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A Global Attractor in a General Diffusive Food Chain Model*

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1. Introduction

Reaction-diffusion equations have been extensively used by applied scientists to model and analyze the population dynamics for interacting species. To model interactions among three species, consider the system

$$
\begin{aligned}
u_t - d_1\Delta u &= u f_1(x,t,u,v,w) \\
v_t - d_2\Delta v &= v f_2(x,t,u,v,w) \qquad \text{in } \Omega\times(0,\infty) \\
w_t - d_3\Delta w &= w f_3(x,t,u,v,w) \\
u = v = w &= 0 \qquad\qquad\qquad\quad \text{on } \partial\Omega\times(0,\infty)
\end{aligned} \tag{1}
$$

In (1), Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary, and $u(x,t)$, $v(x,t)$, $w(x,t)$ represent the population density of species A, B, C, respectively at location $x\in\Omega$ and time $t>0$. The Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

arises because of the random motion of the species within Ω, with diffusion coefficients $d_i>0$ for $i=1,2,3$ and $f_i(x,t,u,v,w)$ is the per capita growth rate for species i at location x, time t and densities $u(x,t)$, $v(x,t)$ and $w(x,t)$. The boundary conditions biologically mean that boundary is lethal for the 3

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species. In order to allow seasonal variation or day-night cycles, we assume that

$$f_i(x, \cdot + T, u, v, w) = f_i(x, \cdot, u, v, w) \tag{2}$$

for all t and some $T > 0$, $i = 1, 2, 3$. System (1) can model a variety of interactions involving three species: a) competition, b) one predator-two prey, c) two predators with one prey, d) food chain, e) mutualism loop. In this article, we are concerned with modeling a *food chain*. To do so, we assume the following conditions on the growth laws $f_i(x, t, u, v, w)$:

$$\begin{array}{ll} \dfrac{\partial f_1}{\partial v} \le 0\ , & \dfrac{\partial f_1}{\partial w} \le 0 \\ \dfrac{\partial f_2}{\partial u} \ge 0\ , & \dfrac{\partial f_2}{\partial w} \le 0 \\ \dfrac{\partial f_3}{\partial u} \ge 0\ , & \dfrac{\partial f_3}{\partial v} \ge 0 \end{array} \tag{3}$$

We investigate the asymptotic behavior of system (1) with conditions (2) and (3). In this vein, recently Feng [2] studied system (1) with Lotka-Volterra type of interaction with no periodicity. There was shown the existence of a global attractor under certain conditions on the eigenvalues of a related elliptic problem via a monotone iteration process of the elliptic associated system.

We give a sufficient condition to ensure the existence of a positive global attractor for our model, given in terms of the spectra of linear differential operators associated with the original reaction-diffusion equations. In so doing, we connect asymptotic coexistence in such a system to the underlying biological assumptions about the model which are expressed in the parameters and coefficients of these operators. The organization of this paper is as follows: in Section 2 we set up our system and give some preliminary results. We first state a threshold type theorem for a general scalar periodic parabolic equation which comes from Avila-Vales [1], and then a theorem comparing the principal eigenvalue theorem coming from Hess [3]. In Section 3 we construct our positive global attractor via an iteration process starting with some solutions to subsystems with one or two species present. We state and prove our main result and also we give some results for extinction. Finally in Section 4, we illustrate our main result. We consider a Lotka-Volterra type of system (1) with periodic coefficients. Since the conditions of our main result are stated in terms of the principal eigenvalue of the scalar periodic-parabolic problem, we can get the conditions using the estimates given in Hess [3, Section II.17].

2. Preliminaries

Consider the equation

$$(4)\qquad \begin{aligned} \frac{\partial u}{\partial t} &= d\,\Delta\, u + uf(x,t,u) && \text{in } \Omega \times (0,\infty) \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned}$$

such that f is T-periodic in t and satisfies

L1 f is continuous in all of its arguments. Also, $f(x,t,u) \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega} \times [0,T])$, where $0 < \beta \le 1$ uniformly for u in bounded subsets of $\mathbb{R}$. Furthermore, we assume that the partial derivatives of f exist and are Hölder continuous, in $\overline{\Omega} \times [0,T] \times \mathbb{R}$.

L2 i) $f(x,t,v) > f(x,t,u)$ if $0 \le v < u$.
ii) $f(x,t,0) > 0$ for some $x \in \Omega$ and $t \in (0,T)$.

L3 $f(x,t,u) < 0$ if $u \ge K$ for some $K > 0$ for all x, t.

