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Nonlinear Flexural Excitations and Drill-String Dynamics

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1. Introduction

In [6] we have shown how a drill-string in a typical rig may be modelled in terms of a space curve with structure. This structure defines the relative orientation of neighbouring cross-sections along the drill-string. Specifying a unit vector (which may be identified with the normal to each cross-section) at each point along the drill-string centroid enables the state of flexure to be related to the angle between this vector and the tangent to the space-curve. Specifying a second vector orthogonal to the first vector (thereby placing it in the plane of the cross-section) can be used to encode the state of bending and twist along the drill-string. Thus a field of two mutually orthogonal unit vectors along the drill-string provides three continuous dynamical degrees of freedom that, together with the continuous three degrees of freedom describing a space-curve relative to some arbitrary origin in space, define a simple Cosserat rod model. The dynamical equations of motion for these continuous six degrees of freedom are presented in section 2. Supplemented with appropriate constitutive relations and boundary conditions the model can fully accommodate the modes of vibration that are traditionally associated with the motion of drill-strings in the engineering literature: namely axial motion along the length of the drill-string, torsional or rotational motion and transverse or lateral motion [3], [4]. This model is well suited to study numerical simulations that offer valuable guidance on the detection and control of destructive vibrational configurations [7]. However situations in which excessive flexural motions become excited provide significant challenges for such simulations. Inspired by [2] we have begun a study of the dynamics of large amplitude flexural motion in the context of extensible and shearable drill-strings under gravity, with a view to controlling the onset of "snap-buckling".

Antman et al [2] have provided general conditions on initial data that permit a certain approximation scheme for a simple Cosserat system to generate a solution that can move irregularly through a family of equilibrium states parametrised by time. In this paper we explicitly construct such families for a class of constituitive relations appropriate for drill-strings.

A series of approximations is discussed leading to an intuitive understanding of such motions and analytic expressions that permit one to estimate the effects of critical dimensionless parameters on the phenomena involved are derived. Such formulae also provide useful benchmarks that can be used to evaluate the efficiency of numerical code designed to simulate more complex vibrational scenarios involving drill-strings.

2. Equations of motion

The general mathematical theory of non-linear elasticity is well established. The general theory of one-dimensional Cosserat continua derived as limits of three-dimensional continua can be consulted in Antman [1]. The theory is fundamentally formulated in the Lagrangian picture in which material elements are labelled by the variable s. The dynamical evolution of a drill-string with mass density, $s \in [0, L_0] \mapsto \rho(s)$, and cross-sectional area, $s \in [0, L_0] \mapsto A(s)$, is governed by Newton's dynamical laws:

$$\begin{aligned} \rho A\, \partial_t^2 \mathbf{r} &= \partial_s \mathbf{n} + \mathbf{F} \\ \partial_t(\rho\, \mathbf{I}(\mathbf{w})) &= \partial_s \mathbf{m} + \partial_s \mathbf{r} \times \mathbf{n} + \mathbf{L} \end{aligned}$$

applied to a triad of orthonormal vectors:

$$s \in [0, L_0] \mapsto \{\mathbf{d_1}(s,t), \mathbf{d_2}(s,t), \mathbf{d_3}(s,t)\}$$

over the space-curve:

$$s \in [0, L_0] \mapsto \mathbf{r}(s,t)$$

at time t where $\mathbf{F}$ and $\mathbf{L}$ denote external force and torque densities respectively and $s \in [0, L_0] \mapsto \rho \mathbf{I}$ is a drill-string moment of inertia tensor. In these field equations the *contact forces* $\mathbf{n}$ and *contact torques* $\mathbf{m}$ are related to the *strains* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ by constitutive relations. The strains are themselves defined in terms of the configuration variables $\mathbf{r}$ and $\mathbf{d}_k$ for $k = 1, 2, 3$ by the relations:

$$\begin{aligned} \partial_s \mathbf{r} &= \mathbf{v} \\ \partial_s\, \mathbf{d}_k &= \mathbf{u} \times\, \mathbf{d}_k \\ \partial_t\, \mathbf{d}_k &= \mathbf{w} \times\, \mathbf{d}_k . \end{aligned} \tag{1}$$

The latter ensures that the triad remains orthonormal under evolution. The last equation identifies

$$\mathbf{w} = \frac{1}{2}\sum_{k=1}^{3} \mathbf{d}_k \times \partial_t \, \mathbf{d}_k$$

with the local angular velocity vector of the director triad.

In view of the comments above about the relative spatial dimensions of the drill-string, the *bottom-hole-assembly* (BHA) and *rotary* may be regarded as heavy mass points with rotary inertia attached to the ends of the drill-string. The effects of the BHA stabilisers can be modelled by boundary conditions [6] which constrain the direction of the tangent to the drill-string at the *drill-bit*. The general model accommodates drill-strings whose characteristics (density, cross-sectional area, rotary inertia) vary with s. If the characteristics change discontinuously at some point (e.g. where the drill-bit interacts with the rock base or drill-strings with different characteristics are joined) conditions on the contact forces and torques on either side, $\pm$, of the point must be satisfied. If a rigid body of mass M_0 and rotary inertia tensor I_0 is also attached to the point then discontinuous contact forces and torques contribute to the equations of motion of such an attachment. In general at a drill-string junction $s = s_0$ the contact forces and couples are subject to the discontinuity conditions:

$$\mathbf{n}^+(s_0,t) - \mathbf{n}^-(s_0,t) + \mathbf{F_0}(s_0,t) = M_0 \;\ddot{\mathbf{r}}\,(s_0,t) \tag{2}$$

$$\mathbf{m}^+(s_0,t) - \mathbf{m}^-(s_0,t) + \mathbf{G_0}(s_0,t) = \; I_0(\dot{\mathbf{w}}(s_0,t)). \tag{3}$$

In these equations $\mathbf{F_0}(s_0,t)$ and $\mathbf{G_0}(s_0,t)$ denote the external forces and torques acting at $s = s_0$.

To close the above equations of motion constitutive relations appropriate to the active drill-string must be specified:

$$\mathbf{n}(s,t) = \hat{\mathbf{n}}(\mathbf{u}(s,t), \mathbf{v}(s,t), \mathbf{u}_t(s,t), \mathbf{v}_t(s,t), \ldots, s)$$
$$\mathbf{m}(s,t) = \hat{\mathbf{m}}(\mathbf{u}(s,t), \mathbf{v}(s,t), \mathbf{u}_t(s,t), \mathbf{v}_t(s,t), \ldots, s).$$

These relations specify a “natural” reference configuration (say at $t = 0$) with strains $\mathbf{U}(s), \mathbf{V}(s)$ such that

$$\hat{\mathbf{n}}(\mathbf{U}(s), \mathbf{V}(s), \ldots, s) = 0$$
$$\hat{\mathbf{m}}(\mathbf{U}(s), \mathbf{V}(s), \ldots, s) = 0.$$

A standard reference configuration has $\partial_s \mathbf{r} = \mathbf{d}_3$, i.e.

$$\mathbf{V}(\mathbf{s}) = \mathbf{d}_3(s, 0).$$

If this standard configuration is such that $\mathbf{r}(s)$ is a space-curve with Frenet curvature $\kappa_0(s)$ and torsion $\tau_0(s)$ and the standard directors are oriented so that $\mathbf{d}_1(s)$ is the unit normal to the space-curve and $\mathbf{d}_2$ the associated unit binormal then

$$\mathbf{U}(s) = \kappa_0(s)\ \mathbf{d}_2(s) + \tau_0(s)\ \mathbf{d}_3(s).$$

This follows immediately from the definition (1) of $\mathbf{u}$ and the Frenet-Serret equations for the space-curve:

$$\begin{aligned}
\partial_s \mathbf{d}_3 &= \kappa_0 \, \mathbf{d}_1 \\
\partial_s \mathbf{d}_1 &= -\kappa_0 \, \mathbf{d}_3 + \tau_0 \, \mathbf{d}_2 \\
\partial_s \mathbf{d}_2 &= -\tau_0 \, \mathbf{d}_1 .
\end{aligned}$$

The use of a curved standard configuration has immediate application to the dynamics of drill-strings in curved bore-holes. In vertical bore-holes a space-curve with zero curvature and torsion defines the natural unstressed configuration of the drill-string. Thus the reference state of the drill-string in the absence of gravity is $\{\mathbf{r}(s,t) = -s\,\mathbf{k},\ \mathbf{d_1}(s,t) = \mathbf{i},\ \mathbf{d_2}(s,t) = -\mathbf{j},\ \mathbf{d_3}(s,t) = -\mathbf{k}\}$ where $s \in [0, L_0]$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a global Cartesian orthonormal frame attached to an origin at the top $(s = 0)$ of the drill-string. The value of L_0 is the physical length of the unstressed drill-string in the absence of gravity.

The simplest constitutive model to consider is based on Kirchhoff constitutive relations with shear deformation. One may exploit the full versatility of the Cosserat model by generalising the Kirchhoff constitutive relations to include viscoelasticity, curved reference states and effects to prohibit total compression. Such a model exhibits a rich dynamical behaviour that accommodates much observed phenomena. However the presence of arbitrary rotations relating the local director frame to the global Cartesian frame renders the equations of motion inherently non-linear.

