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The Band Generated by Homomorphisms on Banach Lattices

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1. Introduction

This paper will consider the closure of the set of operators which may be expressed as a sum of lattice homomorphisms whose range is contained in a Dedekind complete Banach lattice.

Finite sums of lattice homomorphisms have been studied in [2] and [3]. In section 3 we review (and slightly extend) some of these results. In section 4, we consider the band generated by lattice homomorphism, as well as the norm closure of the set of lattice homomorhpisms and additional questions regarding weakly compact operators and the band generated by lattice homomorphisms. We show that an operator is in the band generated by sums of lattice homomorphisms from $C(X)$ to Dedekind complete $C(Y)$ if and only if its transpose maps each point measure at y for y in a residual subset of Y to an atomic measure on X. We give a corresponding result in case E and F are Banach lattices, F is Dedekind complete, and E has a quasi-interior point. Weakly compact operators are also considered, as well as the norm closure of the set of sums of lattice homomorphisms.

2. Preliminaries

For Riesz spaces E and F, $L(E,F)$ will represent the space of order bounded operators from E to F. $T \in L(E,F)$ is a *lattice homomorphism* if T satisfies $Tf \wedge Tg = 0$ for f and g in E which satisfy $f \wedge g = 0$. A band-preserving operator in $L(E,E)$ will be called an *orthomorphism*. Let $C(X)$

and $C(Y)$ represent the Banach lattices of continuous functions on compact Hausdorff spaces X and Y, respectively. Orthomorphisms in $L(C(X), C(X))$ coincide with the set of multiplication operators on $C(X)$, i.e. those operators defined by $Tf(x) = g(x)f(x)$ for some fixed g (See [1, 8.27]). If $Tf = gf$ for $g \in C(X)$, we denote such an orthomorphism T by the symbol $\overline{g}$. If $T \in L(C(X), C(Y))$ is a lattice homomorphism, then there is a positive $g \in C(Y)$ and a function $\phi : Y \to X$ which is continuous when $g(y) > 0$, so that $Tf = \overline{g}\ f \circ \phi$; in other words $Tf(y) = g(y)f(\phi(y))$. (See [1, 7.22]).

For a Riesz space E, $E^{\sim}$ will denote the dual space of (order) bounded linear functionals. If E is a Banach lattice, $E^{\sim}$ coincides with E', the norm dual of E. $E_n^{\sim}$ denotes the subspace of $E^{\sim}$ consisting of order continuous linear functionals. $C'(X)$ may as usual be identified with the set of bounded Baire measures on X, and each $x \in X$ may be considered as a point measure in $C'(X)$. The band of $C'(X)$ of atomic measures, i.e. all $\mu \in C'(X)$ such that $\langle \mu, f \rangle = \sum a_j f(x_j)$ for a sequence $\{a_j\} \subseteq \mathbb{R}$ and a sequence $\{x_j\} \subseteq X$, will be denoted by $C'(X)_a$ (or C'_a). The disjoint complement in $C'(X)$ of *diffuse* measures will be denoted by $C'(X)_d$.

Note that if $E = C(X)$ and $F = C(Y)$, $L(E, F)$ is also the collection of *continuous* operators from E to F. An operator S in $L(C(X), C(Y))$ is determined by its adjoint mapping $S^* : C'(Y) \to C'(X)$ restricted to Y where Y is identified with a subset of $C'(Y)$. Any mapping from Y to $C'(X)$ which is continuous for the $\sigma(C'(X), C(X))$ topology determines an element of $L(C(X), C(Y))$ (see [4, VI.7.1]). The above representation of a lattice homomorphism T is equivalent to the condition that for each $y \in Y$, there are a real number a and an $x \in X$ such that $T^*y = a\hat{x}$ where $\hat{x} \in C'(X)$ is the point measure associated with x, i.e. $a = g(y)$ and $x = \phi(y)$.