In [5] it is shown that under (L1) the linear eigenvalue problem

$$(5)\qquad \begin{aligned} \frac{\partial v}{\partial t} - d\,\Delta\, v - vf(x,t,0) &= \mu v && \text{in } \Omega \times \mathbb{R} \\ v &= 0 && \text{on } \partial\Omega \times \mathbb{R} \\ v \quad & T\text{-periodic in } t \end{aligned}$$

admits a unique $\mu \in \mathbb{R}$ having associated eigenfunction $v \in C^{2+\beta,1+\frac{\beta}{2}}(\overline{\Omega} \times [0,T])$ with $v(x,t) > 0$ for $x \in \Omega$ and $t \in \mathbb{R}$. (μ is called the *principal eigenvalue*. See also [3], [4].) We have the following result.

Lemma 1. ([1, Theorem 4.2]) *Consider equation* (4) *and suppose that f is T-periodic in t and satisfies* (L1)–(L3). *Then* (4) *admits a positive T-periodic solution $u(x,t)$ if and only if μ in* (5) *is negative. Additionally:*

i) *If $\mu < 0$, u is the only such solution. Moreover, if $w(x) \in C^1(\overline{\Omega})$ with $w \gneqq 0$ and U_w denotes the solution of* (4) *with $U_w(x,0) = w(x)$, and if $\varepsilon > 0$ is given, there is $t_0 = t_0(w)$ so that $\|u(x,t) - U_w(x,t)\|_{C^1(\overline{\Omega})} < \varepsilon$ for all $t > t_0$ (i.e., u is globally asymptotically stable with respect to nonnegative initial data).*

ii) *If $\mu > 0$, 0 is globally asymptotically stable with respect to nonnegative initial data.*

LEMMA 2. ([3, Lemma 15.5]) *Let $m_1, m_2 \in C^{\beta,\frac{\beta}{2}}(\overline{\Omega}, \mathbb{R})$ be T-periodic in t and suppose that $m_1 < m_2$. Let μ_i the principal eigenvalues of*

$$\frac{\partial u_i}{\partial t} - d\ \Delta\ u_i - m_i u_i = \mu_i u_i, \quad u_i > 0 \qquad i = 1, 2.$$

Then $\mu_2 < \mu_1$.

3. MAIN RESULT

Consider system (1) and assumptions (2) and (3). We also assume:

H1) $f_i \in C^{\alpha,\frac{\alpha}{2},2}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^3)$ for $i = 1, 2, 3$, where $\alpha \in (0, 1)$.

H2) $\dfrac{\partial f_1}{\partial u} \leq 0$, $\dfrac{\partial f_2}{\partial v} \leq 0$, $\dfrac{\partial f_3}{\partial w} \leq 0$

H3) $f_i(x_i, t_i, 0, 0, 0) > 0$ for some $x_i \in \Omega$ and $t_i \in (0, T)$, for $i = 1, 2, 3$.

H4) There exist k_1, k_2, $k_3 > 0$ so that

$$\begin{array}{ll} f_1(x, t, u, 0, 0) < 0 & \text{if} \quad u \geq k_1 \\ f_2(x, t, 0, v, 0) < 0 & \text{if} \quad v \geq k_2 \\ f_3(x, t, 0, 0, w) < 0 & \text{if} \quad w \geq k_3 \end{array}$$

for any $x \in \overline{\Omega}$ and $t \in [0, T]$

H5)

$$\begin{array}{rcl} f_1(x^{(1)}, t', 0, v, w) & > & 0 \\ f_2(x^{(2)}, t^{(2)}, 0, 0, w) & > & 0 \\ f_2(x'', t'', u, 0, 0) & > & 0 \\ f_3(x^{(2)}, t^{(3)}, u, v, 0) & > & 0 \end{array}$$

where $x^{(1)}$, $x^{(2)}$, x'', $x^{(3)}$ belong to $\overline{\Omega}$, t', $t^{(2)}$, t'', $t^{(3)}$ belong to $\mathbb{R}$ and u, v, w are non-negative.

H6)

$$f_1(x, t, u, v^*, w^*) < 0 \ \text{ if } \ u \geq K_u$$

for some $K_u > 0$, $(x, t) \in \Omega \times (0, \infty)$, w^* is defined below.

$$f_2(x, t, \phi_1, v, 0) < 0 \ \text{ if } \ v \geq K_v$$

for some $K_v > 0$, $(x, t) \in \Omega \times (0, \infty)$, ϕ_1 is defined below.

$$f_2(x, t, 0, v, w^*) < 0 \ \text{ if } \ v \geq K'_v$$

for some $K'_v > 0$, $(x,t) \in \Omega \times (0,\infty), w^*$ is defined below.

$$f_3(x,t,u,v,w) < 0 \quad \text{if} \quad w \geq K_w$$

for some $K_w > 0$, $(x,t) \in \Omega \times (0,\infty), v^*$ is defined below.

In order to define our first step in the iteration process, we restrict system (1) to the cases when $v = w = 0$ and $u = v = 0$. Then we get

$$\begin{aligned} u_t - d_1 \Delta u &= u f_1(x,t,u,0,0) && \text{in } \Omega \times (0,\infty) \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{6}$$

and

$$\begin{aligned} w_t - d_3 \Delta w &= w f_3(x,t,0,0,w) && \text{in } \Omega \times (0,\infty) \\ w &= 0 && \text{on } \partial\Omega \times (0,\infty). \end{aligned} \tag{7}$$

respectively.