The dynamical equations of motion for the drill-string are given in dimensionless form [6] as [1]

[1] The symbols $\mathbf{n}, \mathbf{m}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ and $\rho \mathbf{I}$ are henceforth dimensionless

$$
\begin{aligned}
\ddot{\mathbf{R}} &= \mathbf{n}' - g\mathbf{k} + \mathbf{f} \qquad (4)\\
((\rho\mathbf{I})(\mathbf{w}))_\eta &= \mathbf{m}' + \kappa(\mathbf{R}' \times \mathbf{n}) + \mathbf{l}\\
\mathbf{d}_k{}' &= \mathbf{u} \times \mathbf{d}_k\\
\dot{\mathbf{d}}_k &= \mathbf{w} \times \mathbf{d}_k\\
\mathbf{R}_\sigma &= \mathbf{v} \qquad (5)
\end{aligned}
$$

where $\sigma = s/L_0$, $\partial_\eta \mathbf{R} = \dot{\mathbf{R}}, \partial_\sigma \mathbf{R} = \mathbf{R}'$ etc., κ is the dimensionless constant $EAL_0^2\rho_0/(G(\hat{I}_{11} + \hat{I}_{22}))$ in terms of Young's modulus E for the string, A its cross-sectional area, ρ_0 its uniform density and G its shear modulus. The weight of the drill-string (extracted from the body force density $\mathbf{f}$) is encoded into the dimensionless parameter $g = \hat{g}\rho_0 L_0/E$ where $\hat{g}$ is the acceleration due to gravity and the evolution of the system is given in terms of time t by the dimensionless variable $\eta = ct/L_0$ where $c = \sqrt{E/\rho_0}$.

With

$$
\begin{aligned}
\mathbf{n} &= n_1\,\mathbf{d}_1 + n_2\,\mathbf{d}_2 + n_3\,\mathbf{d}_3\\
\mathbf{m} &= m_1\,\mathbf{d}_1 + m_2\,\mathbf{d}_2 + m_3\,\mathbf{d}_3\\
\mathbf{v} &= v_1\,\mathbf{d}_1 + v_2\,\mathbf{d}_2 + v_3\,\mathbf{d}_3\\
\mathbf{u} &= u_1\,\mathbf{d}_1 + u_2\,\mathbf{d}_2 + u_3\,\mathbf{d}_3\\
\mathbf{w} &= w_1\,\mathbf{d}_1 + w_2\,\mathbf{d}_2 + w_3\,\mathbf{d}_3
\end{aligned}
$$

the classical Kirchoff constitutive relations may be written in dimensionless form as:

$$
\begin{aligned}
n_1 &= \chi v_1\\
n_2 &= \chi v_2\\
n_3 &= v_3 - 1\\
m_1 &= I_{11}u_1 + I_{12}u_2\\
m_2 &= I_{12}u_1 + I_{22}u_2\\
m_3 &= u_3
\end{aligned}
$$

where the dimensionless parameter $\chi = G/E$ is expressed in terms of the ratio of torsional to (Young's) elastic modulus of elasticity and $I_{ij} = \hat{I}_{ij}E/(G(\hat{I}_{11} + \hat{I}_{22}))$ is a set of reduced (dimensionless) inertia components of the drill-string. In the above $\hat{I}_{ij}$ are the components of the rod "inertia tensor", $(\rho\hat{\mathbf{I}})$, with respect to the local director frame $\{\,\mathbf{d}_j\}$. For a cylindrical rod with an annular cross section, outer radius r_O and inner radius r_I the non zero components of $\hat{I}_{ij}$ are

$$\hat{I}_{11} = \hat{I}_{22} = \pi\rho_0(r_O^4 - r_I^4)/4$$
$$\hat{I}_{33} = \pi\rho_0(r_O^4 - r_I^4)/2.$$

The generalisation of these relations to include viscoelasticity and an initially curved drill-string requires the introduction of the dimensionless positive viscoelasticity parameters $\{\alpha_{n1}, \alpha_{n2}, \alpha_{n3}, \alpha_{m1}, \alpha_{m2}, \alpha_{m3}\}$ and the functions $\{\sigma \mapsto U_2(\sigma), \sigma \mapsto U_3(\sigma)\}$ defining the curvature and torsion of the reference space-curve:

$$n_1 = \chi(v_1 + \alpha_{n1}\, v_{1\eta}) \qquad (6)$$
$$n_2 = \chi(v_2 + \alpha_{n2}\, v_{2\eta})$$
$$n_3 = (v_3 + \alpha_{n3}\, v_{3\eta}) - 1$$
$$m_1 = I_{11}(u_1 + \alpha_{m1}\, u_{1\eta}) + I_{12}(u_2 - U_2 + \alpha_{m2}\, u_{2\eta})$$
$$m_2 = I_{12}(u_1 + \alpha_{m1}\, u_{1\eta}) + I_{22}(u_2 - U_2 + \alpha_{m2}\, u_{2\eta})$$
$$m_3 = (u_3 - U_3 + \alpha_{m3}\, u_{3\eta}). \qquad (7)$$

3. Boundary conditions

We assume that the drill-string is connected to effective masses μ_{top} and μ_{bit} modelling the top drive and BHA respectively and point rigid bodies with rotary inertia tensors $\mathbf{J}^{top}$ and $\mathbf{J}^{bit}$ modelling the rotary and drill-bit respectively. The top-drive connection to the drill-string is located at $\sigma = 0$ and the drill-bit is located at $\sigma = 1$. At these points forces $\mathbf{F}^{top}(\eta)$ and $\mathbf{F}^{bit}(\eta)$ and torques $\mathbf{L}^{top}(\eta)$ and $\mathbf{L}^{bit}(\eta)$ act. The nature of these forces depends on the way the boundary conditions are implemented. In some circumstances certain of their components may be prescribed (e.g by the frictional forces or drive torques in evidence) while other components may be determined dynamically by the constraints in evidence (e.g. the way the stabilisers interact with the bore-hole in the BHA). The basic boundary conditions that follow from the discontinuity relations (2), (3) are then:

$$\mu_{top}\ddot{\mathbf{R}}(0,\eta) = \mathbf{n}(0,\eta) - \mu_{top}\, g\mathbf{k} + \mathbf{F}^{top}(\eta)$$
$$\partial_\eta(\mathbf{J}^{top}(\mathbf{w}))(0,\eta) = \mathbf{m}(0,\eta) + \mathbf{L}^{top}(\eta)$$

at $\sigma = 0$ and

$$\mu_{bit}\ddot{\mathbf{R}}(1,\eta) = -\mathbf{n}(1,\eta) - \mu_{bit}\, g\mathbf{k} + \mathbf{F}^{bit}(\eta)$$
$$\partial_\eta(\mathbf{J}^{bit}(\mathbf{w}))(1,\eta) = -\mathbf{m}(1,\eta) + \mathbf{L}^{bit}(\eta)$$

at $\sigma = 1$. The implementation of these boundary conditions can be a matter of some expediency in order to match the model as closely as possible with the way in which an actual drilling process is executed. This is because the process is essentially controlled by varying the magnitude of the tension in the cable attached to the drill-string at the top, the voltage and current in the electric motor that delivers the torque to the drill-string and the manner in which the drill-string itself is lowered through the rotary to accommodate the penetration of the bit into the rock. If the motion of the latter is prescribed so that

$$\mathbf{R}(0,\eta) = \Lambda^T(\eta)\mathbf{k}$$

then the top constraining force is determined from:

$$\mathbf{F}^{top}(\eta) = \mu_{top}\ddot{\mathbf{R}}(0,\eta) - \mathbf{n}(0,\eta) + \mu_{top}\, g\mathbf{k}$$

in terms of the contact force $\mathbf{n}(0,\eta)$. The electric motor is designed to deliver torque that drives the entire drill-string towards a uniform (target) angular speed Ω_0. It also responds to signals from various control feedback devices that are designed to expedite a smooth as possible evolution towards this target state [7]. If one assumes that the rotary is rotationally unconstrained and that the transmission of the torque to the rotary is similarly unconstrained then the torque boundary condition to be satisfied at $\sigma = 0$ is

$$(\mathbf{J}^{top}(\mathbf{w}))_\eta(0,\eta) = \mathbf{m}(0,\eta) + \mathbf{L}^{top}(\eta) \tag{8}$$

for some control torque $\mathbf{L}^{top}(\eta)$.

In practice the rotary is constrained by bearings to rotate in a fixed horizontal plane. In vertical drilling this is orthogonal to the tangent to the drill-string as it passes through the rotary. In these circumstances only the $\mathbf{k}$ component of (8) is imposed. The constraint on the rotary can be modelled by $\mathbf{R}'(0,\eta)\cdot\mathbf{i} = 0$, $\mathbf{R}'(0,\eta)\cdot\mathbf{j} = 0$. These constraints imply the existence of constraining torques in $\mathbf{L}^{top}(\eta)$ in the $\mathbf{i}$ and $\mathbf{j}$ directions which may be determined from the projection of (8) in these directions.

The implementation of realistic boundary conditions at $\sigma = 1$ is considerably more involved and we refer to [6] for details. For the discussion below simplified boundary conditions at both $\sigma = 0$ and $\sigma = 1$ will be employed.