For a positive operator T, suppose that, for each $y \in Y$, there are $x_1, ...,$ $x_n \in X$ and $a_1, a_2, ..., a_n \in \mathbb{R}$ such that $Tf(y) = \sum_{i=1}^n a_i f(x_i)$ for all $f \in C(X)$ (i.e. $T^*y = \sum_{i=1}^n a_i \hat{x}_i$). In [3], we determined that this condition was not sufficient to guarantee that T is a finite sum of lattice homomorphisms, but there is an increasing net of orthomorphisms $\{\overline{g}_\alpha\}$ with supremum the identity operator such that each $\overline{g}_\alpha \circ T$ is such a finite sum. Such an operator is called a *local homomorphism*. However, if n is fixed and $C(Y)$ is Dedekind complete, then T itself is a sum of n lattice homomorphisms. In addition we will show that for any positive $T \in L(C(X), C(Y))$ there is a projection P_n on $C(Y)$ so that $P_n \circ T$ may be written as such a sum.

In section 4, we consider the band generated by lattice homomorphisms and situations in which operators in this band may be expressed as infinite sums. The characterization of finite sums of homomorphisms using mappings from Y to finite sums of point measures on X suggests a similar result using infinite sums of point measures (atomic measures) for the band generated by lattice homomorphisms. We show that such a representation is indeed possible (except on a set of first category). We also describe the projection of weakly compact operators on this band. Our final result concerns the norm closure of the set of sums of lattice homomorphisms.

Bernau, Huijsmans, and de Pagter have also studied the collection of sums of lattice homomorphisms ([2]), giving a disjointness property which characterizes such sums when F is Dedekind complete (providing Theorem 4 below as a corollary).

We begin with two lemmas concerning lattice homomorphisms. The proof of lemma 1 follows directly from the definition of lattice homomorphism.

LEMMA 1. *Let $\{S_\alpha\}$ be a net of operators from a Riesz space E to a Dedekind complete Riesz space F. If $S_\alpha \uparrow S$ and each S_α is a lattice homomorphism, then S is also.*

LEMMA 2. *Let S and T be lattice homomorphisms from a Riesz space E to a Dedekind complete Riesz space F. If T_S is the projection of T onto the band of $L(E, F)$ generated by S, then $S + T_S$ is also a lattice homomorphism.*

Proof. Note that $T_S = \vee_n (T \wedge nS)$ and let $x \wedge y = 0$. Then $T_S x \wedge Sy = (\vee_n (T \wedge nS)x) \wedge Sy = \vee_n ((T \wedge nS)x \wedge Sy)$, but $(T \wedge nS)x \wedge Sy \leq n(Sx \wedge Sy) = 0$, which implies that $(T_S x + Sx) \wedge (T_S y + Sy) \leq T_S x \wedge T_S y + T_S x \wedge Sy + Sx \wedge T_S y + Sx \wedge Sy = 0$. ∎

PROPOSITION 3. *Let T be a bounded operator from a Riesz space E to a Dedekind complete Riesz space F. If T is a sum of lattice homomorphisms, then we may assume these homomorphisms are disjoint.*

Proof. Suppose that $T = S_1 + S_2$ where S_1 and S_2 are (not necessarily disjoint) lattice homomorphisms. Let $R_1 = S_1 + (S_2)_{S_1}$ and $R_2 = S_2 - (S_2)_{S_1}$. Then R_1 and R_2 are lattice homomorphisms, $R_1 \wedge R_2 = 0$ and $R_1 + R_2 = S_1 + S_2$. An induction argument completes the proof. ∎

3. FINITE SUMS OF HOMOMORPHISMS

In [3], we noted that the local homomorphism condition was unnecessary for the characterization of sums of homomorphisms in case $C(Y)$ is Dedekind complete. A proof of this fact (Theorem 4) was given by Bernau, Huijsmans, and de Pagter in [2].

THEOREM 4. *Let X and Y be compact Hausdorff spaces and let $T \in L(C(X), C(Y))$, where $C(Y)$ is Dedekind complete. For a fixed n, the following are equivalent:*

(i) $T = S_1 + ... + S_n$, $S_i \in \text{hom}(C(X), C(Y))$.

(ii) *For each $y \in Y$ there are $x_1, ..., x_n \in X$ and real numbers $a_1, ..., a_n$ such that $T^*y = \sum_{i=1}^n a_i \hat{x}_i$.*

The above may be extended to certain Banach lattices represented by continuous extended real valued functions. For a discussion of such Banach lattices, see [6].

