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On Polarized Surfaces with a Small Generalized Class

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1. INTRODUCTION

Let L be an ample line bundle on a complex connected projective smooth surface X and let $c_2(J_1(L))$ stand for the degree of the second Chern class of the first jet bundle $J_1(L)$ of L. In the classical case when L is very ample, $c_2(J_1(L))$ is the class m of X (embedded via L), i.e. the degree of the dual variety of X. Still in the very ample case, if (X, L) is not a scroll the classification of such pairs is known for $m \leq 25$ (Marchionna and Gallarati [5, p. 195] and [12, Proposition 3.2]). Moreover the only pairs (X, L) with class $m \leq 29$ are polarized ruled surfaces and if $m \leq 11$ we only have $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)), e = 1, 2$ (apart from scrolls).

In this paper L is only assumed to be ample, and, in section 4, we classify pairs (X, L) for small values of $c_2(J_1(L))$ (Theorem 4.1). The situation is very different from the classical case since even for $c_2(J_1(L)) = 5$ we find a non ruled surface. This classification result is gotten by studying the difference $c_2(J_1(L)) - L^2$ in line with the classical papers by Marchionna and Gallarati.

Some of the non ruled surfaces which appear in the classification are more closely described in section 5.

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2. NOTATION AND BACKGROUND MATERIAL

2.1. Throughout the whole paper X will be a complex connected projective smooth surface and L will stand for an ample line bundle on X.

2.2. NOTATION. The Chern classes of a complex vector bundle E on X will be denoted by $c_i(E)$ and $c_2(X)$ will represent the second Chern class of the holomorphic tangent bundle of X. We shall also write $c_2(E)$ for its degree. The canonical line bundle of X will be denoted by K_X, the irregularity of X by $q = q(X)$, its geometric genus by $p_g = p_g(X)$, the Kodaira dimension of X by $\kappa(X)$, the intersection index of two line bundles (or two divisors) A and B on X simply by AB and we shall write $A^2 = AA$. By $A_{|C}$ we mean the line bundle A restricted to a curve C on X. We shall not distinguish between line bundles, invertible sheaves and divisors and thus we shall use additive notation even for line bundles; in particular $A \equiv B$ will mean that two divisors corresponding to A and B respectively are numerically equivalent. If V is a vector subspace of $H^0(X, L)$, $|V|$ will stand for the linear system corresponding to the elements of V; we shall write $|L|$ instead of $|H^0(X, L)|$. Furthermore

$$g = g(L) := 1 + \frac{L^2 + LK_X}{2} \tag{1}$$

will stand for the arithmetic genus of the ample line bundle L.

By $B_{P_1,..,P_r}(X)$ we shall denote X blown up at a finite set $P_1, .., P_r$ of possibly infinitely near points of X.

For a polarized pair (X, L), in case $X = B_{P_1,..,P_r}(\check{X})$ with $\check{X}$ minimal, $\eta : X \to \check{X}$ will always denote the birational morphism onto the minimal model; note that the line bundle $\check{L}$ on $\check{X}$ defined by $L = \eta^* \check{L}$ is ample. The polarized pair $(\check{X}, \check{L})$ will be called a minimal pair of (X, L).

2.3. THE FIRST JET BUNDLE $J_1(L)$ OF L. By $J_1(L)$ we will denote the *first jet bundle* of L. Its detailed construction can be found in [6, Ch. 1] and also [2, 1.6.3]. Locally near a point $x \in X$, a section of $J_1(L)$ is given by a pair (s, ds), where s is a local section of L and ds is the differential of s with respect to local coordinates around x. The map sending (s, ds) to s defines a bundle homomorphism $J_1(L) \to L$ giving rise to the following exact sequence ([2, p.28])

$$0 \to T_X^* \otimes L \to J_1(L) \to L \to 0, \tag{2}$$

where T_X^* stands for the cotangent bundle of X.

The exact sequence (2) immediately gives $c_2(J_1(L)) = c_2(X) + 2K_X L + 3L^2$ and so

$$(3) \qquad c_2(J_1(L)) - L^2 = c_2(X) + 2(K_X L + L^2) = c_2(X) + 4(g-1).$$

In the particular case when L is a very ample line bundle identify X with its image in $\mathbb{P}^N$, $N+1 = \dim H^0(X,L)$, via the embedding associated with L and think of the linear system $|L|$ corresponding to the elements of $H^0(X,L)$ as the hyperplane linear system of X. Consider also the dual variety $D(X)$ of X, i.e., the set of points of $|L|$ corresponding to the tangent hyperplanes and assume that $(X,L) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. It is known that $c_2(J_1(L))$ is the degree of $D(X)$, i.e., the class m of X ([2, Remark 1.6.11] see also [10, (0.4)]). This is the reason why, in the general case when L is only assumed to be ample, $c_2(J_1(L))$ will be called the *generalized class* of (X,L).

