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On the Anelastic Evolution of Second-Grade Materials

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1. Second-grade material points

A *second-grade material point* is characterized by a constitutive equation which depends upon the value of the 2-jet of the configuration at that point. In fact, the constitutive behaviour depends on a pair of 2-jets: (i) the *spatial 2-jet*, from the material body B to its present spatial configuration, and (ii) the *material 2-jet*, from the material body to a standard reference $\mathbb{R}^3$, which we designate as a *reference crystal*. For convenience, the material jet will be understood as going in the opposite direction, from $\mathbb{R}^3$ to B, and evaluated at the origin of $\mathbb{R}^3$. If the material jet remains constant throughout an interval of time, we say that the point does not evolve during that period, and the material response is *elastic* (provided, of course, that the constitutive law depends only on the *present value* of the spatial jet). If, on the other hand, the material jet changes with time, we have a case of *anelastic evolution*, such as in the theory of plasticity. This is, naturally, a point-wise concept. In other words, if a deformation of the reference crystal is given by a 2-jet $\{F^i_\alpha, H^i_{\alpha\beta}\}$, then there exists a fixed stored energy per unit volume of the reference crystal given by:

$$W = W(F^i_\alpha, H^i_{\alpha\beta})$$

With a global chart

$$\kappa_0 : B \to \mathbb{R}^3$$

specified in the body, the constitutive equation at a point $X \in B$ is given by the stored energy per unit volume of the chart as:

$$W_0(F^i_I, H^i_{IJ}, X) = J_P^{-1}\, W(F^i_I P^I_\alpha, H^i_{IJ} P^I_\alpha P^J_\beta + F^i_I Q^I_{\alpha\beta}) \tag{1}$$

where J_P stands for the determinant of P, and $\{P^I_\alpha, Q^I_{\alpha\beta}\}$ is the material jet to the point X.

If, for the different points of the body, the right-hand side of this constitutive law remains the same, except that the material jet $\{P, Q\}$ depends on the point, we say that the body is *uniform* [5, 6]: that is, all points are modeled after one and the same constitutive law. It is only the way in which the reference crystal is *transplanted* into the body, as it were, that changes from point to point.

2. Second-grade symmetries

The reference crystal can be changed by simply specifying a local diffeomorphism of $\mathbb{R}^3$ preserving the origin:

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3 \quad \{Z^\alpha\} \to \phi^A(\{Z^\alpha\}) \quad A, \alpha = 1, 2, 3$$

The 2-jet of this diffeomorphism is given by the quantities:

$$A^A_\alpha = \frac{\partial \phi^A}{\partial Z^\alpha} \;, B^A_{\alpha\beta} = \frac{\partial \phi^A}{\partial Z^\beta \partial Z^\alpha}$$

To any deformation F^i_A, H^i_{AB} of the second reference crystal, there corresponds a unique deformation $F^i_\alpha, H^i_{\alpha\beta}$ of the first, which can be obtained by composition of jets as:

$$F^i_\alpha = F^i_A A^A_\alpha H^i_{\alpha\beta} = H^i_{AB} A^A_\alpha A^B_\beta + F^i_A B^A_{\alpha\beta}$$

a composition law that was already used in Equation (1) above.

A *material symmetry*, $\{G, S\}$, is a change of reference crystal leaving the constitutive law unaffected, viz.:

$$W(F^i H^i) = W(F^i_\rho G^\rho_\alpha, H^i_{\rho\sigma} G^\rho_\alpha G^\sigma_\beta + F^i_\rho S^\rho_{\alpha\beta})$$

These symmetries form a group, whose group operation is the composition of 2-jets, whose neutral element is $\{I, 0\}$, and whose inverse operation is given by:

$$\{G, S\}^{-1} : \; \{(G^{-1})^\alpha_\beta, -S^\mu_{\rho\sigma} (G^{-1})^\rho_\beta (G^{-1})^\sigma_\gamma (G^{-1})^\alpha_\mu\}$$

3. An important example

In [2] we have given a classification of second-grade symmetries. Symmetries of the type $\{G, 0\}$, also called *Toupin symmetries*, are nothing but ordinary first-grade symmetries. An important example of a legitimate second-grade symmetry is provided by the dependence of the constitutive law on the spatial gradient of the density. This is formally the second-order counterpart of fluidity, but it should be remembered that we are not assuming anything as to the first-grade symmetries. In fact, to make matters simpler and more focused, we may assume that the material is a triclinic solid as far as the first-grade behaviour is concerned. A careful calculation [1] shows that a material whose second-grade behaviour is characterized by a dependence on the spatial density gradient enjoys the following symmetry:

$$\{\ \{I, S\} \mid S^{\alpha}_{\alpha\beta} = 0\}$$

that is, the set of all traceless third-order symmetric matrices. Such symmetries form a commutative (additive) group.

