EXTRACTA MATHEMATICAE Vol. **13**, Núm. 3, 283 – 287 (1998)

Some Questions on the Composition Factor Property for Atomic Mappings

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(Research paper presented by F. Montalvo)

AMS *Subject Class.* (1991): 54E40, 54F15, 54B15 *Received December* 10, 1997

All spaces considered in the paper are assumed to be metric, and a mapping means a continuous function. An empty or a one-point space is said to be *degenerate*. Otherwise it is said to be *nondegenerate*. A *continuum* stands for a compact connected space. A continuum is defined to be *decomposable* if it is the union of two its proper subcontinua. Otherwise it is *indecomposable*. A continuum is said to be *hereditarily decomposable* (*hereditarily indecomposable*) provided that each of its nondegenerate subcontinua is decomposable (indecomposable, respectively).

A mapping $f : X \to Y$ between continua is:

(i) *monotone* provided that for each subcontinuum Q of Y the set $f^{-1}(Q)$ is connected;

(ii) *weakly confluent* provided that for each subcontinuum Q of Y there exists a continuum $K \subset X$ such that $f(K) = Q$;

(iii) *confluent* provided that for each subcontinuum Q of Y and for each component K of $f^{-1}(Q)$ we have $f(K) = Q$;

(iv) *atomic* provided that f is surjective and, for each subcontinuum K of X such that $f(K)$ is nondegenerate, $K = f^{-1}(f(K))$.

The notion of an atomic mapping was introduced by R. D. Anderson in [1] to describe special decompositions of continua. Soon, atomic mappings turned out to be important tools in continuum theory and proved to be interesting by themselves, and several their properties have been discovered, e.g. in [4] and [8]. The following fact on atomic mappings is known (see [4, Theorem 1, p. 49] and [8, (4.14), p. 17]).

FACT 1. *Every atomic mapping of a continuum is (hereditarily) monotone.*

We say that a class $\mathfrak{M}$ of mappings has the *composition factor property* provided that for every mappings $f : X \to Y$ and $g : Y \to Z$ the composition is in $\mathfrak{M}$ only if the second mapping is, i.e., if

$$(1) \qquad g \circ f \in \mathfrak{M} \quad \text{implies} \quad g \in \mathfrak{M}.$$

Maćkowiak asked in [8, (5.22), p. 33] if the class of atomic mappings has the composition factor property, and conjectured that it does. He has proved in [8, (5.29), p. 35] the following result.

PROPOSITION 2. *Let X be a continuum, and let the composition $g \circ f$ of mappings $f : X \to Y$ and $g : Y \to Z$ be atomic. If f is weakly confluent, then g is atomic.*

Recall that a continuum X is said to be *arc-like* provided that for each positive ε there is a mapping of X onto an arc whose point-inverses have diameters less than ε. Equivalently, X is arc-like provided that it is the inverse limit of an inverse sequence of arcs (see e.g. [11, (0.23), p. 13]). Since any mapping of a continuum onto an arc-like continuum is weakly confluent, [13, Theorem 4, p. 236], and since any mapping of a continuum onto a hereditarily indecomposable continuum is confluent, [13, Theorem 4, p. 243], thus weakly confluent, the following is a consequence of Proposition 2.

COROLLARY 3. *Let X be a continuum and let the composition $g \circ f$ of mappings $f : X \to Y$ and $g : Y \to Z$ be atomic. If Y is either arc-like or hereditarily indecomposable, then g is atomic.*

The inverse implications to these of Proposition 2 and Corollary 3 are not true, even if all three spaces X, Y and Z are irreducible one-dimensional continua. The next example shows this. Recall that a continuum is X said to be *irreducible* provided that there are two points a and b in X such that no proper subcontinuum of X contains both a and b.

EXAMPLE 4. There are irreducible continua X, Y and Z and surjective mappings $f : X \to Y$ and $g : Y \to Z$ such that Y is hereditarily decomposable and not arc-like, the composition $g \circ f$ and g are both atomic, while f is not weakly confluent:

In the Euclidean plane $\mathbb{R}^2$ let X be the $\sin(1/x)$-curve consisting of its limit segment $L = \{(0,y) \in \mathbb{R}^2 : y \in [-1,1]\}$ and of the half-ray $H = \{(x,y) \in \mathbb{R}^2 : y = \sin(1/x),\ x \in (0,1]\}$, i.e., $X = L \cup H$. Identify the two end points $(0,-1)$ and $(0,1)$ of L into one point p and let $f : X \to Y$ be the identification mapping. Thus $f(L) = C$ is a circle containing p, and the restriction $f|H$ is a homeomorphism. Therefore the continuum Y is the union of a circle C (so it is not arc-like) and of the ray $f(H)$ that approximates C, hence it is an irreducible continuum. The mapping f is not weakly confluent since for a small arc $A \subset C$ containing the point p in its relative interior (i.e., so that p is not an end point of A) none of the two components of $f^{-1}(A)$ is mapped onto A under f.