4. The equations for small flexural excitations of drill-strings

Under certain conditions the drill-string in the above model exhibits instabilities in torsional, axial and lateral modes of excitation [6]. Small amplitude lateral excitations arise as perturbations (about a vertical static configuration) of the form

$$
\begin{aligned}
\mathbf{R}(\sigma,\eta) &= \epsilon X(\sigma,\eta)\ \mathbf{i} + (\frac{g\,\sigma^2}{2} - \sigma + \epsilon Z(\sigma,\eta))\ \mathbf{k} \\
\mathbf{d}_1 &= \mathbf{i} + \epsilon\Theta(\sigma,\eta)\mathbf{k} \\
\mathbf{d}_2 &= -\mathbf{j} \\
\mathbf{d}_3 &= -\mathbf{k} + \epsilon\Theta(\sigma,\eta)\mathbf{i}.
\end{aligned}
$$

Substituting these into (4)-(5), (6)-(7) with $\epsilon \ll 1$ yields equations for axial, lateral and flexural waves:

$$
\frac{\partial^2}{\partial\eta^2}Z(\sigma,\eta) - \frac{\partial^2}{\partial\sigma^2}Z(\sigma,\eta) - \alpha_{n3}\frac{\partial^3}{\partial\eta\partial\sigma^2}Z(\sigma,\eta) = 0
$$

$$
\begin{aligned}
&\Theta(\sigma,\eta)\,(\sigma g-1)\,\kappa\ (\chi+\sigma g-\chi\,\sigma g) - \kappa\,\chi\,\alpha_{n1}\,(\sigma g-1)^2\,\frac{\partial}{\partial\eta}\Theta(\sigma,\eta) \\
&\quad -\alpha_{n1}\kappa\,\chi\ (\sigma g-1)\,\frac{\partial^2}{\partial\eta\partial\sigma}X(\sigma,\eta) + I_{22}\,\alpha_{m2}\frac{\partial^3}{\partial t\partial\sigma^2}\Theta(\sigma,\eta) \\
&+I_{22}\,\frac{\partial^2}{\partial\sigma^2}\Theta(\sigma,\eta) - I_{22}\,\frac{\partial^2}{\partial\eta^2}\Theta(\sigma,\eta) + \kappa\ (\chi+\sigma g-\chi\,\sigma g)\,\frac{\partial}{\partial\sigma}X(\sigma,\eta) = 0
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&\Theta(\sigma,\eta)g\ (\chi-1) + \chi\,\frac{\partial^2}{\partial\sigma^2}X(\sigma,\eta) + \chi\,\alpha_{n1}\left(\frac{\partial}{\partial\eta}\Theta(\sigma,\eta)\right)g \\
&+\alpha_{n1}\chi\ (\sigma g-1)\,\frac{\partial^2}{\partial\eta\partial\sigma}\Theta(\sigma,\eta) - (\chi+\sigma g-\chi\,\sigma g)\,\frac{\partial}{\partial\sigma}\Theta(\sigma,\eta) \\
&-\frac{\partial^2}{\partial\eta^2}X(\sigma,\eta) + \chi\,\alpha_{n1}\frac{\partial^3}{\partial\eta\partial\sigma^2}X(\sigma,\eta) = 0
\end{aligned}
\tag{10}
$$

If the viscoelastic damping and gravitational weight terms proportional to g are neglected equations (9) and (10) may be decoupled resulting in the traditional Euler-Bernoulli-Kelvin beam equation for small lateral deflections

[5]:

$$
\begin{aligned}
& I_{22}\,\chi\,\frac{\partial^4}{\partial\sigma^4}X(\sigma,\eta)+\chi\,\kappa\,\frac{\partial^2}{\partial\eta^2}X(\sigma,\eta) \\
& \quad - I_{22}\;(1+\chi)\,\frac{\partial^4}{\partial\sigma^2\partial\eta^2}X(\sigma,\eta)+I_{22}\,\frac{\partial^4}{\partial\eta^4}X(\sigma,\eta)=0.
\end{aligned}
$$

These equations have been extensively analysed in the literature and their solutions are applicable to a wide range of systems that can exhibit perturbative instabilities. They arise naturally from the general theory based on the equations in section (2) and when supplemented with (linear or static) boundary conditions enable one to study the spectra of the linear flexural modes of the system. However we are mainly interested here in the non-perturbative, non-linear modes associated with non-static boundary conditions such as those that occur at the ends of a realistic drill-string in operation [6]. In the next section we seek an approximation scheme which permits an analytic determination of such non-perturbative flexural modes.

5. The equations for large amplitude motion of light elastic drill-strings with heavy attachments

For the configuration $\mathbf{R}(\sigma,\eta),\mathbf{d}_k(\sigma,\eta)$, with *strain* variables $\mathbf{v},\mathbf{u}$ and angular velocity $\mathbf{w}$ defined by

$$
\mathbf{R}'=\mathbf{v},\qquad \mathbf{d}_k'=\mathbf{u}\times\mathbf{d}_k, \tag{11}
$$

$$
\dot{\mathbf{d}}_k=\mathbf{w}\times\mathbf{d}_k,
$$

the equations of motion of the drill-string under gravity are given by

$$
\ddot{\mathbf{R}}=\mathbf{n}'-g\mathbf{k}+\mathbf{f}, \tag{12}
$$

$$
\partial_\eta[\rho\mathbf{I}\cdot\mathbf{w}]=\mathbf{m}'+\kappa\mathbf{R}'\times\mathbf{n}+\mathbf{l}, \tag{13}
$$

in terms of external body forces $\mathbf{f}$ and torques $\mathbf{l}$. At $\sigma=0$, the variables $\mathbf{R},\mathbf{w}$ satisfy the equation of motion of a rigid body with heavy effective reduced mass μ_0 and effective reduced inertia $\mathbf{J}_0(\eta)$,

$$
\begin{aligned}
\mu_0\ddot{\mathbf{R}}_0(\eta)&=\mathbf{n}_0(\eta)-\mu_0 g\mathbf{k}+\mathbf{f}_0(\eta),\\
\partial_\eta[\mathbf{J}_0(\eta)\cdot\mathbf{w}_0(\eta)]&=\mathbf{m}_0(\eta)+\mathbf{l}_0(\eta),
\end{aligned} \tag{14}
$$

where $\mathbf{R}_0(\eta) = \mathbf{R}(0,\eta), \mathbf{n}_0(\eta) = \mathbf{n}(0,\eta), \mathbf{m}_0(\eta) = \mathbf{m}(0,\eta), \mathbf{w}_0(\eta) = \mathbf{w}(0,\eta)$. Similarly at $\sigma = 1$, these variables satisfy the equation of motion of a rigid body with heavy effective reduced mass μ_1 and effective reduced inertia $\mathbf{J}_1(\eta)$:

$$\begin{aligned} \mu_1\ddot{\mathbf{R}}_1(\eta) &= -\mathbf{n}_1(\eta) - \mu_1 g\mathbf{k} + \mathbf{f}_1(\eta), \\ \partial_\eta[\mathbf{J}_1(\eta)\cdot\mathbf{w}_1(\eta)] &= -\mathbf{m}_1(\eta) + \mathbf{l}_1(\eta) \end{aligned} \tag{15}$$

where $\mathbf{R}_1(\eta) = \mathbf{R}(1,\eta), \mathbf{n}_1(\eta) = \mathbf{n}(1,\eta), \mathbf{m}_1(\eta) = \mathbf{m}(1,\eta), \mathbf{w}_1(\eta) = \mathbf{w}(1,\eta)$. We now choose ($\mathbf{f} = \mathbf{l} = 0$, $\mathbf{f}_1 = 0 = \mathbf{l}_1$) and consider the approximation where the left-hand sides of (12) and (13) may be neglected at all time compared with the forces and torques on the right-hand sides. This approximation has been discussed in [2] and effectively means that the dynamical motion at an end of the drill-string dominates the forces and torques that drive the system. Following [2] we refer to this approximation in terms of "light elastic drill-strings with heavy attachments". It is expected to be a reasonably approximation as long as the weight of the drill-string is not excessive compared with that of the BHA.

Then (12) may readily be integrated with respect to σ,

$$\mathbf{n}(\sigma,\eta) = g(\sigma-1)\mathbf{k} + \mathbf{n}_1(\eta) = g\sigma\mathbf{k} + \mathbf{n}_0(\eta), \tag{16}$$

while (13) becomes

$$\mathbf{m}' = -\kappa\mathbf{R}' \times \mathbf{n}. \tag{17}$$

To proceed further one needs to specify explicit constitutive relations for the drill-string. For this analysis we adopt the Kirchoff relations for a uniform cylindrically symmetric drill-string, neglect all forms of damping and refer to [2] for an account of the role played by viscoelastic damping in the dynamics and mathematical formulation of this approximation as the first term in an asymptotic expansion:

$$\mathbf{n} = \mathbf{v} - \mathbf{d}_3, \qquad \mathbf{m} = \tfrac{1}{2}\mathbf{u} + \tfrac{1}{2}u_3\mathbf{d}_3, \tag{18}$$

where we have set $\chi = 1$. Thus the drill-string can undergo extension and shear (cf. [2]). Although $\chi = 1$ is rather special it greatly simplifies formulae (as the analysis in section (10) demonstrates). From (18) and use of (16), (11) we find,

$$\mathbf{R}' = \mathbf{n}_1(\eta) + g(\sigma-1)\mathbf{k} + \mathbf{d}_3, \tag{19}$$

and from (17) with use of (19) we find,

$$\kappa^{-1}\mathbf{m}' = -\mathbf{R}' \times \mathbf{n} = \mathbf{n}_1(\eta) \times \mathbf{d}_3 + g(\sigma - 1)\mathbf{k} \times \mathbf{d}_3. \tag{20}$$

From (18), $\mathbf{u} = 2\mathbf{m} - m_3\mathbf{d}_3$, and since $u_3 = m_3$ equation (11) is reduced to,

$$\mathbf{d}_3' = \mathbf{u} \times \mathbf{d}_3 = 2\mathbf{m} \times \mathbf{d}_3. \tag{21}$$

The strategy is to solve equations (20), (21) for $\mathbf{m}(\sigma,\eta)$ and $\mathbf{d}_3(\sigma,\eta)$ in terms of $\mathbf{n}_1(\eta)$ or $\mathbf{n}_0(\eta)$ and then solve (19) for $\mathbf{R}(\sigma,\eta)$. Next one may substitute $\mathbf{m}(\sigma,\eta)$ and $\mathbf{d}_3(\sigma,\eta)$ in (11) to solve

$$\begin{aligned}\mathbf{d}_1' &= \mathbf{u} \times \mathbf{d}_1 = 2\mathbf{m} \times \mathbf{d}_1 - m_3\mathbf{d}_3 \times \mathbf{d}_1,\\ \mathbf{d}_2' &= \mathbf{u} \times \mathbf{d}_2 = 2\mathbf{m} \times \mathbf{d}_2 - m_3\mathbf{d}_3 \times \mathbf{d}_2,\end{aligned}$$

for $\mathbf{d}_1$ and $\mathbf{d}_2$. Finally the η dependence of the solutions is determined by fixing the evolution of $\mathbf{n}_1(\eta)$ from the boundary conditions. The imposition of the boundary conditions means that we are dealing with a nonlinear eigenvalue problem.