According to Lemma 1, each of the previous equations possesses a unique globally attracting positive T-periodic solution ϕ_1 and ϕ_3, respectively if the principal eigenvalues of

$$\begin{aligned} \frac{\partial v}{\partial t} - d_1 \Delta v - v f_1(x,t,0,0,0) &= \mu_{(1,0,0,0)} v && \text{in } \Omega \times (0,\infty) \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial w}{\partial t} - d_3 \Delta w - w f_3(x,t,0,0,0) &= \mu_{(3,0,0,0)} w && \text{in } \Omega \times (0,\infty) \\ w &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned}$$

are both negative.

Similarly denote by v^*, w^*, u_*, v_* the unique globally attracting positive-periodic solutions to

$$\begin{aligned} v_t - d_2 \Delta v &= v f_2(x,t,\phi_1,v,0) && \text{in } \Omega \times (0,\infty) \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{8}$$

$$\begin{aligned} w_t - d_3 \Delta w &= w f_3(x,t,\phi_1,v^*,w) && \text{in } \Omega \times (0,\infty) \\ w &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{9}$$

$$
\begin{aligned}
u_t - d_1\,\Delta\, u &= u f_1(x,t,u,v^*,w^*) && \text{in } \Omega\times(0,\infty)\\
u &= 0 && \text{on } \partial\Omega\times(0,\infty),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
v_t - d_2\,\Delta\, v &= v f_2(x,t,0,v,w^*) && \text{in } \Omega\times(0,\infty)\\
v &= 0 && \text{on } \Omega\times(0,\infty).
\end{aligned}
\tag{11}
$$

Lemma 1 implies that these solutions exist if

$$
\mu_{(2,\phi_1,0,0)} < 0, \quad \mu_{(3,\phi_1,v^*,0)} < 0
$$

$$
\mu_{(1,0,v^*,w^*)} < 0, \quad \text{and} \quad \mu_{(2,0,0,w^*)} < 0,
$$

where these numbers denote the principal eigenvalues for the equations

$$
\begin{aligned}
\frac{\partial f}{\partial t} - d_2\,\Delta\, f \ - f_2(x,t,\phi_1,0,0)f &= \mu_{(2,\phi_1,0,0)} f\\
\frac{\partial g}{\partial t} - d_3\,\Delta\, g - f_3(x,t,\phi_1,v^*,0)g &= \mu_{(3,\phi_1,v^*,0)} g\\
\frac{\partial h}{\partial t} - d_1\,\Delta\, h - f_1(x,t,0,v^*,w^*)h &= \mu_{(1,0,v^*,w^*)} h\\
\frac{\partial F}{\partial t} - d_2\,\Delta\, F \ - f_2(x,t,0,0,w^*)F &= \mu_{(2,0,0,w^*)} F
\end{aligned}
\tag{12}
$$

in $\Omega\times(0,\infty)$ and $f=0,\ \ g=0,\ \ h=0,\ \ F=0$ on $\partial\Omega\times(0,\infty)$, respectively. We construct now via the following monotone iteration process a "rectangle" which encloses the positive global attractor. We start this process with

$$
\left(\overline{U}^{(0)},\overline{V}^{(0)},\overline{W}^{(0)}\right) = (\phi_1, v^*, w^*) \quad \text{and} \quad \left(\underline{U}^{(0)},\underline{V}^{(0)},\underline{W}^{(0)}\right) = (u_*, v_*, \phi_3)\,.
$$

Then we solve for the positive T–periodic solutions for the following boundary value problems:

$$
\begin{aligned}
\overline{U}_t^{(k)} - d_1\Delta\overline{U}^{(k)} &= \overline{U}^{(k)} f_1\left(x,t,\overline{U}^{(k)},\underline{V}^{(k-1)},\underline{W}^{(k-1)}\right)\\
\overline{V}_t^{(k)} - d_2\Delta\overline{V}^{(k)} &= \overline{V}^{(k)} f_2\left(x,t,\overline{U}^{(k-1)},\overline{V}^{(k)},\underline{W}^{(k-1)}\right)\\
\overline{W}_t^{(k)} - d_3\Delta\overline{W}^{(k)} &= \overline{W}^{(k)} f_3\left(x,t,\overline{U}^{(k-1)},\overline{V}^{(k-1)},\overline{W}^{(k)}\right)\\
\underline{U}_t^{(k)} - d_1\Delta\underline{U}^{(k)} &= \underline{U}^{(k)} f_1\left(x,t,\underline{U}^{(k)},\overline{V}^{(k-1)},\overline{W}^{(k-1)}\right)\\
\underline{V}_t^{(k)} - d_2\Delta\underline{V}^{(k)} &= \underline{V}^{(k)} f_2\left(x,t,\underline{U}^{(k-1)},\underline{V}^{(k)},\overline{W}^{(k-1)}\right)\\
\underline{W}_t^{(k)} - d_3\Delta\underline{W}^{(k)} &= \underline{W}^{(k)} f_3\left(x,t,\underline{U}^{(k-1)},\underline{V}^{(k-1)},\underline{W}^{(k)}\right)
\end{aligned}
\tag{13}
$$