2.4. SPECIAL SURFACES. A pair (X,L) is said a *scroll* if X has a $\mathbb{P}^1$-bundle structure and for any fibre f we have $L_{|f} = \mathcal{O}_{\mathbb{P}^1}(1)$; a *conic bundle* (X,L) is a $\mathbb{P}^1$-bundle, possibly blown up at a finite set of distinct points, such that $L_{|f} = \mathcal{O}_{\mathbb{P}^1}(2)$ for the general fibre f; the pair (X,L) is said to be a *Del Pezzo pair* if $L = -K_X$.

A minimal model $\neq \mathbb{P}^2$ of a ruled surface will be denoted by X_e. It is a $\mathbb{P}^1$-bundle $p : X_e \to C$ over a nonsingular curve C of genus $q = h^1(\mathcal{O}_X)$ and $X_e = \mathbb{P}(F)$ for a rank-2 locally free sheaf F on C. We can choose F in such a way that there exists a section C_0 of p of minimal self intersection $C_0^2 = -e = \deg F$ for which $\mathrm{Num}(X_e) = \mathbb{Z}[C_0] \oplus \mathbb{Z}[f]$, f being a fibre of p. We shall write $L \equiv aC_0 + bf$ to mean that any divisor on X_e corresponding to the line bundle L is numerically equivalent to $aC_0 + bf$. Recall that ([4, Propositions 2.18, 2.21]) the ampleness of L is equivalent to

$$(4) \qquad a > 0 \text{ and } b > ae \text{ if } e \geq 0, \;\; a > 0 \text{ and } b > \frac{ae}{2} \text{ if } e < 0.$$

Remark. By recalling the topological Euler characteristic of the minimal models of the surfaces, we immediately get:

- if $\kappa(X) = -\infty$ and the minimal model is $\mathbb{P}^2$, then $c_2(X) \geq 3$;
- if $\kappa(X) = -\infty$ and the minimal model is not $\mathbb{P}^2$, then $c_2(X) \geq 4(1-q)$;
- if $\kappa(X) = 0$ and the minimal model is a $K3$ surface, then $c_2(X) \geq 24$;

- if $\kappa(X) = 0$ and the minimal model is an Enriques surface, then $c_2(X) \geq 12$;
- if $\kappa(X) = 0$ and the minimal model is either bielliptic or abelian, then $c_2(X) \geq 0$;
- if $\kappa(X) = 1$, then $c_2(X) \geq 12\chi(\mathcal{O}_X)$ with $\chi(\mathcal{O}_X) \geq 0$.

3. Preliminary results

In the classical case when L is very ample Marchionna proved the inequality $m - L^2 \geq -1$ for the class m of X and the cases $m - L^2 = -1, 0$ were characterized by Marchionna himself and by Gallarati ([5, p. 195]). In view of what we said in 2.3 the following known facts generalize those clasical results in the wider context of ample divisors.

THEOREM 3.1. ([9, Proposition A.1]) *Let things be as in* 2.1. *Then* $c_2(J_1(L)) \geq L^2 - 1$. *Moreover*

1) $c_2(J_1(L)) = L^2 - 1$ *if and only if* $(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)), e = 1, 2$;

2) $c_2(J_1(L)) = L^2$ *if and only if* (X, L) *is a scroll.*

More recently the first author pointed out the following facts ([10, Proposition 1.4 and Theorem 1.6])

