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The Quasi-Invertible Manifold of a JB^*-Triple †

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1. Introduction

The work of E. Cartan in classifying Hermitian symmetric spaces [1] showed that every n-dimensional non-compact symmetric space B has a compact symmetric space 'dual' M with $B \subset \mathbb{C}^n \subset M$. In one dimension, for example, we have $\Delta \subset \mathbb{C} \subset \bar{\mathbb{C}}$ where Δ is the open unit disc and $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Other examples of compact symmetric spaces are given by Grassmann manifolds. Indeed, Loos [6] gave an alternative description of the compact symmetric spaces using a Grassmann-like construction defined in terms of a Jordan theoretic quasi-inverse. The Jordan structure involved is that of a JB^*-triple (although Loos phrased his construction in terms of the more general Jordan pairs). In infinite dimensions, the work of Kaup [4, 5] shows that JB^*-triples precisely characterise the bounded symmetric domains (the infinite dimensional analogues of the non-compact symmetric spaces) and that the non-compact/compact duality of finite dimensions is replaced by a duality between bounded symmetric domains and simply connected symmetric manifolds of compact type. In summary each bounded symmetric domain B may be realised as the open unit ball of a JB^*-triple Z and there is a unique simply connected symmetric manifold of compact type M_K associated with Z so that we have $B \subset Z \subset M_K$.

The quasi-inverse construction mentioned above can be made for an arbitrary JB^*-triple Z to give a complex manifold M_Q modeled on Z [3, 6] which we term the quasi-invertible manifold of Z. In this note, we report on the relationship between the compact type symmetric manifold, M_K, and the quasi-invertible manifold, M_Q, associated to a JB^*-triple Z and indicate how

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this relationship may be used to prove facts about both manifolds. Additional details and proofs will appear in [7].

In [2], it is shown that whenever Z sits densely in M_Q (we say Z has the density property), the quasi-invertible manifold is a homogeneous manifold. In fact, the density property has a much stronger consequence, namely that M_Q turns out to be a symmetric manifold of compact type, and so the universal covering manifold of M_Q is precisely M_K.

Throughout, we use 'manifold' to mean 'complex Banach manifold'. A biholomorphic map g on a manifold M carrying a tangent norm is a symmetry at the point $m \in M$ if $g^{-1} = g$, m is an isolated fixed point of g and g is an isometry with respect to the tangent norm on M. A manifold is called symmetric if there is a symmetry at every point. We refer to [9] for a comprehensive introduction to symmetric manifolds.

By Kaup [5], every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a JB^*-triple, and for each JB^*-triple there is a unique simply connected symmetric manifold of compact type, M_K. A JB^*-triple is a complex Banach space Z with a real trilinear mapping $\{\cdot,\cdot,\cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables x and z, and is complex anti-linear in y.

(ii) The map $x \mapsto \{x, x, z\}$, denoted $x \Box x$, is Hermitian, $\sigma(x \Box x) \geq 0$ and $\|x \Box x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the operator spectrum.

(iii) The product satisfies the following "triple identity"

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

Important algebraic operators in the theory are $x \Box y$, where $x \Box y(z) = \{x, y, z\}$, the quadratic operator Q_x given by $Q_x(y) = \{x, y, x\}$ and the Bergmann operator $B(x, y) = I - 2x \Box y + Q_x Q_y$. The Bergmann operators are used in the construction of the quasi-invertible manifold as follows. The pair $(x, y) \in Z \times Z$ is said to be quasi-invertible if $B(x, y)$ is an invertible operator in $\mathcal{L}(Z)$. If (x, y) is quasi-invertible, let

$$x^y = B(x, y)^{-1}(x - Q_x y)$$

and call x^y the quasi-inverse of x with respect to y. On $Z \times Z$ define the equivalence relationship $\sim$ by $(x, y) \sim (x_1, y_1)$ if, and only if, $(x, y - y_1)$ is quasi-invertible and $x_1 = x^{y - y_1}$. The equivalence class containing (x, y) is denoted by $(x : y)$. For each y in Z, let $U_y = \{(x : y) : x \in Z\}$ and define $\phi_y : U_y \to Z$ by $\phi_y(x : y) = x$. Let $M_Q = {}^{Z \times Z}/_{\sim}$ be the set of all equivalence classes. Then, with respect to the charts $\{(U_y, \phi_y, Z) : y \in Z\}$, M_Q has the structure of a connected complex Banach manifold [6].

2. JB^*-TRIPLES WITH THE DENSITY PROPERTY

In this section, we assume that the JB^*-triple Z has the density property, that is, the open subset $\{(x : 0) : x \in Z\}$ of M_Q (identified with Z) is dense in M_Q. Consequently, [2, Theorem 5.3] the translation map $t_c : z \mapsto z + c$ on Z extends to a biholomorphic map on M_Q which we also denote by t_c. The linear operator $B(z, -z)^{\frac{1}{2}}$ is defined on Z via the functional calculus ($B(z, -z)$ has strictly positive spectrum [5]) and extends to a biholomorphic map on M_Q via $B(z, -z)^{\frac{1}{2}}(x : y) = (B(z, -z)^{\frac{1}{2}}x : B(z, -z)^{-\frac{1}{2}}y)$. We may therefore consider for any $z \in Z$, the biholomorphic map on M_Q given by $g_z = t_z B(z, -z)^{\frac{1}{2}} \tilde{t}_z$, where $\tilde{t}_z : M_Q \to M_Q$ is the biholomorphic 'quasi-inverse' map $(x : y) \mapsto (x : y + z)$.