It is convenient to use the Euler angles θ, ϕ, ψ to relate the director frame to the global Cartesian frame $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$\begin{aligned}\mathbf{d}_1 &= (-\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi)\mathbf{i} + (\cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi)\mathbf{j}\\ &\qquad - (\sin\theta\cos\phi)\mathbf{k},\\ \mathbf{d}_2 &= (-\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi)\mathbf{i} + (\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi)\mathbf{j}\\ &\qquad + (\sin\theta\sin\phi)\mathbf{k},\\ \mathbf{d}_3 &= (\cos\psi\sin\theta)\mathbf{i} + (\sin\psi\sin\theta)\mathbf{j} + (\cos\theta)\mathbf{k}.\end{aligned}$$

Then with

$$\begin{aligned}\mathbf{R} &= X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k},\\ \mathbf{n}_1 &= n_{1x}(\eta)\mathbf{i} + n_{1y}(\eta)\mathbf{j} + n_{1z}(\eta)\mathbf{k},\\ \mathbf{m} &= m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k},\end{aligned}$$

we have

$$\begin{aligned}\mathbf{n}_1(\eta) \times \mathbf{d}_3 =& \mathbf{i}[n_{1y}(\eta)\cos\theta - n_{1z}(\eta)\sin\psi\sin\theta]\\ &+ \mathbf{j}[n_{1z}(\eta)\cos\psi\sin\theta - n_{1x}(\eta)\cos\theta]\\ &+ \mathbf{k}[n_{1x}(\eta)\sin\psi\sin\theta - n_{1y}(\eta)\cos\psi\sin\theta],\end{aligned}$$

$$\mathbf{k} \times \mathbf{d}_3 = -\mathbf{i} \sin\psi \sin\theta + \mathbf{j} \cos\psi \sin\theta$$

and equation (20) yields

$$\begin{aligned} \kappa^{-1} m_x' &= n_{1y}(\eta) \cos\theta - n_{1z}(\eta) \sin\psi \sin\theta - g(\sigma - 1) \sin\psi \sin\theta \\ \kappa^{-1} m_y' &= n_{1z}(\eta) \cos\psi \sin\theta - n_{1x}(\eta) \cos\theta + g(\sigma - 1) \cos\psi \sin\theta \\ \kappa^{-1} m_z' &= n_{1x}(\eta) \sin\psi \sin\theta - n_{1y}(\eta) \cos\psi \sin\theta. \end{aligned} \tag{22}$$

Similarly equation (21) reduces to

$$\begin{aligned} \psi' &= -2m_z + 2(m_x \cos\psi + m_y \sin\psi) \cot\theta, \\ \theta' &= 2(m_y \cos\psi - m_x \sin\psi). \end{aligned} \tag{23}$$

Differentiating (23) with respect to σ and using (22) gives

$$\kappa^{-1} \psi'' = 2(-n_{1x}(\eta) \sin\psi + n_{1y}(\eta) \cos\psi)(\sin\theta)^{-1} - \tfrac{1}{2}\psi'\theta' \cot\theta \tag{24}$$

$$\begin{aligned} \kappa^{-1} \theta'' =& 2n_{1z}(\eta) \sin\theta - 2(n_{1x}(\eta) \cos\psi + n_{1y}(\eta) \sin\psi) \cos\theta \\ &+ \psi'(m_x \cos\psi + m_y \sin\psi) + 2g(\sigma - 1) \sin\theta. \end{aligned} \tag{25}$$

These equations, supplemented with boundary conditions, permit an analysis of the general motion of the light drill-string as a dynamic space curve.

6. The equations for large planar flexural motion of light elastic drill-strings with heavy attachments

In order to proceed we now restrict to planar motion by seeking solutions with $\psi = 0$, $\phi = 0$ and $\mathbf{R} \cdot \mathbf{j} = 0$:

$$\begin{aligned} \mathbf{R} &= X\mathbf{i} + Z\mathbf{k}, \\ \mathbf{d}_1 &= \cos\theta \mathbf{i} - \sin\theta \mathbf{k}, \\ \mathbf{d}_2 &= \mathbf{j}, \\ \mathbf{d}_3 &= \sin\theta \mathbf{i} + \cos\theta \mathbf{k}. \end{aligned} \tag{26}$$

Hence from equation (11), (17):

$$\begin{aligned} \mathbf{w} &= \dot{\theta}\, \mathbf{j}, \\ \mathbf{u} &= \theta'\, \mathbf{j}, \\ \mathbf{m} &= \tfrac{1}{2}\theta'\, \mathbf{j}, \\ \mathbf{n}_1(\eta) &= n_{1x}(\eta)\, \mathbf{i} + n_{1z}(\eta)\, \mathbf{k}. \end{aligned} \tag{27}$$

We introduce new variables $A(\eta), \gamma(\eta)$ in place of $n_{1x}(\eta), n_{1z}(\eta)$ by writing

$$\mathbf{n}_1(\eta) = -(2\kappa)^{-1}A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}),$$

where, without loss of generality, we set $A(\eta) \geq 0$ and $-\pi \leq \gamma(\eta) \leq \pi$. For planar motion, equation (24) is satisfied and (25) reduces to

$$\begin{aligned} \theta'' =& 2\kappa[n_{1z}(\eta)\sin\theta - n_{1x}(\eta)\cos\theta] + 2\kappa g(\sigma - 1)\sin\theta \\ =& - A(\eta)^2\sin(\theta - \gamma(\eta)) + 2\kappa g(\sigma - 1)\sin\theta \end{aligned} \tag{28}$$

for $\theta(\sigma, \eta)$. We may integrate equation (19) using (26) to get

$$\begin{aligned} \mathbf{R}(\sigma, \eta) = \mathbf{R}(0, \eta) &- (\kappa\beta)^{-1}\sigma A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}) \\ &+ g\mathbf{k}(\tfrac{1}{2}\sigma^2 - \sigma) + \textstyle\int_0^\sigma d\sigma'(\sin\theta(\sigma', \eta)\mathbf{i} + \cos\theta(\sigma', \eta)\mathbf{k}). \end{aligned} \tag{29}$$

Clearly

$$\begin{aligned} \mathbf{R}(1, \eta) = \mathbf{R}(0, \eta) &- (\kappa\beta)^{-1}A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}) \\ &+ (-\tfrac{1}{2})g\mathbf{k} + \textstyle\int_0^1 d\sigma'(\sin\theta(\sigma', \eta)\mathbf{i} + \cos\theta(\sigma', \eta)\mathbf{k}). \end{aligned} \tag{30}$$

From (14), (15) these must satisfy the boundary equations:

$$\begin{aligned} \mu_0\ddot{\mathbf{R}}_0(\eta) &= \mathbf{n}_0(\eta) - \mu_0\, g\,\mathbf{k} + \mathbf{f}_0, \\ \mu_1\ddot{\mathbf{R}}_1(\eta) &= -\mathbf{n}_1(\eta) - \mu_1\, g\,\mathbf{k}. \end{aligned} \tag{31}$$

These equations constitute a non-linear eigenvalue problem of some complexity. In principle one can use Greens function techniques to express solutions to (28) in terms of Airy functions. However the analytic integration of such functions in (29) is not possible and recourse to a numerical investigation of a two point boundary value problem becomes necessary. We seek a further simplification in order to progress analytically.

7. Planar modes for a drill-string with negligible weight

The need for integrals of Airy functions can be eliminated if we set $g = 0$. With this simplification we may extract non-linear modes analytically if we adopt as boundary conditions $\mathbf{d}_1(0, \eta) = \mathbf{i}, \mathbf{d}_2(0, \eta) = \mathbf{j}, \mathbf{d}_3(0, \eta) = \mathbf{k}, \mathbf{J}_1(\eta) = 0$. Thus the drill-string is assumed clamped at the upper end and the BHA has zero rotary inertia. The former implies $\theta_0(\eta) = 0$ from (26) while the latter implies $\theta_1'(\eta) = 0$ from (15) and (27). Thus (28) gives the boundary value problem

$$\theta'' + A(\eta)^2 \sin(\theta - \gamma(\eta)) = 0, \tag{32}$$

$$\theta(0,\eta) = 0, \qquad \theta'(1,\eta) = 0. \tag{33}$$

We shall see below that given $\mathbf{R}(0,\eta)$ equations (30) and (31) enable one to determine $\mathbf{n}_1(\eta)$ and hence (via (16)) $\mathbf{n}_0(\eta)$.