in $\Omega \times (0,\infty)$ and

$$\overline{U}^{(k)} = \overline{V}^{(k)} = \overline{W}^{(k)} = 0 = \underline{U}^{(k)} = \underline{V}^{(k)} = \underline{W}^{(k)} \quad \text{on} \quad \partial\Omega$$

for $k = 1, 2, \ldots$. It is easy to see that $\underline{U}^{(0)} \leq \overline{U}^{(0)}$, $\underline{V}^{(0)} \leq \overline{V}^{(0)}$ and $\underline{W}^{(0)} \leq \overline{W}^{(0)}$, which we denote by

$$\left(\underline{U}^{(0)}, \underline{V}^{(0)}, \underline{W}^{(0)}\right) \leq \left(\overline{U}^{(0)}, \overline{V}^{(0)}, \overline{W}^{(0)}\right),$$

using a comparison theorem for parabolic equations (Theorem 5.1 in [2] and Theorem 10.1 in [7]) and the food chain conditions.

We need to prove that $\overline{U}^{(1)}$, $\overline{V}^{(1)}$, $\overline{W}^{(1)}$, $\underline{U}^{(1)}$, $\underline{V}^{(1)}$, and $\underline{W}^{(1)}$ exist. First of all, observe that

$$\overline{W}^{(1)} = \overline{W}^{(0)} \quad \text{and} \quad \underline{U}^{(1)} = \underline{U}^{(0)}.$$

Let us prove that $\overline{U}^{(1)}$ exists. According to Lemma 1, this is true if in

$$\Psi_t - d_1 \,\Delta\, \Psi - f_1(x,t,0,v_*,\phi_3)\Psi = \mu_{(1,0,v_*,\phi_3)}\Psi$$

we have that the eigenvalue $\mu_{(1,0,v_*,\phi_3)} < 0$.

The food chain conditions imply that $f_1(x,t,0,v_*,\phi_3) > f_1(x,t,0,v^*,w^*)$. Therefore,

$$\mu_{(1,0,v_*,\phi_3)} < \mu_{(1,0,v^*,w^*)}.$$

We assumed that

$$\mu_{(1,0,v^*,w^*)} < 0$$

to have the existence of $\underline{U}^{(0)}$. Hence $\overline{U}^{(1)}$ exists. Similarly, we prove that $\overline{V}^{(1)}$, $\underline{V}^{(1)}$ and $\underline{W}^{(1)}$ exist since

$$\mu_{(2,0,0,\overline{W}^{(0)})} < 0$$

for both $\overline{V}^{(1)}$ and $\underline{V}^{(1)}$ and $\mu_{(3,0,0,0)} < 0$ for $\underline{W}^{(1)}$. Now we want to establish the monotonicity relation

$$\begin{aligned}\left(\underline{U}^{(0)}, \underline{V}^{(0)}, \underline{W}^{(0)}\right) \leq \left(\underline{U}^{(1)}, \underline{V}^{(1)}, \underline{W}^{(1)}\right) &\leq \left(\overline{U}^{(1)}, \overline{V}^{(1)}, \overline{W}^{(1)}\right) \\ &\leq \left(\overline{U}^{(0)}, \overline{V}^{(0)}, \overline{W}^{(0)}\right).\end{aligned}$$

Using the comparison theorem as above and food chain conditions, we prove that

$$\overline{U}^{(1)} \le \overline{U}^{(0)} \quad \text{and} \quad \overline{V}^{(1)} \le \overline{V}^{(0)}$$

as well as

$$\underline{V}^{(0)} \le \underline{V}^{(1)} \quad \text{and} \quad \underline{W}^{(0)} \le \underline{W}^{(1)}.$$

In order to prove the inequality in the middle, let us prove now that

$$\underline{V}^{(1)} < \overline{V}^{(1)}.$$

We have seen that

$$\overline{U}^{(0)} > \underline{U}^{(0)} \quad \text{and} \quad \underline{W}^{(0)} < \overline{W}^{(0)}.$$

Then

$$f_2\left(x, t, \overline{U}^{(0)}, \overline{V}^{(1)}, \underline{W}^{(0)}\right) > f_2\left(x, t, \underline{U}^{(0)}, \overline{V}^{(1)}, \overline{W}^{(0)}\right)$$

hence

$$\overline{V}_t^{(1)} - d_2\Delta\overline{V}^{(1)} > \overline{V}^{(1)} f_2\left(x, t, \underline{U}^{(0)}, \overline{V}^{(1)}, \overline{W}^{(0)}\right).$$

Therefore, again by the comparison theorem we have

$$\underline{V}^{(1)} < \overline{V}^{(1)}.$$

Similarly, we prove that $\underline{W}^{(1)} \le \overline{W}^{(1)}$ and $\underline{U}^{(1)} \le \overline{U}^{(1)}$.