Let P be the subset $\{g_c : c \in Z\}$ of biholomorphic maps on M_Q. Since $(x : y) = g_y g_u(0 : 0)$ where $u = B(y, -y)^{\frac{1}{2}}x - y$, M_Q is homogeneous under the group generated by P and these mappings can be used to endow M_Q with a symmetric structure. We recall that the "Möbius" maps $\{g_c : c \in Z\}$ are used in [5] to construct the compact type symmetric manifold of Z. On the JB^*-triple Z, the surjective linear isometries coincide with the algebraic isomorphisms [5] and we denote these by $\texttt{Aut}(Z)$. Every element of the group $K := \texttt{Aut}(Z)$ easily extends to give a biholomorphic map on M_Q [6].

PROPOSITION 2.1. *Let Z be a JB^*-triple with the density property. There exists a norm γ on the tangent bundle of the quasi-invertible manifold M_Q such that every element of P and every element of K is a γ-isometry. Moreover, the group of all biholomorphic γ-isometries is precisely $G := K\hat{P}$ where $\hat{P}$ is the group generated by P.*

The tangent norm γ in Proposition 2.1 is given by

$$\begin{aligned}\gamma((x : y), v) &:= \|(g_{-u}g_{-y})'(x : y)v\| \\ &= \|B(u, -u)^{-\frac{1}{2}}B(u, y)B(y, -y)^{-\frac{1}{2}}v\|\end{aligned}$$

for $(x : y) \in M_Q$ and $v \in T_{(x:y)}M_Q$ where $u = \mathbf{u}(x, y) := B(y, -y)^{\frac{1}{2}}x - y$. The main obstacle to be overcome in the proof of Proposition 2.1 is that of showing γ is well-defined. This turns out to be a consequence of the following lemma.

Lemma 2.2. *For a and b in Z with $g_a(b) \in Z$, we have*

$$B(g_a(b), -g_a(b)) = B(a,-a)^{\frac{1}{2}} B(b,a)^{-1} B(b,-b) B(a,b)^{-1} B(a,-a)^{\frac{1}{2}}.$$

Equipped with Proposition 2.1, one can then show that M_Q is a symmetric Banach manifold modeled on Z. Moreover, it follows from Kaup's classification and the fact that the group of all biholomorphic isometries of M_Q has the form $G = K\hat{P}$ that M_Q must be of compact type. Taking the universal covering manifold of M_Q ensures it is simply connected and we have:

Corollary 2.3. *If the JB^*-triple Z has the density property then the quasi-invertible manifold M_Q is a compact type symmetric Banach manifold and its associated JB^*-triple is Z. The unique compact type symmetric manifold of Z is therefore the universal covering manifold of M_Q.*

3. The general case

Henceforth, we no longer assume that the JB^*-triple in question has the density property. The unique simply connected compact type symmetric manifold M_K of a JB^*-triple Z is defined by Kaup [5] to be the universal covering manifold of a previously constructed symmetric manifold N. By Kaup's construction, the group $G = K\hat{P}$ from above also acts transitively on this symmetric manifold N. As the definition and construction of M_K and N in [5] are quite intricate, we do not reproduce them here. Referring therefore to [5], we show how to embed M_Q into N, thereby relating M_Q and M_K.

Lemma 3.1. *Let x, y, x' and y' be elements of an arbitrary JB^*-triple, Z. Suppose $(x : y) = (x' : y')$ in M_Q. Let $u = \mathbf{u}(x,y)$ and $u' = \mathbf{u}(x',y')$. Then, in the symmetric manifold N of [5], $g_y(u) = g_{y'}(u')$.*

This lemma allows one to unambiguously define the map $J\colon M_Q \to N$, $J(x : y) = g_y(\mathbf{u}(x,y))$ for x and y in Z. Moreover, one can show that J is injective, holomorphic and its range is the open subset $\{g_a(b) : a, b \in Z\}$ of N. Also, $J^{-1}\colon J(M_Q) \to M_Q$ is holomorphic and so M_Q is biholomorphically equivalent to an open submanifold of the symmetric manifold N. This means that M_Q inherits some of the properties of the compact type symmetric manifold N. For example, the natural tangent norm of N induces a tangent norm on any open submanifold, in particular on M_Q. It has been proved that M_K (and hence N) has constant positive holomorphic curvature [8] and since holomorphic curvature is a local property, the inherited norm on M_Q must also have the same property.

COROLLARY 3.2. *The quasi-invertible manifold, M_Q, carries a tangent norm.*

COROLLARY 3.3. *M_Q has constant positive holomorphic curvature.*

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