Define further $g : Y \to Z = [0,1]$ as a mapping that shrinks C to 0 and is a homeomorphism on the rest, i.e., on $f(H)$. Thus g is atomic by its definition. The composition $g \circ f : X \to [0,1]$ shrinks L to 0 and is a homeomorphism on H, so it is atomic. The proof is complete.

Later, in [9, Chapter 1, Example, p. 7] Maćkowiak has proved that the assumption of weak confluence of f is essential in Proposition 2, and therefore the class of atomic mappings does not have the composition factor property. Constructing the needed example he considers the Knaster irreducible continuum X that can be mapped onto $[0,1]$ under an open and monotone mapping h such that $h^{-1}(t)$ is nondegenerate for every $t \in [0,1]$ (see [7]). Thus the continuum X has a continuous decomposition into tranches $h^{-1}(t)$. Next he defines an intermediate continuum Y, a mapping $f : X \to Y$ (which is not weakly confluent) and a monotone mapping g from Y onto an arc, such that $g \circ f$ is atomic and g is not atomic.

Another example answering Maćkowiak question in the negative has been constructed in [5, Example 2, p. 140]. Now the circle of pseudo-arcs (as constructed in [2, Section 7, p. 189-191]) is taken as the continuum X. Again X admits a continuous monotone decomposition into nondegenerate continua (each of which is a pseudo-arc), now with a simple closed curve Z as the decomposition space.

A common property for the domain continua X used in the two examples is that each of them contains (the first one, in fact, is) an irreducible continuum for which the Kuratowski decomposition into tranches is continuous, and each element of this decomposition is nondegenerate. It is known that if an irreducible continuum admits a continuous decomposition into tranches, and if all tranches are nondegenerate, then the continuum must contain a dense family of indecomposable tranches containing indecomposable subcontinua of arbitrarily small diameters, [12]. However, a continuum having the discussed property was constructed in [10] that contains no nondegenerate hereditarily indecomposable continuum. Thus the following questions concerning composition factor property for atomic mappings arise in a natural way.

QUESTION 5. Does composition factor property for atomic mappings hold if the domain X of the first mapping f (a) does not contain any irreducible continuum which admits a continuous decomposition into nondegenerate tranches? (b) is hereditarily decomposable?

Note that, by the above mentioned result of [12], an affirmative answer to (b) implies that an answer to (a) is also affirmative.

Besides weak confluence of f in Proposition 2, another condition under which implication (1) holds for the class $\mathfrak{M}$ of atomic mappings was given by Hosokawa in [6, Theorem 3.8, p. 36].

PROPOSITION 6. *Let X be a continuum, and let the composition $g \circ f$ of mappings $f : X \to Y$ and $g : Y \to Z$ be atomic. If the set*

$$\{z \in Z : (g \circ f)^{-1}(z) \text{ is nondegenerate}\}$$

contains no subset that is dense in itself, then g is atomic.

Remark 7. Note that, alike for Proposition 2, also the assumption of Proposition 6 is sufficient, but not necessary, for g to be atomic. Indeed, if X is the above mentioned Knaster irreducible continuum X that can be mapped onto $[0, 1]$ under an open and monotone mapping h such that $h^{-1}(t)$ is nondegenerate for every $t \in [0, 1]$ (see [7]), then h is atomic by its definition (see [9, Example, p. 7]). Taking $g : [0, 1] \to [0, 1]$ as the identity, we see that $g \circ h$ and g are atomic, while the set $\{z \in [0, 1] : (g \circ h)^{-1}(z) \text{ is nondegenerate}\}$ is the whole $[0, 1]$, so it is dense in itself.

Recall also that each atomic mapping from an arcwise connected continuum onto a nondegenerate one is a homeomorphism (see [4, Corollary 9, p. 53, Theorems 5, 6, p. 52]; cf. also [8, (6.3), p. 51]). Thus if the domain X of f is arcwise connected (if it is locally connected, in particular), then atomicity of the composition $g \circ f$ implies that it is a homeomorphism, whence g is, and therefore implication (1) trivially holds.

In the light of Propositions 2 and 6, as well as of the examples mentioned above, the following questions can be asked.

QUESTION 8. What are sufficient *and* necessary conditions on (a) the domain continuum X of f (b) the mapping f, under which implication (1) holds for the class $\mathfrak{M}$ of atomic mappings?

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