Inductively we can prove that

$$\begin{aligned}\left(\underline{U}^{(k-1)}, \underline{V}^{(k-1)}, \underline{W}^{(k-1)}\right) &\le \left(\underline{U}^{(k)}, \underline{V}^{(k)}, \underline{W}^{(k)}\right) \le \left(\overline{U}^{(k)}, \overline{V}^{(k)}, \overline{W}^{(k)}\right) \\ &\le \left(\overline{U}^{(k-1)}, \overline{V}^{(k-1)}, \overline{W}^{(k-1)}\right)\end{aligned}$$

in Ω, for $k = 1, 2, \ldots$. This relation guarantees the existence of

$$\begin{aligned}(\overline{U}, \overline{V}, \overline{W}) &= \lim_{k\to\infty}\left(\overline{U}^{(k)}, \overline{V}^{(k)}, \overline{W}^{(k)}\right) \\ (\underline{U}, \underline{V}, \underline{W}) &= \lim_{k\to\infty}\left(\underline{U}^{(k)}, \underline{V}^{(k)}, \underline{W}^{(k)}\right).\end{aligned}$$

Now we can state the main theorem of this paper.

THEOREM 1. *Consider the reaction-diffusion system* (1), (2), (3), *along with conditions* H1–H6. *Also assume that*

i) $\mu_{(1,0,0,0)} < 0$ *and* $\mu_{(3,0,0,0)} < 0$ *where these numbers denote the principle eigenvalues of the linearized problems*

$$\begin{aligned} \frac{\partial v}{\partial t} - d_1 \Delta v - v f_1(x,t,0,0,0) &= \mu_{(1,0,0,0)} v \qquad && \text{in } \Omega \times (0,\infty) \\ v &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial w}{\partial t} - d_3 \Delta w - w f_3(x,t,0,0,0) &= \mu_{(3,0,0,0)} w \qquad && \text{in } \Omega \times (0,\infty) \\ w &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned}$$

ii) $\mu_{(2,\phi_1,0,0)} < 0$, $\mu_{(3,\phi_1,v^*,0)} < 0$, $\mu_{(1,0,v^*,w^*)} < 0$, $\mu_{(2,0,0,w^*)} < 0$ *where these numbers denote the principal eigenvalues for the equations*

$$\begin{aligned} \frac{\partial f}{\partial t} - d_2 \,\Delta\, f \;- f_2(x,t,\phi_1,0,0) f &= \mu_{(2,\phi_1,0,0)} f \\ \frac{\partial g}{\partial t} - d_3 \,\Delta\, g - f_3(x,t,\phi_1,v^*,0) g &= \mu_{(3,\phi_1,v^*,0)} g \\ \frac{\partial h}{\partial t} - d_1 \,\Delta\, h - f_1(x,t,0,v^*,w^*) h &= \mu_{(1,0,v^*,w^*)} h \\ \frac{\partial F}{\partial t} - d_2 \,\Delta\, F \;- f_2(x,t,0,0,w^*) F &= \mu_{(2,0,0,w^*)} F \end{aligned}$$

in $\Omega \times (0,\infty)$ *and*

$$f = 0, \quad g = 0, \quad h = 0, \quad F = 0 \quad \text{on } \partial\Omega \times (0,\infty)$$

respectively. Then the reaction-diffusion system (1) *possesses a positive global attractor, namely,*

$$S = [\underline{U}, \overline{U}] \times [\underline{V}, \overline{V}] \times [\underline{W}, \overline{W}].$$

Proof. We prove first that if (u,v,w) is a solution to (1), then there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned} (1+\varepsilon_0)\phi_1(x,t) &\geq u(x,t) \geq u_*(x,t) + \varepsilon_0 \\ v^*(x,t) &\geq v(x,t) \geq v_*(x,t) \\ w^*(x,t) &\geq w(x,t) \geq \phi_3(x,t) + \varepsilon_0, \end{aligned}$$

For t large enough and $x \in \Omega$.

Let $\widetilde{u}$ be a solution to (6) such that

$$u(x,0) = \widetilde{u}(x,0).$$

We have that

$$\widetilde{u}_t - d_1\,\Delta\,\widetilde{u} = \widetilde{u} f_1(x,t,\widetilde{u},0,0) \geq \widetilde{u} f_1(x,t,\widetilde{u},v,w) \quad \text{in} \quad \Omega \times (0,\infty).$$

Therefore, $\widetilde{u}(x,t) \geq u(x,t)$ in $\Omega \times (0,\infty)$ by the comparison theorem.