We choose to solve the boundary value problem (32) (33) for $\theta(\sigma,\eta)$ in terms of $A(\eta)$ and $\gamma(\eta)$. Let $\alpha(\sigma,\eta) = \theta(\sigma,\eta) - \gamma(\eta)$, then $\alpha'' = \theta''$, and equation (28) becomes

$$\alpha'' + A(\eta)^2 \sin\alpha = 0 \tag{34}$$

with boundary conditions

$$\alpha(0,\eta) = -\gamma(\eta), \qquad \alpha'(1,\eta) = 0. \tag{35}$$

By multiplying (34) with α', we get for some $M(\eta)$ a first integral for α:

$$\tfrac{1}{2}(\alpha')^2 - A(\eta)^2 \cos\alpha = M(\eta). \tag{36}$$

Using the boundary condition at $\sigma = 1$, equation (36) can be written

$$\alpha' = \varepsilon\sqrt{2}A(\eta)\sqrt{\cos\alpha - \cos\alpha_1(\eta)}, \tag{37}$$

where $\varepsilon = \pm 1$. One can glean a qualitative picture of the eigen-modes from the (instantaneous) phase portrait of equation (34). At $\sigma = 0$, $\alpha'(0,\eta) = \varepsilon\sqrt{2}A(\eta)\sqrt{\cos\alpha(0,\eta) - \cos\alpha_1(\eta)}$, therefore the sign ε in (37) corresponds to the sign of $\alpha'(0,\eta)$. The stationary points $\alpha' = 0$, $\alpha'' = 0$ are given by

$$\alpha = 0, \pm\pi.$$

Typical solutions of the boundary-value problem (34), (35) correspond to the trajectories $\sigma \in [0,1] \mapsto$ AB, CD, ABCD, CDAB, ABCDAB,.. in the phase plane (α,α') (Figure 1) depending on $\alpha'(0,\eta)$.

In order to relate the integration of (37) to standard elliptic integrals let $k(\eta) = |\sin\frac{\alpha_1(\eta)}{2}|$, $(k(\eta) \in (0,1))$ and transform α to ξ with $\sin\xi(\sigma,\eta) = k(\eta)^{-1}\sin\frac{\alpha(\sigma,\eta)}{2}$. Using the following identities

$$\begin{aligned}
&\cos\alpha - \cos\alpha_1(\eta) = 2k(\eta)^2\cos^2\xi,\\
&\sin\alpha = 2k(\eta)\sin\xi\sqrt{1 - k(\eta)^2\sin^2\xi},\\
&\sin\alpha d\alpha = 4k(\eta)^2\sin\xi\cos\xi d\xi
\end{aligned}$$

equation (37) reduces to

$$d\sigma = \frac{\varepsilon d\xi}{A(\eta)\sqrt{1-k(\eta)^2\sin^2\xi}} \tag{38}$$

and the boundary conditions for ξ become

$$\sin\xi_0(\eta) = -k(\eta)^{-1}\sin\tfrac{1}{2}\gamma(\eta), \qquad \xi_1(\eta) = \varepsilon(2j+1)\tfrac{\pi}{2},$$

where $j = 0, 1, 2....$ Thus if we integrate (38) from 0 to 1,

$$\begin{aligned} 1 = \int_0^1 d\sigma' &= \frac{\varepsilon}{A(\eta)}\Big(F(\varepsilon(2j+1)\frac{\pi}{2}, k(\eta)) - F(\xi_0(\eta), k(\eta))\Big) \\ &= \frac{\varepsilon}{A(\eta)}\left((2j+1)\varepsilon K(k(\eta)) - F(\xi_0(\eta), k(\eta))\right) \end{aligned} \tag{39}$$

where F is the elliptic integral $F(\xi, k) = \int_0^\xi dz(1-k^2\sin^2 z)^{-1/2}$ and $K(k) = F(\pi/2, k)$ is the complete elliptic integral. Hence

$$A_j^\varepsilon(\eta) \equiv A(\alpha_1(\eta), \gamma(\eta)) = ((2j+1)K(k(\eta)) - \varepsilon F(\xi_0(\eta), k(\eta))) \tag{40}$$

where $\alpha_1(\eta) = \alpha(1, \eta)$, $k(\eta) = |\sin\frac{\alpha_1(\eta)}{2}|$, $\sin\xi_0(\eta) = -k(\eta)^{-1}\sin\frac{1}{2}\gamma(\eta)$. Thus we have a two index (ε, j) family of solutions α_j^ε to (34), (35) in terms of the three dynamical parameters $\{A(\eta), \alpha_1(\eta), \gamma(\eta)\}_j^\varepsilon$constrained by relation (40).

Each member of this family is described by a surface (Figure 2) in a space with coordinates $[A, \alpha_1, \gamma]$ or equivalently $[A, \theta_1, \gamma]$. For fixed η a point on such a surface generates an allowed flexural mode. Once the η dependence of the coordinates of this point is known, the evolution of this flexural mode is generated by a curve in this surface.

The dependence of ξ on σ can be obtained by integrating (38) from 0 to σ

$$\begin{aligned} \sigma = \int_0^\sigma d\sigma' &= \frac{\varepsilon}{A(\eta)}\left(F(\xi, k(\eta)) - F(\xi_0(\eta), k(\eta))\right) \\ &= \frac{\varepsilon}{A(\eta)}\big(2j\varepsilon K(k(\eta)) + F(\bar{\xi}, k(\eta)) - F(\xi_0(\eta), k(\eta))\big) \end{aligned} \tag{41}$$

where $\bar{\xi} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $\xi = \bar{\xi} + \varepsilon j\pi$.

Equation (41) can be written

$$\varepsilon\,\mathrm{sn}(A_j^\varepsilon(\eta)\sigma + \varepsilon F(\xi_0(\eta), k(\eta)), k(\eta)) = \sin(\xi) = k(\eta)^{-1}\sin\tfrac{1}{2}\alpha,$$

where sn is a Jacobi elliptic function. Finally, in terms of θ for each ε, j:

$$\begin{aligned}\theta(\sigma,\eta) = \alpha(\sigma,\eta) + \gamma(\eta) =& 2\arcsin(k(\eta)\mathrm{sn}(A(\eta)\sigma \\ &+ \varepsilon F(\xi_0(\eta), k(\eta)), k(\eta))) + \gamma(\eta).\end{aligned}$$

By putting this solution in (27) we obtain expressions for $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{w}, \mathbf{u}, \mathbf{m}, \mathbf{n}$,. Similarly (29) yields $\mathbf{R}$. All these solutions are given in terms of $A(\eta)$,$\gamma(\eta)$ and $\alpha_1(\eta)$ satisfying (40).

8. Reduction to a dynamical system

Having obtained analytic expressions for the instantaneous flexural eigenmodes it is of some interest to reduce the complete dynamical evolution of the drill-string to a dynamical system. We concentrate on the evolution of a particular mode labelled by j and ε (and drop these labels henceforth) and solve for $\gamma(\eta)$ and $\alpha_1(\eta)$. From the evolution equation,

$$\mu_1\ddot{\mathbf{R}}(1,\eta) = -\mathbf{n}_1(\eta) = (2\kappa)^{-1}A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}) \tag{42}$$

while equation (31) yields

$$\begin{aligned}\mathbf{R}_1(\eta) &= \mathbf{R}_0(\eta) - (2\kappa)^{-1}A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}) \\ &\quad + \int_0^1 d\sigma'(\sin\theta(\sigma',\eta)\mathbf{i} + \cos\theta(\sigma',\eta)\mathbf{k}) \\ &= X_1(\eta)\mathbf{i} + Z_1(\eta)\mathbf{k}.\end{aligned}$$

We choose $\mathbf{R}_0(\eta) = 0$ and evaluate the integral to get

$$\begin{aligned}\cos\gamma(\eta)X_1(\eta) - \sin\gamma(\eta)Z_1(\eta) &= \int_0^1 d\sigma' \sin\alpha(\sigma',\eta) \\ &= \varepsilon\frac{2k(\eta)\ \cos\xi(0,\eta)}{A(\eta)} \equiv f_1(\eta), \\ \sin\gamma(\eta)X_1(\eta) + \cos\gamma(\eta)Z_1(\eta) &= -(2\kappa)^{-1}A(\eta)^2 + \int_0^1 d\sigma' \cos\alpha(\sigma',\eta) \\ = -(2\kappa)^{-1}A(\eta)^2 - 1 &+ 2\frac{(2j+1)E(k) - \varepsilon E(\xi(0,\eta),k)}{A(\eta)} \\ &= -(2\kappa)^{-1}A(\eta)^2 - 1 + f_2(\eta) \equiv f_3(\eta)\end{aligned}$$

in terms of the abbreviations f_1, f_2, f_3. These equations can be solved for

$$X_1(\eta) = f_1(\eta)\cos\gamma(\eta) + f_3(\eta)\sin\gamma(\eta), \tag{43}$$

$$Z_1(\eta) = -f_1(\eta)\sin\gamma(\eta) + f_3(\eta)\cos\gamma(\eta). \tag{44}$$

But the equation of motion (42) reads

$$\begin{aligned} \mu_1 \ddot{X}_1(\eta) &= A(\eta)^2 \sin\gamma(\eta) \\ \mu_1 \ddot{Z}_1(\eta) &= A(\eta)^2 \cos\gamma(\eta) \end{aligned} \tag{45}$$

and, from (43), (44), we note that $X_1(\eta), Z_1(\eta)$ are functions of $\alpha_1(\eta)$ and $\gamma(\eta)$. Hence after differentiating them twice with respect to η, the equations (45) are reduced to two second order ordinary equations for $\alpha_1(\eta)$ and $\gamma(\eta)$ of the form:

$$\begin{aligned} \ddot{\gamma}(\eta) &= \mathcal{F}_\gamma(\alpha_1, \dot{\alpha}_1, \gamma, \dot{\gamma}) \\ \ddot{\alpha}_1(\eta) &= \mathcal{F}_\alpha(\alpha_1, \dot{\alpha}_1, \gamma, \dot{\gamma}). \end{aligned}$$

These equations determine the evolution of each non-linear planar flexural eigen-mode
$\{X(\sigma,\eta), Z(\sigma,\eta), \theta(\sigma,\eta)\}$ within this framework.

