon$ for some $f \in S_{X^*}$ and some $\varepsilon > 0$. Then, for every $g \in \operatorname{conv} \mathcal{B}_0$ there exists $z_g \in S_X$ such that $|(f - g)(z_g)| > \varepsilon$.

Let us perform the following inductive procedure. For a set $H \subset X^*$ and a subspace $L \subset X$, we denote by $H_{|L}$ the set $\{h_{|L} : h \in H\}$ of all restrictions to L of elements of H.

1) Let $\{x_i\}_1^\infty \subset S_X$ be such that $f(x_i) \to 1$. Put $Y_1 = \overline{\operatorname{span}}\, \{x_i\}_1^\infty$. Since $\mathcal{B}_{0|Y_1}$ is obviously a boundary for Y_1, by Theorem 1 and Fact, there exists a

countable set $B_1 \subset \mathcal{B}_0$ such that $B_{1|Y_1}$ is a boundary for Y_1. Let D_1 be a countable dense subset of $\operatorname{conv} B_1$.

2) Suppose we already have separable subspaces $Y_1 \subset \cdots \subset Y_n$, countable subsets $B_1 \subset \cdots \subset B_n$ of $\mathcal{B}_0$, and countable dense sets D_k in $\operatorname{conv} B_k$ for $k = 1, \ldots, n$. Put $Y_{n+1} = \overline{\operatorname{span}}\,(Y_n \cup \{z_g : g \in D_n\})$. As above, take a countable set $B_{n+1} \subset \mathcal{B}_0$ such that $B_{n+1} \supset B_n$ and $B_{n+1|Y_{n+1}}$ is a boundary for Y_{n+1}. Let D_{n+1} be any countable dense subset of $\operatorname{conv} B_{n+1}$.

Let us put $Y = \overline{\bigcup_{n=1}^{\infty} Y_n}$, $A = \bigcup_{n=1}^{\infty} B_n$ and $D = \bigcup_{n=1}^{\infty} D_n$. Then Y is separable, A is countable, and D is a countable dense subset of $\operatorname{conv} A$.

We claim that $A_{|Y}$ is a boundary for Y. Indeed, since Y is polyhedral, by Theorem 1 each true face F of B_Y contains in its relative interior a point y that belongs to some Y_n. By our construction, there exists $h \in B_n \subset A$ such that $h(y) = 1$. Thus the face F is all contained in $h^{-1}(1)$.

Since for each $g \in D$ the point z_g belongs to S_Y, we have

$$\operatorname{dist}(f_{|Y}, \operatorname{conv} A_{|Y}) = \mathit{dist}(f_{|Y}, D_{|Y}) \geq \inf_{g \in D} |(f-g)(z_g)| \geq \varepsilon.$$

This contradiction with Rodé's theorem proves that B_{X^*} is the closed convex hull of $\mathcal{B}_0$. Consequently, we have $\operatorname{card} \mathcal{B}_0 \leq \operatorname{dens} X \leq \operatorname{dens} X^* \leq \operatorname{card} \mathcal{B}_0$ (the first inequality follows from Theorem 1, and the second one holds for any normed space). ∎

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