Lemma 1 tells us that

$$\lim_{t\to\infty} \|\phi_1(\cdot,t) - \widetilde{u}(\cdot,t)\|_{C^1(\Omega)} = 0.$$

Therefore, given $\varepsilon > 0$ there exists $T_0(\varepsilon) > 0$ such that when $t \geq T_0(\varepsilon)$,

$$0 < u(x,t) \leq \widetilde{u}(x,t) \leq \ (1+\varepsilon)\phi_1(x,t), \qquad x \in \Omega.$$

In order to prove that $v^*(x,t) \geq v(x,t)$, we need to consider the following equation

$$\Phi_t - d_2 \Delta\Phi = \Phi f_2(x,t,(1+\varepsilon)\phi_1,\Phi,0).$$

Pick $\varepsilon_0 > 0$ such that the eigenvalue $\mu_{(2,(1+\varepsilon_0)\phi_1,0,0)} < 0$, which is true since $\lim_{\varepsilon\to 0} \mu_{(2,(1+\varepsilon)\phi_1,0,0)} = \mu_{(2,\phi_1,0,0)}$ by lemma 15.7 in [3]. Then, there exists a T-periodic positive globally attracting solution $V^*_{\varepsilon_0}$ of

$$\Phi_t - d_2 \Delta\Phi = \Phi f_2(x,t,(1+\varepsilon_0)\phi_1,\Phi,0).$$

Condition 3 implies that

$$\left(V^*_{\varepsilon_0}\right)_t - d_2 \Delta V^*_{\varepsilon_0} = V^*_{\varepsilon_0} f_2(x,t,(1+\varepsilon)\phi_1, V_{\varepsilon_0},0) \geq V^*_{\varepsilon_0} f_2(x,t,u,V^*_{\varepsilon_0},w)$$

Therefore $V^*_{\varepsilon_0}(x,t) \geq v(x,t)$ in $\Omega \times (T_0,\infty)$, if $\varepsilon_0 \to 0$ then $V^*_{\varepsilon_0} \to v^*$, so $v^*(x,t) \geq v(x,t)$ in Ω and t big enough. To take care of the inequality $w^*(x,t) \geq w(x,t)$ we use our previous inequalities along with condition 3 to have that

$$f_3(x,t,(1+\varepsilon)\phi_1,v^*,w) \geq f_3(x,t,u,v,w) \qquad \text{in } \Omega \text{ and } t \text{ big enough}$$

As before pick ε_0 such that $\mu_{(3,(1+\varepsilon_0)\phi_1,v^*,0)} < 0$ then there exists ${w^*}_{\varepsilon_0}$ solution to

$$\Phi_t - d_3\Phi = \Phi f_3(x,t,(1+\varepsilon)\phi_1,v^*,\Phi)$$

then ${w^*}_{\varepsilon_0}(x,t) \geq w(x,t)$ in (T_0,∞), moreover if $\varepsilon_0 \to 0$ then $w^*_{\varepsilon_0} \to w^*$ then $w^* \geq w$ in $\Omega \times (\widetilde{T},\infty)$.

Next, we prove that $u(x,t) \geq u_*(x,t)$, $x \in \Omega$ and t big enough. Let $\widehat{u}$ be a solution to (10) such that $u(x,0) = \widehat{u}(x,0)$. We have that

$$u_t - d_1\Delta u \geq u f_1(x,t,u,v^*,w^*) \qquad \text{in } \Omega \times (T_0,\infty)$$

Therefore $u(x,t) \geq \widehat{u}(x,t)$ in $\Omega \times (T_0,\infty)$ again by comparison theorem. Lemma 1 says that for $\varepsilon_0 > 0$ exists T_1 such that $\|u_*(\cdot,t) - \widehat{u}(\cdot,t)\|_{C^1(\Omega)} \leq \varepsilon_0$ then $u(x,t) \geq u_*(x,t) + \varepsilon_0$ for $t > \max\{T_0,T_1\}$.

Now to prove $v(x,t) \geq v_*(x,t)$ in $\Omega \times (T_0,\infty)$ we have that

$$(v_*)_t - d_2 v_* = v_* f_2(x,t,0,v_*,w^*) \leq v_* f_2(x,t,u,v_*,w)$$

then $v_*(x,t) \leq v(x,t)$ in $\Omega \times (T_0,\infty)$.

Finally, let $\tilde{w}$ solution to (7) such that $w(x,0) = \tilde{w}(x,0)$ then

$$\tilde{w}_t - d_3\Delta\tilde{w} = \tilde{w} f_3(x,t,0,0,\tilde{w}) \leq \tilde{w} f_3(x,t,u,v,\tilde{w})$$

then $\tilde{w}(x,t) \leq w(x,t)$ in $\Omega \times (0,\infty)$ by comparison. Then for ε_0 there exists $T_3 > 0$ such that $w(x,t) \geq \phi_3(x,t) + \varepsilon_0$ in $\Omega \times (T_3,\infty)$.