9. Quaternion formulation

In numerical simulations the use of Euler angle parametrisations of rotation group elements must be done with caution in order to obviate the intrusion of coordinate singularities in the evolution. An alternative procedure is to use a quaternion parametrisation which shifts the location of unwanted singularities. It is therefore not without relevance to recast the above solution into these variables. If $(q_1, q_2, q_3, q_4)(\sigma, \eta)$, (with $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$), denote the quaternions then the constraints of orthonormality among the directors can be taken into account by writing:

$$\begin{aligned} \mathbf{d}_1 &= (q_1^2 - q_2^2 - q_3^2 + q_4^2)\mathbf{i} + 2(q_1q_2 + q_3q_4)\mathbf{j} + 2(q_1q_3 - q_2q_4)\mathbf{k}, \\ \mathbf{d}_2 &= 2(q_1q_2 - q_3q_4)\mathbf{i} + (-q_1^2 + q_2^2 - q_3^2 + q_4^2)\mathbf{j} + 2(q_2q_3 + q_1q_4)\mathbf{k}, \\ \mathbf{d}_3 &= 2(q_1q_3 + q_2q_4)\mathbf{i} + 2(q_2q_3 - q_1q_4)\mathbf{j} + (-q_1^2 - q_2^2 + q_3^2 + q_4^2)\mathbf{k}. \end{aligned}$$

The restriction to planar motion follows with with $q_1 = q_3 = 0$,and $\mathbf{R}\cdot\mathbf{j} = 0$:

$$\begin{aligned}\mathbf{R} &= X\,\mathbf{i} + Z\,\mathbf{k},\\ \mathbf{d}_1 &= (q_4^2 - q_2^2)\mathbf{i} - 2q_2q_4\mathbf{k},\\ \mathbf{d}_2 &= \mathbf{j},\\ \mathbf{d}_3 &= 2q_2q_4\mathbf{i} + (q_4^2 - q_2^2)\mathbf{k}.\end{aligned} .$$

The relation to the previously used Euler angle θ is given by

$$q_2 = \sin\frac{\theta}{2}, q_4 = \cos\frac{\theta}{2}.$$

Hence from equation (11), (17):

$$\begin{aligned}\mathbf{m} &= (q_2'q_4 - q_4'q_2)\,\mathbf{j},\\ \mathbf{n}_1(\eta) &= n_{1x}(\eta)\,\mathbf{i} + n_{1z}(\eta)\,\mathbf{k}.\end{aligned}$$

As before, introduce new variables $A(\eta), \gamma(\eta)$ in place of $n_{1x}(\eta), n_{1z}(\eta)$ by writing

$$\mathbf{n}_1(\eta) = -A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}),$$

where, without lose of generality $A(\eta) \geq 0$ and $-\pi \leq \gamma(\eta) \leq \pi$.

Equations (20), (21) now reduce to:

$$q_4' = -\tfrac{1}{2}m_y q_2, \tag{46}$$

$$q_2' = \tfrac{1}{2}m_y q_4, \tag{47}$$

$$m_y' = -n_x(q_4^2 - q_2^2) + n_z 2q_2q_4.$$

It is convenient to introduce new variables Q_2, Q_4

$$\begin{pmatrix} Q_2 \\ Q_4 \end{pmatrix} = \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} q_2 \\ q_4 \end{pmatrix}$$

so

$$\begin{aligned}m_y' &= (A^2\sin\gamma)(q_4^2 - q_2^2) - (A^2\cos\gamma)(2q_2q_4) = -2A^2Q_2Q_4,\\ m_y &= q_2'q_4 - q_4'q_2 = Q_2'Q_4 - Q_4'Q_2.\end{aligned}$$

From (46), (47)

$$q_4'' = -\tfrac{1}{2}m_y' q_2 - \tfrac{1}{4}m_y^2 q_4,$$
$$q_2'' = \tfrac{1}{2}m_y' q_4 - \tfrac{1}{4}m_y^2 q_2,$$

and

$$Q_2'' = \tfrac{1}{2}m_y' Q_4 - \tfrac{1}{4}m_y^2 Q_2 = -A^2 Q_2 Q_4^2 - (\tfrac{Q_2'}{Q_4})^2 Q_2 \tag{48}$$

with boundary conditions

$$q_2(0,\eta) = 0, \quad q_4(0,\eta) = 1,$$
$$q_2'(1,\eta) = q_4'(1,\eta) = 0,$$

$$Q_2(0,\eta) = -\sin\tfrac{\gamma(\eta)}{2}, \quad Q_4(0,\eta) = \cos\tfrac{\gamma(\eta)}{2},$$
$$Q_2'(1,\eta) = Q_4'(1,\eta) = 0.$$

Multiplying (48) by $2Q_2'/(A^2Q_4^2)$ and integrating gives

$$\frac{{Q_2'}^2}{A(\eta)^2 Q_4^2} + Q_2^2 = Q_2(1,\eta)^2.$$

Hence

$$Q_2' = \varepsilon A(\eta) Q_4 \sqrt{Q_2(1,\eta)^2 - Q_2^2}$$

where $\varepsilon = \pm 1$. Let $k(\eta) \equiv Q_2(1,\eta)$ and $\hat{Q}(\sigma,\eta) = Q_2(\sigma,\eta)/k(\eta)$ then

$$1 = \int_0^1 d\sigma' = \frac{\varepsilon}{A(\eta)}(F(\hat{Q}(1,\eta),k(\eta)) - F(\hat{Q}(0,\eta),k(\eta))$$
$$= \frac{1}{A(\eta)}[(2j+1)K(k(\eta)),\eta) - \varepsilon F(\hat{Q}_0(\eta),k(\eta))]$$

where F is the elliptic integral $F(x,k) = \int_0^x dz(1-k^2z^2)^{-1/2}(1-z^2)^{-1/2}$, $\hat{Q}_0(\eta) \equiv \hat{Q}(0,\eta)) = Q_2(0,\eta)/k(\eta) = -k(\eta)^{-1}\sin\tfrac{1}{2}\gamma(\eta)$. Hence for each $\epsilon = \pm 1$ and $j = 0, 1, 2...$,

$$A_j^{\varepsilon}(\eta) = \Big((2j+1)K(k(\eta)) - \varepsilon F(\hat{Q}_0(\eta),k(\eta))\Big)$$

defines a surface in a space with coordinates $\{A, k, \gamma\}$.

10. The case $\chi \neq 1$

In section 5 we commented that a simplification arises if one takes $\chi = 1$ in the constitutive equations for the drill-string. In this section we relax this condition.

With $\chi \neq 1$,

$$\begin{aligned} \mathbf{n} &= \chi v_1 \mathbf{d}_1 + \chi v_2 \mathbf{d}_2 + (v_3 - 1)\mathbf{d}_3, \\ \mathbf{m} &= \tfrac{1}{2} u_1 \mathbf{d}_1 + \tfrac{1}{2} u_2 \mathbf{d}_2 + u_3 \mathbf{d}_3. \end{aligned} \tag{49}$$

which reduces to (18) when $\chi = 1$. In the case of a planar motion, $v_2 = u_1 = u_3 = 0$, and (16) and (49) gives

$$\begin{aligned} \mathbf{n}(\sigma,\eta) &= \chi v_1(\sigma,\eta)\mathbf{d}_1(\sigma,\eta) + \Big(v_3(\sigma,\eta) - 1\Big)\mathbf{d}_3(\sigma,\eta) \\ &= g(\sigma-1)\mathbf{k} + \mathbf{n}_1(\eta) \\ &= g(\sigma-1)\mathbf{k} - (2\kappa)^{-1}A(\eta)^2(\sin\gamma(\eta)\mathbf{i} + \cos\gamma(\eta)\mathbf{k}), \end{aligned} \tag{50}$$

where $\mathbf{d}_1$, $\mathbf{d}_3$ are given in (26). Thus

$$\begin{aligned} \chi v_1 \cos\theta + (v_3 - 1)\sin\theta &= -(2\kappa)^{-1}A(\eta)^2 \sin\gamma(\eta), \\ -\chi v_1 \sin\theta + (v_3 - 1)\cos\theta &= g(\sigma - 1) - (2\kappa)^{-1}A(\eta)^2\cos\gamma(\eta). \end{aligned}$$