PROPOSITION 3.2. *Let things be as in* 2.1 *and assume that* $c_2(J_1(L)) - L^2 > 0$ *and* $g \geq 2$. *Then* $c_2(J_1(L)) - L^2 \geq 2g$ *equality holding if and only if either*

a) $X = X_{-1}$ *is a minimal ruled surface of invariant* $e = -1$, $q = 1$, $g = 2$ *and* $L \equiv 3C_0 - f$;

b) $X = X_e$ *is a minimal ruled surface of invariant* e, $-q \leq e \leq 0$, $g = 2q > 0$ *and* $L \equiv 2C_0 + (e+1)f$;

c) X *is a minimal surface endowed with an elliptic fibration* $X \to \mathbb{P}^1$, $q = 1$, $p_g = 0$, $g = 2$, $L^2 = 1$ *(more details on the multiple fibres can be found in* [1, Theorem 1.5]*)*;

d) X *is the Jacobian of a smooth curve* C *of genus* $2, L$ *is numerically equivalent to* C *embedded in* X, $g = 2$, $L^2 = 2$, $h^0(L) = 1$;

e) X *is the product of two elliptic curves,* L *is numerically equivalent to the sum of the factors,* $g = 2$, $L^2 = 2$, $h^0(L) = 1$; *or*

f) X *is a minimal hyperelliptic surface; if* $\check{f}$ *stands for the reduced component of the curve of maximal multiplicity of the rational pencil of elliptic curves and if* $\check{C}$ *is a fibre of the Albanese map, then* nL *is numerically equivalent to* $r\check{C} + s\check{f}$, r, s *integers,* $rs = n = 1, 2, 3$, $g = 2$, $L^2 = 2$, $h^0(L) = 1$.

In the sequel when classifying pairs (X, L) with small values of the difference $c_2(J_1(L)) - L^2$, we shall need the following

PROPOSITION 3.3. *Let things be as in* 2.1 *and* $g \leq 1$, *then the pair* (X, L) *is as in the following table:*

$c_2(J_1(L)) - L^2$	(X, L)	g	L^2
-1	$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)), e = 1, 2$	0	1, 4
0	scroll	$q = 0, 1$	
1, 2	do not exist		
3	$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$	1	9
4	$(X_0, -K_{X_0})$ with $q = 0$	1	8
$3 + r$	$X = B_{P_1,..,P_r}(\mathbb{P}^2)$, $L = -K_X$, $r = 1, \ldots, 8$	1	$9 - r$

Proof. By recalling the structure of pairs (X, L) with $g \leq 1$ ([8, Corollaries 2.3 and 2.3]), the result immediately follows from Theorem 3.1. ∎

4. POLARIZED PAIRS WITH GENERALIZED CLASS $c_2(J_1(L)) \leq 9$

Let things be as in 2.1. In the classical case when L is very ample the pairs (X, L) with class $m = c_2(J_1(L)) \leq 29$ are all polarized ruled surfaces and if $m = c_2(J_1(L)) \leq 11$ we only have scrolls or $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e))$, $e = 1, 2$ ([12, Proposition 3.2 and Theorem 3.4]). The situation becomes far richer when L is only assumed to be ample, as shown by the following theorem which is the main result of the paper.

THEOREM 4.1. *Let things be as in 2.1 and assume that (X, L) is not a scroll. If $c_2(J_1(L)) \leq 9$, then the pair (X, L) is as in the following table:*

$c_2(J_1(L))$	(X, L)	g
0	$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$	0
3	$(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$	0
5	as in 3.2 c)	2
6	as in 3.2 d), 3.2 e), 3.2 f), or the minimal pair is as in 3.2 d), 3.2 f) and $X = B_P(\check{X})$	2
6	the minimal pair is $\check{X} = X_{-1}, q = 1, \check{L} \equiv 5C_0 - f$ and $X = B_P(\check{X})$	2
6	the minimal pair is $\check{X} = X_0, q = 1, \check{L} \equiv 5C_0 + f$ and $X = B_P(\check{X})$	2
7	as in 3.2 a), or the minimal pair is as in 3.2 a) and $X = B_{P_1,..,P_s}(\check{X}), s = 1, 2$	2
7	the minimal pair is $\check{X} = X_0, q = 1, \check{L} \equiv 3C_0 + f$ and $X = B_P(\check{X})$	2
7	the minimal pair is $\check{X} = X_e, e = -1, 0, q = 1, \check{L} \equiv 3C_0 + (e+1)f$ and $X = B_{P_1 P_2}(\check{X})$	2
8	as in 3.2 b) with q=1, or the minimal pair is as in 3.2 b) with $q = 1$ and $X = B_{P_1,..,P_s}(\check{X}), s = 1, 2, 3$	2
9	Xminimal elliptic surface,$\chi(\mathcal{O}_X) = 0, L^2 = 1$	3

In order to get this result we start studying the difference $c_2(J_1(L)) - L^2$ in line with the classical papers by Marchionna and Gallarati and their generalizations quoted above. In view of the ampleness of L, Theorem 4.1 will be immediately proved once we have classified the pairs with $c_2(J_1(L)) - L^2 \leq 8$.