Now we want to establish that for $k = 1,2,\ldots,$

$$\begin{aligned} \overline{U}^{(k)}(x,t) &\geq u(x,t) \geq \underline{U}^{(k)}(x,t) \\ \overline{V}^{(k)}(x,t) &\geq v(x,t) \geq \underline{V}^{(k)}(x,t) \\ \overline{W}^{(k)}(x,t) &\geq w(x,t) \geq \underline{W}^{(k)}(x,t), \end{aligned} \tag{14}$$

for $x \in \Omega$ and t big enough.

Assume that the above inequality is true for $k-1$. We prove that $\overline{U}^{(k)}(x,t) \geq u(x,t)$.

To do this, let $\widetilde{U}$ be a solution to (13) (first equation) such that

$$u(x,0) = \widetilde{U}(x,0).$$

By hypothesis of induction, we have that

$$v(x,t) \geq \underline{V}^{(k-1)}(x,t) \qquad \text{and} \qquad w(x,t) \geq \underline{W}^{(k-1)}(x,t)$$

so $\widetilde{U}_t - d_1\Delta\widetilde{U} = \widetilde{U} f_1\left(x,t,\widetilde{U},\underline{V}^{(k-1)},\underline{W}^{(k-1)}\right) \geq \widetilde{U} f_1\left(x,t,\widetilde{U},v,w\right)$. Then $u(x,t) \leq \widetilde{U}(x,t)$ for $x \in \Omega$ and t big enough.

Lemma 1 implies that for $\varepsilon > 0$, there exists a t big enough such that

$$\|\widetilde{U}(\cdot,t) - \overline{U}^{(k)}(\cdot,t)\|_{C^1(\Omega)} \leq \varepsilon$$

Therefore,

$$u(x,t) \leq \widetilde{U}(x,t) \leq \overline{U}^{(k)}(x,t) \quad \text{for} x \in \Omega \text{ and } t \text{ big enough.}$$

In a similar fashion, we get that

$$\overline{V}^{(k)}(x,t) \geq v(x,t) \quad \text{and} \quad \overline{W}^{(k)}(x,t) \geq w(x,t) \quad \text{for } x \in \Omega \text{ and } t \text{ big enough.}$$

Now we prove $w(x,t) \geq \underline{W}^{(k)}(x,t)$. We know that

$$u(x,t) \geq \underline{U}^{(k-1)}(x,t) \quad \text{and} \quad v(x,t) \geq \underline{V}^{(k-1)}(x,t)$$

then

$$w_t - d_3\Delta w = wf_3(x,t,u,v,w) \geq wf_3\left(x,t,\underline{U}^{(k-1)},\underline{V}^{(k-1)},w\right)$$

hence $w(x,t) \geq \underline{W}^{(k)}$. Therefore, $w(x,t) \geq \underline{W}^{(k)}(x,t)$ for $x \in \Omega$ and t big enough.

Similarly we prove the remaining inequalities, so we have

$$\begin{aligned}
\overline{U}^{(k)}(x,t) &\geq u(x,t) \geq \underline{U}^{(k)}(x,t)\\
\overline{V}^{(k)}(x,t) &\geq v(x,t) \geq \underline{V}^{(k)}(x,t)\\
\overline{W}^{(k)}(x,t) &\geq w(x,t) \geq \underline{W}^{(k)}(x,t),
\end{aligned}$$

for $x \in \Omega$ and t big enough.

Letting k tend to ∞, we get

$$\left(\overline{U},\overline{V},\overline{W}\right) \geq (u,v,w) \geq (\underline{U},\underline{V},\underline{W}) \quad \text{in} \quad C^1(\overline{\Omega})\ .$$

Therefore, S is a global attractor of system (1). ∎

Now we have some extinction results, which we state in the following.

THEOREM 2. *Consider system* (1) *with conditions* (2) *and* (3).

i) *Assume conditions* (H1)–(H6) *and* $\mu_{(1,0,0,0)} > 0$, $\mu_{(2,0,0,0)} > 0$, $\mu_{(3,0,0,0)} > 0$, *then every nonnegative solution to* (1) *converges to* $(0,0,0)$ *in* $C(\overline{\Omega})$ *as* $t \to \infty$.

ii) *Suppose that* $\mu_{(1,0,0,0)} > 0$, $\mu_{(2,0,0,0)} > 0$ *and* $\mu_{(3,0,0,0)} < 0$, *then every nonnegative solution to* (1) *converges to* $(0,0,\phi_3)$ *in* $C(\overline{\Omega})$ *as* $t \to \infty$.

Proof. i) Let $(u(x,t), v(x,t), w(x,t))$ be a solution to (1) and $\widetilde{u}(x,t)$ a solution to (6) such that $\widetilde{u}(x,0) = u(x,0)$. Again the comparison theorem implies that $u(x,t) < \widetilde{u}(x,t)$.