Solving these equations for v_1 and v_3

$$\begin{aligned} v_1 &= \chi^{-1}[-(2\kappa)^{-1}A^2\sin\gamma\cos\theta + (2\kappa)^{-1}A^2\cos\gamma\sin\theta - g(\sigma-1)\sin\theta], \\ v_3 &= 1 - (2\kappa)^{-1}A^2\sin\gamma\sin\theta - (2\kappa)^{-1}A^2\cos\gamma\cos\theta + g(\sigma-1)\cos\theta \end{aligned}$$

yields

$$\mathbf{R}' = v_1\mathbf{d}_1 + v_3\mathbf{d}_3. \tag{51}$$

where $\mathbf{d}_1$ and $\mathbf{d}_3$ are given by (26). From (50), (51), the torque equation (17) is then expressed in terms of θ as

$$\begin{aligned} \theta'' = &- A(\eta)^2\sin(\theta-\gamma) + 2\kappa g(\sigma-1)\sin\theta \\ &+ \tfrac{1}{2}(1-\chi^{-1})[(2\kappa)^{-1}A(\eta)^4\sin(2\theta-2\gamma) - 2g(\sigma-1)A(\eta)^2\sin(2\theta-\gamma) \\ &+ 2\kappa g^2(\sigma-1)^2\sin 2\theta]. \end{aligned}$$

With $g = 0$ this reduces to

$$\alpha'' = -A(\eta)^2 \sin\alpha - \tfrac{1}{4}\kappa^{-1}(\chi^{-1} - 1)A(\eta)^4 \sin 2\alpha \tag{52}$$

with boundary conditions

$$\alpha(0, \eta) = -\gamma(\eta), \qquad \alpha_\sigma(1, \eta) = 0. \tag{53}$$

As before the stationary points in the phase plane are given by $\alpha' = 0$ and real roots of

$$\alpha'' = A(\eta)^2 \sin\alpha[-1 - (2\kappa)^{-1}(\chi^{-1} - 1)A(\eta)^2 \cos\alpha] = 0.$$

For $(2\kappa)^{-1}A(\eta)^2(\chi^{-1} - 1) > 1$ (or $(2\kappa)^{-1}A(\eta)^2(\chi^{-1} - 1) < -1$), there are real roots at

$$\alpha = 0, \pi, \pm\cos^{-1}\frac{-2\kappa}{(\chi^{-1} - 1)A(\eta)^2}$$

while for $-1 < (2\kappa)^{-1}A(\eta)^2(\chi^{-1} - 1) < 1$, they occur only at

$$\alpha = 0, \pi.$$

and we recall that the parameter $A(\eta)$ depends on the initial configuration $\mathbf{R}_1(\eta)$ and $\dot{\mathbf{R}}_1(\eta)$.

For χ near 1, the phase portrait (figure 3) is similar to the one in figure 1. When $2\kappa/(\chi^{-1} - 1) < A(\eta)^2$ the phase portrait changes structure (see figure 4).

Equation (52) admits a first integral

$$(\alpha')^2 - 2A(\eta)^2 \cos\alpha - \tfrac{1}{4}\kappa^{-1}(\chi^{-1} - 1)A(\eta)^4 \cos 2\alpha = M(\eta), \tag{54}$$

where using the boundary condition (53), $M(\eta) = -2A(\eta)^2 \cos\alpha_1(\eta) - \frac{1}{4}\kappa^{-1}(\chi^{-1} - 1)A(\eta)^4 \cos 2\alpha_1(\eta)$. Equation (54) can be written as

$$\begin{aligned}\alpha' = \varepsilon[2A(\eta)^2(\cos\alpha - \cos\alpha_1(\eta))\\ + \tfrac{1}{4}\kappa^{-1}(\chi^{-1} - 1)A(\eta)^4(\cos 2\alpha - \cos 2\alpha_1(\eta))]^2,\end{aligned} \tag{55}$$

where $\varepsilon = \pm 1$.

For the case $A(\eta)^2 > 2\kappa/(\chi^{-1} - 1)$, we can transfer this equation to a standard elliptic integral form, with

$$l(\eta) = \sqrt{\frac{-4\kappa + (\chi^{-1} - 1)A(\eta)^2(1 - \cos\alpha_1(\eta))}{4\kappa + (\chi^{-1} - 1)A(\eta)^2(1 + \cos\alpha_1(\eta))}}$$

and $m(\eta) = l(\eta)^{-1}|\tan\frac{\alpha_1(\eta)}{2}|^{-1}$, $\sin\zeta(\sigma,\eta) = l(\eta)\tan\frac{\alpha(\sigma,\eta)}{2}$, then equation (55) is reduced to

$$\int_0^\sigma d\sigma' = \int_{\zeta_0(\eta)}^{\zeta(\sigma,\eta)} \frac{\varepsilon}{A(\eta)\sqrt{[-2+(2\kappa)^{-1}(\chi^{-1}-1)A(\eta)^2(1-\cos\alpha_1(\eta))](1-\cos\alpha_1(\eta))}} \cdot \frac{d\zeta}{\sqrt{1-m(\eta)^2\sin^2\zeta}} \tag{56}$$

which is standard form.

The boundary conditions for ζ are

$$\sin\zeta_0(\eta) = -l(\eta)\tan\tfrac{1}{2}\gamma(\eta), \qquad \zeta_1(\eta) = \varepsilon(2j+1)\tfrac{\pi}{2},$$

where $j = 0, 1, 2....$

If we integrate (56) from $\sigma = 0$ to 1,

$$1 = \int_0^1 d\sigma' = \frac{\varepsilon\ ((2j+1)\varepsilon K(k(\eta)) - F(\xi_0(\eta), m(\eta)))}{A(\eta)\sqrt{[-2+(2\kappa)^{-1}(\chi^{-1}-1)A(\eta)^2(1-\cos\alpha_1(\eta))](1-\cos\alpha_1(\eta))}}.$$

where F is the elliptic integral defined previously with $K(m) = \int_0^{\pi/2} d\xi(1-m^2\sin^2\xi)^{-1/2}$. This is the generalisation of (39) to the case $\chi \neq 1$. From the solution (56) to (52) the equations for the displacement of the entire rod follow along the same lines as given in section(8). The role of the initial conditions in controlling a stable evolution through the system of quasi-stable equilibrium configurations is critical. We show in [8] how the critical conditions arise for small planar flexural motions in the context of the approximations discussed here.

11. Conclusions

Analytic expressions for the shear deformation have been obtained describing a family of planar flexural modes along a light rod subject to clamped boundary conditions at one end but with a heavy attached mass without rotary inertia free to move at the other, in a limit in which the dynamics of the rod is driven by the end conditions. The dynamics of these modes can be generated from an analysis of the phase portraits of a simple differential equation together with the structure of birfurcation surfaces in a 3-parameter space describing end-stress and end-shear deformation. These motions arise as approximate solutions of a simple Cosserat model which may be used to simulate the dynamical properties of the active components of a drilling assembly and offer a useful means of gaining a broad insight into the delicate dynamics associated with such structures.

Acknowledgements

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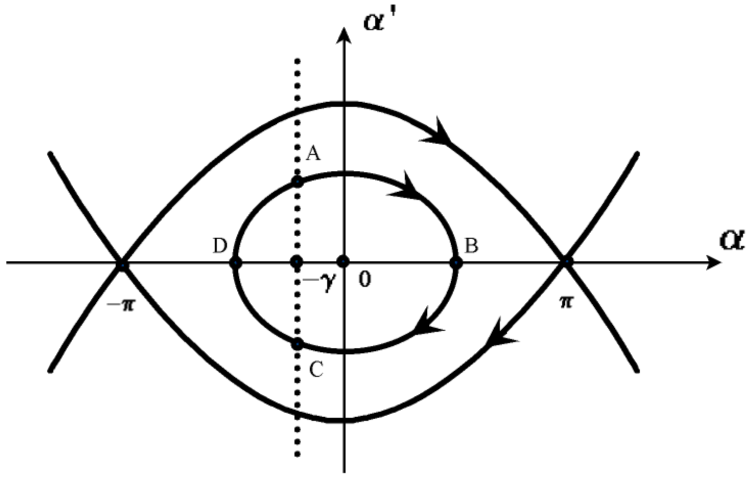

Figure 1: Phase Portrait for planar flexural modes α of a drill-string with $\chi = 1$. The boundary condition at $\sigma = 1$ corresponds to a point with $\alpha' = 0$ while the boundary condition at $\sigma = 0$ lies on the line (shown dotted) with $\alpha = -\gamma$. Non-linear eigen-modes correspond to trajectories that connect such points as σ varies from 0 to 1. Thus solutions of the boundary-value problem (34), (35) correspond to instantaneous Cosserat drill-string mode shapes given by the trajectories $\sigma \in [0, 1] \mapsto$ AB, CD, ABCD, CDAB, ABCDAB, ..., depending on the value of $\alpha'(0, \eta)$.

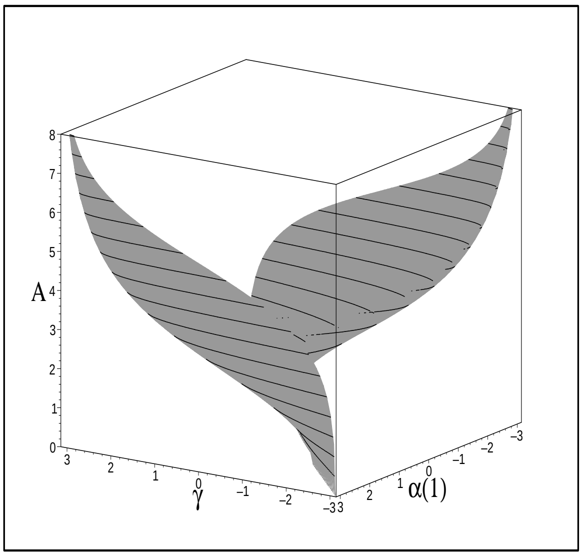

Figure 2: Segment of the bifurcation sheet ($j = 0$) describing the first flexural mode in the space (A, α_1, γ). The variables A, γ determine the instantaneous contact force at either end of a Cosserat drill-string with $\chi = 1$ while $\alpha_1 = \theta_1 - \gamma$ determines the corresponding degree of flexure at the end $\sigma = 1$.