The study of the difference $c_2(J_1(L)) - L^2$ is contained in the following two theorems, where we deal separatedly with the non ruled and the ruled cases. More detailed information on the cases not completely described in the following tables will be given in section 5.

THEOREM 4.2. *Let things be as in 2.1 with $g \geq 2$ and $c_2(J_1(L)) - L^2 \leq 8$. If $\kappa(X) \neq -\infty$, then the pair (X, L) is as in the following table:*

$c_2(J_1(L)) - L^2$	(X, L)	g	L^2
$0,1,2,3$	do not exist		
4	3.2 c), 3.2 d), 3.2 e), 3.2 f)	2	$1,2,2,2$
5	the minimal pair of (X, L) is as in 3.2 d), 3.2 f) (with $r = 1$ and $n = 2,3$) and $X = B_P(\check{X})$	2	$1,1$
$6,7$	do not exist		
8	X minimal abelian or bielliptic surface	3	4
8	X minimal elliptic surface, $\chi(\mathcal{O}_X) = 0$	3	$1,2,3$

Proof. Let $\Delta = c_2(J_1(L)) - L^2$. As $\kappa(X) \neq -\infty$, by Theorem 3.1 we have $\Delta \geq 1$, so by Proposition 3.2 we stick to considering the cases $g = 2,3,4$. We proceed according to the values of the Kodaira dimension $\kappa(X)$ of X, having noted that $\kappa(X) < 2$ by [10, Proposition 2.1.].

Step 1. Let $k(X) = 1$. By Remark and (3) we have $\Delta \geq 4(g-1)$ and $\chi(\mathcal{O}_X) = 0$, therefore either $g = 2$ or $g = 3$. In the latter case $\Delta = 8, c_2(X) = 0$ and so X is minimal; moreover, a multiple of K_X being effective, $1 \leq K_X L = 4 - L^2$, and so we are in the last case of the table. If $g = 2$, [1, Theorem 1.5] and the fact that $\chi(\mathcal{O}_X) = 0$ say that X is as in 3.2 c).

Step 2. Let $\kappa(X) = 0$. By Remark and (3) we have $\Delta \geq 4(g-1)$, $g = 2,3$ and the minimal model of X is either bielliptic or abelian. If $g = 3$, then $\Delta = 8$, $c_2(X) = 0$ and so X is minimal; since K_X is numerically trivial , we have $L^2 = 4$ and so we fall into the case listed at the last but one line in the table. If $g = 2$, [1, Theorem 2.7] says that we can only have cases 3.2 d), 3.2 e) and 3.2 f) and the blow up of 3.2 d) or 3.2 f) at a single point. Since $\Delta = c_2(X) + 4$, we get $\Delta = 4,5$, according to whether X is minimal or not. ∎

THEOREM 4.3. *Let things be as in* 2.1 *with* $g \geq 2$ *and* $c_2(J_1(L)) - L^2 \leq 8$. *If* $\kappa(X) = -\infty$ *then the pair* (X, L) *is as in the following table:*

$c_2(J_1(L)) - L^2$	(X, L)	g	L^2
0	scroll	$q \geq 2$	
$1,2,3$	do not exist		
4	as in 3.2 a), 3.2 b) with $q = 1$	2	$3,4$
$4+s, s=1,2$	the minimal pair of (X, L) is as in 3.2 a) and $X = B_{P_1,\ldots,P_s}(\check{X})$	2	$2,1$
$4+s, s=1,2,3$	the minimal pair of (X, L) is as in 3.2 b) with $q = 1$ and $X = B_{P_1,\ldots,P_s}(\check{X})$	2	$3,2,1$
5	the minimal pair of (X, L) is $\check{X} = X_{-1}, q = 1, \check{L} \equiv 5C_0 - f$ and $X = B_P(\check{X})$	2	1
5	the minimal pair of (X, L) is $\check{X} = X_0, q = 1, \check{L} \equiv 3C_0 + f$ and $X = B_P(\check{X})$	2	2
5	the minimal pair of (X, L) is $\check{X} = X_0, q = 1, \check{L} \equiv 5C_0 + f$ and $X = B_P(\check{X})$	2	1
6	the minimal pair of (X, L) is $\check{X} = X_e, e = -1, 0, q = 1, \check{L} \equiv 3C_0 + (e+1)f$ and $X = B_{P_1P_2}(\check{X})$	2	$1,1$
8	$X = X_e$ with $e = 0,1,2$, $q = 0$ and $L \equiv 2C_0 + (3+e)f$	2	12
8	$X = X_e$ with $e = -1,0,1, q = 1$ and $L \equiv 2C_0 + (2+e)f$	3	8
8	$X = X_0$ with $q = 1$ and $L \equiv 3C_0 + f$	3	6
8	$X = X_{-1}$ with $q = 1$ and $L \equiv 5C_0 - 2f$	3	5
8	as in 3.2 b) with $q = 2$	4	4