Lemma 1 implies that $\widetilde{u}(x,t) \to 0$ uniformly in Ω as $t \to \infty$. Then $u(x,t) \to 0$ uniformly in Ω as $t \to \infty$. Similarly, we get that both $v(x,t)$ and $w(x,t)$ converge to 0 uniformly in Ω as $t \to \infty$. Therefore, $(u(x,t), v(x,t), w(x,t)) \to (0,0,0)$ uniformly in Ω as $t \to \infty$.

ii) Let $(u(x,t), v(x,t), w(x,t))$ be a solution to (1). Arguing as in i) we have that
$u(x,t) \to 0$ and $v(x,t) \to 0$ uniformly in Ω. Now since $\mu_{(3,0,0,0)} < 0$, we have that $\underline{W}^{(0)}$ exists. Then, arguing as in Theorem 1, we get that $w(x,t)$ converges uniformly to ϕ_3 in Ω. ∎

4. Applications

In this final section we give an example to illustrate Theorem 1. Consider the following reaction-diffusion system

$$
\begin{aligned}
u_t - d_1\Delta u &= u\left(a_1(x,t) - b_{11}(x,t)v - b_{12}(x,t) - b_{13}(x,t)w\right) \\
v_t - d_2\Delta v &= v\left(a_2(x,t) + b_{21}(x,t)u - b_{22}(x,t)v - b_{23}(x,t)w\right) \quad \text{in } \Omega \times (0,\infty) \\
w_t - d_3\Delta w &= w\left(a_3(x,t) + b_{31}(x,t)u + b_{32}(x,t)v - b_{33}(x,t)w\right) \\
u = v &= w = 0 \qquad \text{on } \partial\Omega \times (0,\infty)
\end{aligned}
$$

The functions a_i and b_{ij} are smooth non-negative periodic in t functions and Ω is a bounded smooth domain in $\mathbb{R}^n$. As we mentioned in the Introduction, Feng [2] considered this system with non-negative constants a_i and b_{ij}. In order to apply Theorem 1 to this system, we need to consider the following eigenvalue problems associated when two species are absent,

$$
\begin{aligned}
&\varphi_t - d_1\Delta\varphi - a_1(x,t)\varphi = \mu_{a_1}\varphi && \text{in } \Omega \times (0,\infty) \\
&\varphi = 0 && \text{on } \partial\Omega \times (0,\infty)
\end{aligned}
$$

and

$$
\begin{aligned}
&\psi_t - d_3\Delta\psi - a_3(x,t)\psi = \mu_{a_3}\psi && \text{in } \Omega \times (0,\infty) \\
&\psi = 0 && \text{on } \partial\Omega \times (0,\infty).
\end{aligned}
$$

In the case of one specie absent,

$$
\begin{aligned}
f_t - d_1\,\Delta f - \Big(a_1(x,t) - b_{12}(x,t)\overline{V}^{(0)}(x,t) - b_{13}\overline{W}^{(0)}(x,t)\Big)\, f& \\
= \mu_{a_1 - b_{12}\overline{V}^{(0)} - b_{13}\overline{W}^{(0)}} f& \\
g_t - d_2\,\Delta g - \Big(a_2(x,t) + b_{21}(x,t)\overline{V}^{(0)}(x,t)\Big)\, g = \mu_{a_2 + b_{21}\overline{V}^{(0)}} g& \\
h_t - d_2\,\Delta h - \Big(a_2(x,t) - b_{23}(x,t)\overline{W}^{(0)}(x,t)\Big)\, h = \mu_{a_2 - b_{23}\overline{W}^{(0)}} h& \\
F_t - d_3\,\Delta F - \Big(a_3(x,t) + b_{31}(x,t)\overline{U}^{(0)}(x,t) + b_{32}\overline{V}^{(0)}(x,t)\Big)\, F& \\
= \mu_{a_3 + b_{31}\overline{U}^{(0)} + b_{32}\overline{V}^{(0)}} F&
\end{aligned}
$$

in $\Omega \times (0,\infty)$ and

$$f = 0, \quad g = 0, \quad h = 0, \quad F = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty)$$

Assuming the previous principal eigenvalues to be negative as well as conditions H1–H6, we get a global attractor for our system.

Remark. For the various estimates of the principal eigenvalue of periodic-parabolic problems we refer to [4]. We have formulated the existence of a global attractor in terms of the eigenvalue sign of some periodic-parabolic eigenvalue problem, raising the question of how temporal periodicity and spatial heterogeneity interact to mediate coexistence. Thinking about such problems, we are led immediately to analyze the relative contributions of space and time to eigenvalues of the form $\mu_{g(x,t)}$, where $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is T-periodic in time. Once again we refer to [4] for a valuable initial investigation of this topic, but there remains much to be done in this direction in order for us to be able to find answers regarding the interplay between temporal and spatial effects.

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