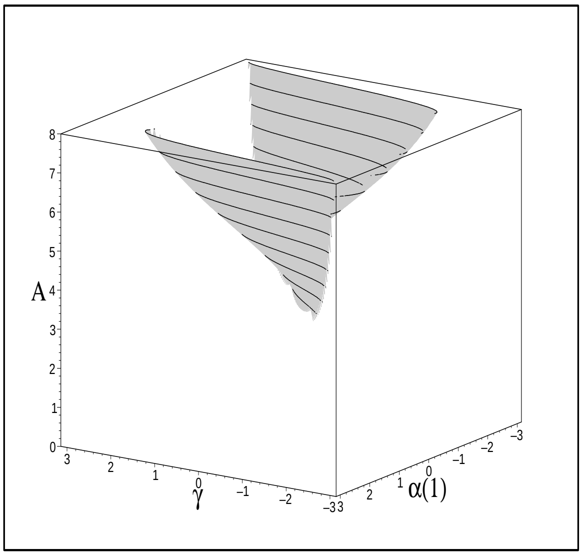

Figure 3: Segment of the the bifurcation sheet ($j = 1$) describing the next flexural mode in the space (A, α_1, γ). The variables A, γ determine the instantaneous contact force at either end of a Cosserat drill-string with $\chi = 1$ while $\alpha_1 = \theta_1 - \gamma$ determines the corresponding degree of flexure at the end $\sigma = 1$.

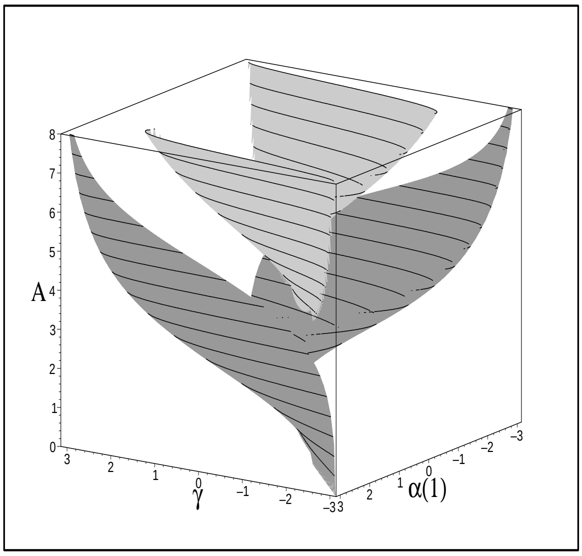

Figure 4: Segments of bifurcation sheets $j = 0, 1$ (distinguished by a graded grey scale) describing the first two flexural modes in the space (A, α_1, γ). The sheet with $j = 1$ lies above that with $j = 0$. The variables A, γ determine the instantaneous contact force at either end of a Cosserat drill-string with $\chi = 1$ while $\alpha_1 = \theta_1 - \gamma$ determines the corresponding degree of flexure at the end $\sigma = 1$.

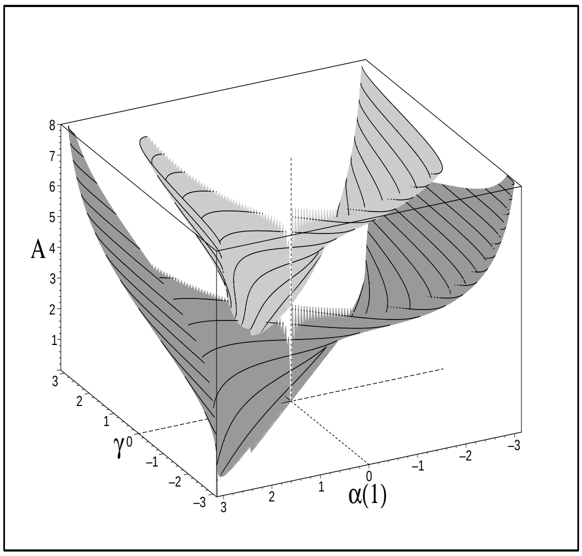

Figure 5: An alternative view of the bifurcation sheets $j = 0, 1$, (distinguished by a graded grey scale) describing the first two flexural modes in the space (A, α_1, γ) showing how they relate to the section $\gamma = 0$ in Figure 6. The sheet with $j = 1$ lies above that with $j = 0$. The variables A, γ determine the instantaneous contact force at either end of a Cosserat drill-string with $\chi = 1$ while $\alpha_1 = \theta_1 - \gamma$ determines the corresponding degree of flexure at the end $\sigma = 1$.

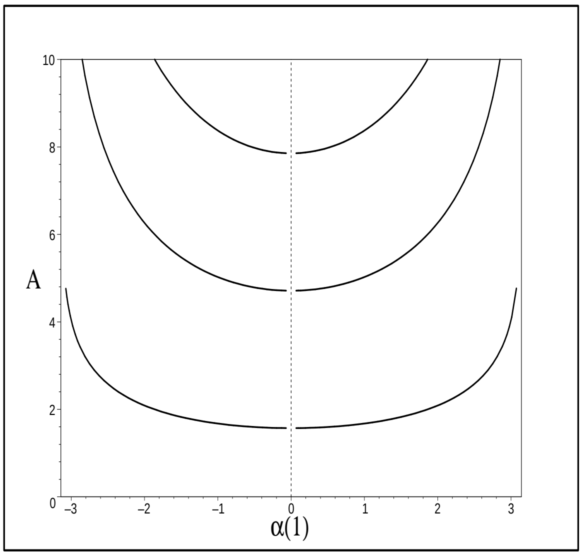

Figure 6: Bifurcation loci determined by the elliptic functions described in the text. The lowest two curves correspond to the section $\gamma = 0$ through the bifurcation sheets ($j = 0, 1$, $\varepsilon = \pm 1$) shown in Figure 5. The variable A determines the instantaneous contact force at either end of a Cosserat drill-string with $\chi = 1$ while $\alpha_1 = \theta_1$ determines the corresponding degree of flexure at the end $\sigma = 1$.

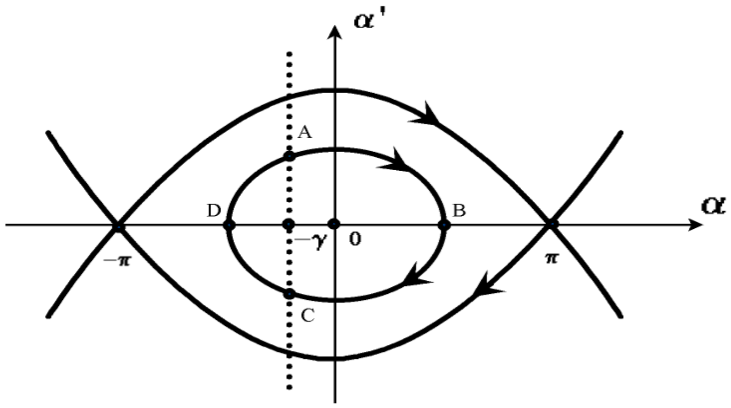

Figure 7: Phase Portrait for planar flexural modes α of a Cosserat drill-string with $\chi \neq 1$ and $A(\eta)^2 < 2\kappa/(\chi^{-1} - 1)$ The boundary condition at $\sigma = 1$ corresponds to a point with $\alpha' = 0$ while the boundary condition at $\sigma = 0$ lies on the line (shown dotted) with $\alpha = -\gamma$. Non-linear eigen-modes correspond to trajectories that connect such points as σ varies from 0 to 1.

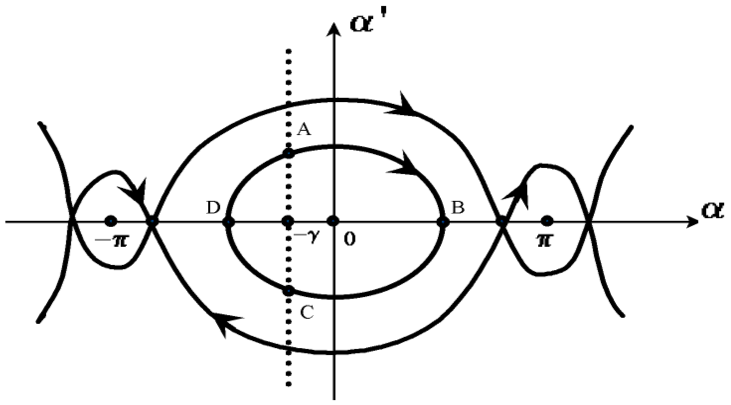

Figure 8: Phase Portrait for planar flexural modes α of a Cosserat drill-string with $\chi \neq 1$ and $A(\eta)^2 > 2\kappa/(\chi^{-1} - 1)$. The boundary condition at $\sigma = 1$ corresponds to a point with $\alpha' = 0$ while the boundary condition at $\sigma = 0$ lies on the line (shown dotted) with $\alpha = -\gamma$. Non-linear eigen-modes correspond to trajectories that connect such points as σ varies from 0 to 1.