THEOREM 5. *Let E and F be Banach lattices. Suppose that E has a quasi-interior point e, F is Dedekind complete, and $T \in L(E, F)$. For a fixed n, the following are equivalent:*

(i) *$T = S_1 + ... + S_n$, where each S_i is a lattice homomorphism.*

(ii) *If X is a representation space for E (with $e = 1_X$) and Y a representation space for $T(E)$ with $T(e)$ finite on Y, then there exists a dense open subset G of Y so that for each $y \in G$ there are $x_1, ..., x_n \in X$ and non-negative real numbers $a_1, ..., a_n$ such that*

$$Tf(y) = \sum_{i=1}^n a_i f(x_i).$$

Proof. For (i) $\Rightarrow$ (ii), recall that each S_i can be viewed as a weighted composition operator (see Theorem 1 in [5]), i.e. $S_i f(y) = r(y) f(\phi(y))$ for a real valued function r and an X valued function ϕ defined for each y in a dense open subset H_i of Y. Thus (ii) is satisfied for the dense open subset $G = \bigcap_{i=1}^n H_i$. For (ii) $\Rightarrow$ (i), consider the collection of all Z_α which are clopen in Y and subsets of G. Now $\bigcup Z_\alpha = G$ and $\{Z_\alpha\}$ is directed by inclusion. The restriction of each $P_{Z_\alpha} \circ T$ to $C(X)$ satisfies (ii) of the previous theorem so that (on $C(X)$) $P_{Z_\alpha} \circ T = \sum_{i=1}^n S_{i,\alpha}$ for lattice homomorphisms $S_{i,\alpha}$. Now $\bigvee_\alpha P_{Z_\alpha}$ is the identity operator on $C(Y)$. For $S_i = \bigvee_\alpha S_{i,\alpha}$ and T restricted to $C(X)$

(again denoted by T), we note that

$$T = \bigvee P_{z_\alpha} \circ T = \bigvee_\alpha \sum_{i=1}^n S_{i,\alpha} = \sum_{i=1}^n \bigvee_\alpha S_{i,\alpha} = \sum_{i=1}^n S_i .$$

Since each S_i extends to a lattice homomorphism on E, we conclude that T is a finite sum of lattice homomorphisms. ∎

THEOREM 6. *Let $T : C(X) \to C(Y)$ be positive with $C(Y)$ Dedekind complete. Then for any positive integer n there is a projection P_n on $C(Y)$ such that $P_n \circ T$ is a sum of n homomorphisms and $Q \circ T$ is not such a sum for any non-zero projection on $C(Y)$ satisfying $Q \wedge P_n = 0$.*

Proof. Let $Z = \{y \in Y : T^*y = a_1\hat{x}_1 + ... + a_n\hat{x}_n$ for some $a_i \geq 0$ and $x_i \in X\}$. Note that Z is closed and let Z^0 denote the interior of Z (implying that Z^0 is clopen). Letting $P_n = P_{Z^0}$ (the projection on Z^0), note that $P_n \circ T$ is a sum of n homomorphisms by Theorem 4.

Let Q be a projection on $C(Y)$ with $Q \wedge P_n = 0$, and suppose that Q is the projection determined by the clopen set $A \subseteq Y$. Since $A \cap Z^0 = \emptyset$, there is a $y_0 \in A \sim Z$. Since $(Q \circ T)^*y_0 = T^*y_0$ cannot be written in the form $a_1\hat{x}_1 + ... + a_n\hat{x}_n$, we conclude that $Q \circ T$ is not the sum of n lattice homomorphisms. ∎

4. THE BAND GENERATED BY LATTICE HOMOMORPHISMS

For Dedekind complete $C(Y)$, finite sums of lattice homomorphisms in $L(C(X), C(Y))$ and the vector lattice generated by these sums are characterized by the adjoint mapping restricted to Y in Theorem 4. We consider now the related question for the *band* generated by lattice homomorphisms. A parallel condition may be given for this band, using atomic measures rather than finite sums of point measures as in Theorem 4. The principle result is Theorem 10, where this condition is established, along with the generalization in Theorem 11.

First, however, we give a result for the following special case.