Proof. Let $\Delta = c_2(J_1(L)) - L^2$. Notice first that X is not $\mathbb{P}^2$ otherwise $L = \mathcal{O}_{\mathbb{P}^2}(n)$ and by (3) one gets $3 + 4(g-1) \leq 8$, which implies $g \leq 2$, and so $g = 2$, a contradiction. Hence X dominates a ruled surface X_e. By Remark and (3) we have $\Delta \geq 4(1-q) + 4(g-1) = 4(g-q)$ and so $g \leq 2 + q$. We now recall that, since (X, L) is not a scroll and X is a ruled surface $\neq \mathbb{P}^2$, then $g \geq 2q$ ([10, Lemma 1.3]) and so we get $2q \leq g \leq q + 2$ which says $q = 0, 1, 2$. If $q = 2$, then $g = 4, \Delta = 8 = 2g$, so that Proposition 3.2 leads to the last case in the table. If $q = 1$, then either $g = 2$ or $g = 3$. In case $(q, g) = (1, 2)$, the corresponding cases in the table come from [1, Theorem 3.3]. In case $(q, g) = (1, 3)$, condition $\Delta \leq 8$, by (3), gives $\Delta = 8$ and $c_2 = 0$. Hence X is a minimal elliptic ruled surface X_e. Letting $L \equiv aC_0 + bf$, the genus formula gives $4 = (a-2)(b-ae) + a(b-e)$; the ampleness conditions (4) are enough to conclude. It only remains to consider the case $(q, g) = (0, 2)$. Condition $\Delta \leq 8$, by (3), gives $\Delta = 8$ and $c_2 = 4$. Hence X is a rational ruled surface X_e. Letting again $L \equiv aC_0 + bf$, the genus formula gives $2 = (a-2)(b-ae) + a(b-2-e)$; the ampleness conditions are enough to conclude also in this case. ∎

5. Special cases and remarks on the generalized class

In this section we describe some polarized pairs met in the previous section.

5.1. Abelian surfaces. Let $X = \mathbb{C}^2/\Lambda$, for a lattice Λ, be a two dimensional complex torus. As is known [7, Ch. 2] to any line bundle L on X one can associate a $\mathbb{Z}-$linear alternating 2-form E on Λ; moreover there exists a basis of Λ such that E is represented by

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \text{ with } D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ d_1, d_2 \geq 0,\ d_1|d_2,\ if\, d_1 > 0$$

and the pair d_1, d_2 is uniquely determined by L. Finally we recall that $L^2 = 2d_1d_2$ and, by Riemann-Roch, $\chi(L) = d_1d_2$. In case L ample, L is called a *polarization of type* (d_1, d_2). All this immediately gives that, for a minimal abelian surface X, (X, L) is as in Theorem 4.2 if and only if L is a polarization of type $(1, 2)$. Notice that if a (not necessarily ample) line bundle L defines the pair $(d_1, d_2) = (1, 2)$, then, by Riemann-Roch and Serre duality, either L or $-L$ is effective, hence one can always define a polarization of type $(1, 2)$. Moreover ([7, Ch. 10]) either

i) $(X, L) = (E_1 \times E_2, {p_1}^*L_1 \otimes {p_2}^*L_2)$, with E_i an elliptic curve polarized by the bundle L_i of type $i (i = 1, 2)$, or

ii) $|L|$ has no fixed component and exactly four base points, its general member is smooth and the number of the singular curves in that pencil is $c_2(J_1(L)) = 12$.