4. The Eshelby tensors

With the stored energy function given, one can proceed to calculate the *Piola-Kirchhoff stresses* by differentiation in a given chart with respect to the spatial jet:

$$T_i{}^I = \frac{\partial W_0}{\partial F^i_I} \qquad T_i{}^{IJ} = \frac{\partial W_0}{\partial H^i_{IJ}}$$

Similarly, by differentiation with respect to the material jet [3] we obtain the *hybrid Eshelby tensors*:

$$b^{\alpha}_I = \frac{\partial W_0}{\partial P^I_{\alpha}} \qquad b^{\alpha\beta}_I = \frac{\partial W_0}{\partial Q^I_{\alpha\beta}}$$

If we adopt a reference crystal instantaneously coinciding with the chart, we obtain the purely configurational Eshelby tensors [4]:

$$b^J_I = b^{\alpha}_I P^J_{\alpha} + B_I{}^{\alpha\beta} Q^J_{\alpha\beta} = -W_0 \delta^I_J + T_i{}^J F^i_I + 2T_i{}^{JL} H^i_{IL}$$

$$b^{JK}_I = b^{\alpha\beta}_I P^J_{\alpha} P^K_{\beta} = T_i{}^{JK} F^i_I$$

On the other hand, if we adopt a chart whose 2-jet at a point instantaneously coincides with a given fixed reference crystal, we obtain the purely material Eshelby tensors at that point as:

$$b_{0\alpha}^{\ \ \beta} = J_P(b_K^I P^{-1}{}_I^{\ \beta} P_\alpha^K + b_K^{IJ} Q_{\alpha\rho}^K P^{-1}{}_I^{\ \beta} P^{-1}{}_J^{\ \rho})$$

$$b_{0\alpha}^{\ \ \beta\gamma} = J_P b_K^{IJ} P^{-1}{}_I^{\ \beta} P^{-1}{}_J^{\ \gamma} P_\alpha^K$$

5. Evolution laws

We consider possible evolution laws given by first-order differential equations and driven by the present values of the transplant maps and by the Eshelby tensors, recognizing that other possible driving forces could be considered. We have, therefore,

$$\dot{P}_\rho^L = f_\rho^L(b_I^{\ J}, b_I^{\ JK}, P_\alpha^I, Q_{\alpha\beta}^I) \quad \text{and} \quad \dot{Q}_{\rho\sigma}^L = g_{\rho\sigma}^L(b_I^{\ J}, b_I^{\ JK}, P_\alpha^I, Q_{\alpha\beta}^I)$$

where a superimposed dot denotes time derivatives. The functions f and g cannot be completely arbitrary, but must be subjected to certain restrictions which we presently determine. The first such restriction emanates from the fact that a proper evolution law must, in some sense, be independent of the chart. Choosing, then, a chart that instantaneously coincides with the reference crystal, namely, one for which $\{P, Q\} = \{P_0, Q_0\} = \{I, 0\}$, we obtain as time derivatives the values:

$$\dot{P}_{0\ \alpha}^{\ \beta} = P^{-1}{}_I^{\ \beta} \dot{P}_\alpha^I \quad \text{and} \quad \dot{Q}_{0\ \alpha\beta}^{\ \gamma} = P^{-1}{}_I^{\ \gamma}(\dot{Q}_{\alpha\beta}^I - Q_{\rho\beta}^I \dot{P}_{0\ \alpha}^{\ \rho} - Q_{\alpha\rho}^I \dot{P}_{0\ \beta}^{\ \rho})$$

These quantities represent some kind of *inhomogeneity velocity gradients* at the reference crystal level. Invoking the above stated invariance, we conclude that the evolution laws must be given by some function f_ρ^μ and $g_{\rho\sigma}^\mu$ connecting those gradients to the material Eshelby tensors in the form:

$$\dot{P}_{0\ \rho}^{\ \mu} = f_\rho^\mu(b_{0\ \alpha}^{\ \beta}, b_{0\ \alpha}^{\ \beta\gamma}) \quad \text{and} \quad \dot{Q}_{0\ \rho\sigma}^{\ \mu} = g_{\rho\sigma}^\mu(b_{0\ \alpha}^{\ \beta}, b_{0\ \alpha}^{\ \beta\gamma})$$

Further reductions of the evolution laws can be obtained by considerations of material symmetry. These reductions are of two kinds. The first represents the fact that two evolution laws that differ by constant or time-varying members of the symmetry group of the crystal of reference must be regarded

as equivalent. This leads to the following *principle of actual evolution*: for an evolution law to represent an actual physical evolution, rather than just a rearrangement within the symmetry group, at least one of the constitutive functions f or g must have a non-vanishing component lying outside of the Lie algebra of the symmetry group. In our example, this translates into the statement that the evolution law must prescribe a non–zero trace for g.

The second kind of restrictions arising from material symmetry, consists of imposing conditions of invariance under symmetry transformations of the reference crystal. For our example it appears [1] that the functions f cannot depend on their first argument, whereas there are no further restrictions imposed on the functions g. Further restrictions on the evolutions laws may stem from the application of the thermodynamic dissipation inequality.

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