THEOREM 7. *Let $T : C(X) \to C(Y)$ be positive with $C(Y)$ Dedekind complete and $C_n^\sim(Y)$ separating on $C(Y)$. Then T is in the band generated by the lattice homomorphisms if and only if $T = \sum_{i=1}^\infty R_i$ (order) where $R_i \wedge R_j = 0$ for $i \neq j$ and each R_i is a homomorphism.*

Proof. The implication ($\Leftarrow$) is clear. For ($\Rightarrow$), consider $T, \{T_\alpha\}$ where $T, T_\alpha: C(X) \to C(Y)$. $0 \le T_\alpha \uparrow T$ and each T_α is a finite sum of lattice homomorphisms. For each $\mu \in (C_n^\sim(Y))_+$, let P_μ be the projection onto the band $(C'_\mu)_\perp^d$ in $C(Y)$ dual to μ (C'_μ is the band in C' generated by μ, $(C'_\mu)_\perp$ consists of all $f \in C(Y)$ satisfying $\langle \mu, |f| \rangle = 0$, and $(C'_\mu)_\perp^d$ is the disjoint complement of $(C'_\mu)_\perp$ in $C(Y)$.) $P_\mu \circ T_\alpha$ is then also a sum of lattice homomorphisms. Since $P_\mu \circ T_\alpha \uparrow P_\mu \circ T$, we may find an increasing sequence $\{T_n\} \subseteq \{T_\alpha\}$ such that $P_\mu \circ T_n \uparrow P_\mu \circ T$ by choosing T_n to satisfy $\langle \mu, (T - T_n)1 \rangle < 2^{-n}$. Suppose that

$$P_\mu \circ T_1 = T_{1,1} + \ldots + T_{1,m_1}$$

where the $T_{1,i}$ are disjoint lattice homomorphisms. We may then express $P_\mu \circ T_2$ as

$$P_\mu \circ T_2 = T_{2,1} + \ldots + T_{2,m_2}$$

where $m_2 \ge m_1$, $T_{2,i}$ are disjoint and

$$T_{2,j} = (P_\mu \circ T_2)_{T_{i,j}}\,.$$

For $j \le m_1$. In general, let

$$P_\mu \circ T_i = T_{i,1} + \ldots + T_{i,m_i}$$

where $T_{i,j}$ are disjoint (in j) homomorphisms which satisfy

$$T_{i,j} = (P_\mu \circ T_i)_{T_{i-1,j}}$$

for $j \le m_{i-1}$. Let $R_{j,\mu} = \vee_i T_{i,j}$, and note that $R_{j,\mu}$ is a lattice homomorphism. If $\mathcal{A}$ is a maximal collection of pairwise disjoint elements of $(C_n^\sim(Y))_+$, let $R_j = \vee_{\mu \in \mathcal{A}} R_{j,\mu}$, from which we conclude that the R_j are disjoint homomorphisms and $T = \vee_j R_j = \sum_{j=1}^\infty R_j$ since $\mathcal{A}$ is separating on $C(Y)$. ∎

We now turn to the question of the relation between the band generated by lattice homomorphisms and $\sigma(C', C)$ continuous mappings from Y to $C'(X)$. In the following, for $\mu \in C'$, μ^a will be the atomic part of μ (the projection onto $C'(X)_a$), and $\mu^d = \mu - \mu^a$ its disjoint complement (the projection onto $C'(X)_d$).

LEMMA 8. *Let $S: C(X) \to C(Y)$ be positive. Suppose $(S^* y)^a = \sum b_i \hat{x}_i$ with x_i distinct. Given $\epsilon > 0$ let $\phi_\epsilon : Y \to C'_a$ be given by $\phi_\epsilon(y) = \sum (b_i - \epsilon)^+ \hat{x}_i$ and define $S_\epsilon f(y) = \langle \phi_\epsilon(y), f \rangle$. Then $S_\epsilon f(y)$ is upper semicontinuous on Y when $f \ge 0$.*

Proof. Suppose net $y_\alpha \to y_0$ and $S_\epsilon f(y_\alpha) \to \overline{\lim}_{y\to y_0} S_\epsilon f(y)$. We must show that $S_\epsilon f(y_0) \geq \lim_\alpha S_\epsilon f(y_\alpha)$. For convenience, let $S^*(y_\alpha) = \mu_\alpha$ and $\mu_\alpha^\epsilon = \phi_\epsilon(y_\alpha)$. By taking subnets if necessary we may assume $\mu_\alpha \to \mu$, $\mu_\alpha^a \to \rho$, $\mu_\alpha^d \to \eta$, $\mu_\alpha^\epsilon \to \nu$ in the $\sigma(C', C)$ topology for some μ, ρ, η and ν. We will complete the proof by showing $\nu \leq \phi_\epsilon(y_0)$.