5.2. Bielliptic surfaces. As is known, any bielliptic surface X is isomorphic to $(A \times B)/G$, where A and B are elliptic curves, G is finite group acting componentwise on $A \times B$. There are seven types of bielliptic surfaces, as pointed out by Bagnera-De Franchis; some information is given in the following table ([11]).

type	G	basis (A', B') of $\mathrm{Num}(X)$
1	$\mathbb{Z}_2$	$(A/2, B)$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(A/2, B/2)$
3	$\mathbb{Z}_4$	$(A/4, B)$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$(A/4, B/2)$
5	$\mathbb{Z}_3$	$(A/3, B)$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$(A/3, B/3)$
7	$\mathbb{Z}_6$	$(A/6, B)$

Let L be any divisor on X numerically equivalent to $mA' + nB'$, with $m, n \in \mathbb{Z}$ and where (A', B') denotes the basis as above. By intersecting L with A and B and by using the Nakai-Moishezon criterion, one sees that L is ample if and only if $m > 0, n > 0$. Moreover, since $AB = \mathrm{ord}\ G$, the above table shows that $A'B' = 1$ for any type and so $L^2 = 2mn$. Therefore, for any minimal bielliptic surface X, (X, L) is as in Theorem 4.2 if and only if L is numerically equivalent either to $A' + 2B'$ or to $2A' + B'$.

5.3. Elliptic surfaces. As is known, any minimal elliptic surface X with $\chi(\mathcal{O}_X) = 0$ admits an elliptic fibration onto a smooth curve, whose singular fibers are multiple fibers. For any surface X like that, (X, L) is as in Theorem 4.2 if and only if L is an ample line bundle with $K_X L \leq 3$. By using the canonical bundle formula one can work out the possible multiplicities of the multiple fibers and the genus of the base curve. This has already been done in cases $K_X L = 1, 2$ in [3].

In Theorem 4.1 some values of $c_2(J_1(L))$ are common to certain pairs (X, L) and their minimal pairs $(\check{X}, \check{L})$. This is a general fact stemming from

Remark. Let $\sigma : X \to X_1$ be the morphism contracting down an exceptional curve E and L_1 the ample line bundle on X_1 defined by $L = \sigma^*(L_1) - rE$. Recalling (3) we get

$$\begin{aligned} c_2(J_1(L)) &= c_2(X) + 2K_X L + 3L^2 \\ &= c_2(X_1) + 1 + 2(K_{X_1}L_1 + r) + 3({L_1}^2 - r^2)^2 \\ &= c_2(J_1(L_1)) + 1 + 2r - 3r^2 \end{aligned}$$

which, for $r = 1$, gives $c_2(J_1(L)) = c_2(J_1(L_1))$.

References

[1] Beltrametti, M., Lanteri, A., Palleschi, M., Algebraic surfaces containing an ample divisor of arithmetic genus two, *Arkiv for matematik,* **25** (1987), 189–210.

[2] Beltrametti, M.C., Sommese, A.J., "The Adjunction Theory of Complex Projective Varieties", de Gruyter, Berlin-New York, 1995.

[3] De Cataldo, M.A., Palleschi, M., Polarized surfaces of positive Kodaira dimension with canonical bundle of small degree, *Forum Math.,* **4** (1992), 217–229.

[4] Hartshorne, R., "Algebraic Geometry", GTM **52**, Springer Verlag, New York - Heidelberg-Berlin, 1997.

[5] Kleiman, S.L., Tangency and Duality, in Vancouver Conference in Algebraic Geometry, *Canad. Math. Soc. Conf. Proc.,* **6** (1986), 163–225.

[6] Kumpera, A., Spencer, D., "Lie equations, Vol.I: general theory", Ann. of Math. Stud.,**73** Princeton, N. J., 1972.

[7] Lange, H., Birkenhake, Ch., "Complex Abelian Varieties", GTM **302**, Springer Verlag, New York-Heidelberg-Berlin, 1992.

[8] Lanteri, A., Palleschi, M., About the adjunction process for polarized algebraic surfaces, *J. Reine Angew. Math.,* **352** (1984), 15–23.

[9] Lanteri, A., Palleschi, M., Sommese, A.J., On the discriminant locus of an ample and spanned line bundle, *J. Reine Angew. Math.,* **477** (1996), 199–219.

[10] Palleschi, M., On the second chern class of the first jet bundle of an ample line bundle on a surface, preprint, (1996).

[11] Serrano, F., Divisors on bielliptic surfaces, *Math. Z.,* **203** (1990), 527–533.

[12] Turrini, C., Verderio, E., Projective surfaces of small class, *GeometriæDedicata,* **47** (1993), 1–14.