There is a fixed N so that μ_α^ϵ is the sum of at most N non-zero point measures ($N \leq ||S||/\epsilon$). Thus ν is the sum of at most N non zero point measures; $\nu = \sum_1^K a_i\hat{x}_i$ with $K \leq N$. If $\mu_\alpha^\epsilon = \sum_1^{K_\alpha} a_{i,\alpha}\hat{x}_{i,\alpha}$, let $\mu'_\alpha = \sum_1^{K_\alpha}(a_{i,\alpha}+\epsilon)\hat{x}_{i,\alpha}$. We have then $\mu_\alpha^\epsilon \leq \mu'_\alpha \leq \mu_\alpha^a$. By again taking a subnet, we may assume $\mu'_\alpha \to \mu'$ $(\sigma(C', C))$. μ' is also the sum of at most N point measures, and $\sum_1^K (a_i+\epsilon)\hat{x}_i \leq \mu' \leq \rho^a$, where $\nu = \sum_1^K a_i\hat{x}_i$ as above. If we let $\rho^a = \sum c_i\hat{z}_i$ and $\rho_\epsilon^a = \sum(c_i-\epsilon)^+\hat{z}_i$, then we may conclude $\nu \leq \rho_\epsilon^a$. Since $\mu = \rho + \eta$, we have $\phi_\epsilon(y_0) \geq \rho_\epsilon$, and the proof is complete. ∎

LEMMA 9. *Let $C(Y)$ be Dedekind complete. For f upper semicontinuous on Y, let $\ell(f) = \wedge\{g \in C(Y) \mid g(y) \geq f(y)\ \forall y \in Y\}$ (infimum in $C(Y)$). Then ℓ has the following properties.*

(i) *$\ell(f) \leq f$ (as functions on Y).*

(ii) *$\ell(f) = \vee\{h \in C(Y) \mid h(y) \leq f(y)\ \forall y \in Y\}$.*

(iii) *If f_1 and f_2 are upper semicontinuous then $\ell(f_1+f_2) = \ell(f_1)+\ell(f_2)$.*

(iv) *$\{y \mid f(y) > \ell(f)(y)\}$ is of first category.*

Proof. Property (i) follows from the fact that f is the pointwise infimum of the set in the definition of $\ell(f)$, and this exceeds the infimum in $C(Y)$. This implies property (ii); since $g \geq h$ for every g in the definition of $\ell(f)$ and h in the set in statement (ii). To prove (iii) suppose that $g_1, g_2 \in C(Y)$ and that $g_1 \geq f_1$, $g_2 \geq f_2$. Hence $g_1+g_2 \geq f_1+f_2$ and $g_1+g_2 \geq \ell(f_1+f_2)$. Taking the infimum over all possible g_1 yields $\ell(f_1)+g_2 \geq \ell(f_1+f_2)$, and repeating for all g_2 implies $\ell(f_1)+\ell(f_2) \geq \ell(f_1+f_2)$. If we consider $h_1, h_2 \in C(Y)$ with $h_1 \leq f_1$ and $h_2 \leq f_2$ and modify the preceding argument using property (ii), we conclude $\ell(f_1)+\ell(f_2) \leq \ell(f_1+f_2)$, which proves (iii).

For (iv), note that $\{y \mid f(y) > \ell(f)(y)\} = \cup_j\{y \mid f(y) - \ell(f)(y) > \frac{1}{j}\}$. ∎

THEOREM 10. *Let $C(Y)$ be Dedekind complete. Then the following are equivalent:*

(i) *$T \in L(C(X), C(Y))$ is in the band generated by lattice homomorphisms.*

(ii) *$T^*y \in C'(Y)_a$ for y outside a set of first category.*

(iii) *For each $y \in Y$ outside a set of first category, there are sequences $\{a_j\} \subseteq \mathbb{R}$ and $\{x_j\} \subseteq X$, so that $Tf(y) = \sum_1^\infty a_j f(x_j)$.*

Proof. It is clear that (ii) and (iii) are equivalent.

For (i) $\Rightarrow$ (ii), we may assume $T \geq 0$. Let $0 \leq T_\alpha \uparrow T$ where $T_\alpha = T_{\alpha 1}, +\dots + T_{\alpha,n}$ and each $T_{\alpha,j}$ is a homomorphism. Note that $||T^*y|| = T1(y)$ and that, for $\epsilon > 0$, $\{y \in Y \mid ||(T^*y)^d|| \geq \epsilon\} \subseteq \{y \in Y \mid T1(y) - T_\alpha 1(y) \geq \epsilon\}$, since $(T_\alpha^* y)^d = 0$. (Recall that $\mu^d = \mu - \mu^a$, where μ^a is the atomic part of μ.) If $s(y) = \vee_\alpha T_\alpha 1(y)$ (pointwise) then $\{y \mid ||(T^*y)^d|| \geq \epsilon\} \subseteq \{y \mid T1(y) - s(y) \geq \epsilon\}$. The set $\{y \mid T1(y) - s(y) < \epsilon\}$ is open; it is also dense, since $T_\alpha 1 \uparrow T1$. Since $\{y \mid ||T^*y)^d|| > 0\} = \cap_j \{y \mid ||(T^*y)^d|| > \frac{1}{j}\}$, the proof is complete.

For (ii) $\Rightarrow$ (i), again assume $T \geq 0$, let $\epsilon > 0$ and consider T_ϵ as in the Lemma 8. Each $T_\epsilon f$ is an upper semicontinuous function on Y when $f \geq 0$. Let $\tilde{T}_\epsilon f = \ell(T_\epsilon f)$ when $f \geq 0$ and $\tilde{T}_\epsilon f = \ell(T_\epsilon(f^+)) - \ell(T_\epsilon(f^-))$ for arbitrary f. $\tilde{T}_\epsilon$ is then a positive (bounded) transformation from $C(X)$ to $C(Y)$, and property (iii) of Lemma 9 implies that $\tilde{T}_\epsilon$ is linear. $\tilde{T}_\epsilon^* y$ is the sum of at most N point measures (N as in the lemma), so that $\tilde{T}_\epsilon$ is a finite sum of homomorphisms. Let $\epsilon_j = \frac{1}{j}$ in the following. Note that $\{y \mid (T^*y)^a > 0\} = \cup_j \{y \mid T_{\epsilon_j} 1(y) > 0\}$ and that $\{y \mid T_{\epsilon_j} 1(y) > 0\} = \{y \mid \tilde{T}_{\epsilon_j} 1(y) > 0\} \cup \{y \mid T_{\epsilon_j} 1(y) > \tilde{T}_{\epsilon_j} 1(y)\}$. The definition of $\tilde{T}_{\epsilon_j}$ implies that $\{y \mid T_{\epsilon_j} 1(y) > \tilde{T}_{\epsilon_j} 1(y)\}$ is first category, so that $\tilde{T}_{\epsilon_j} 1 \neq 0$ for some j, since $\{y \mid (T^*y)^a > 0\}$ is not of first category. Since every T in (ii) thus dominates a lattice homomorphism (or sum of homomorphisms), and T minus this homomorphism also satisfies (ii), we conclude that T is in the band generated by lattice homomorphisms. ∎

THEOREM 11. *Let E and F be Banach lattices. Suppose that E has a quasi-interior point e, F is Dedekind complete, and $T \in L(E, F)$.*

Let X be a representation space for E (with $e = 1_X$) and let Y be a representation space for $T(E)$ with $T(e)$ finite on Y Suppose that for each $y \in Y$ outside a set of first category, there are sequences $\{a_j\} \subseteq \mathbb{R}$ and $\{x_j\} \subseteq X$, so that $Tf(y) = \sum_1^\infty a_j f(x_j)$. Then $T \in L(E, F)$ is in the band generated by lattice homomorphisms.

Proof. Assume $T \geq 0$ and restrict T to $C(X) \subseteq E$. The previous theorem implies that we may find T_α on $C(X)$ so that $T_\alpha \uparrow T$ and each T_α is a finite sum of lattice homomorphisms. If $T_\alpha = S_1 + \dots + S_n$, where each S_j is a lattice homomorphism on $C(X)$, the homomorphisms S_j may be extended to homomorphisms on E in such a way that the extended $T_\alpha = S_1 + ... + S_n \leq T$. (Extend S_1 to a homomorphism on E with $S_1 \leq T$. Since $S_2 \leq T - S_1$ on $C(X)$, extend S_2 to E so that $S_2 \leq T - S_1$. Repeat the above through $S_n \leq T - S_1 - ... - S_{n-1}$.) Thus, $T = \vee T_\alpha$ in $L(E,F)$, where each T_α is a sum of homomorphisms in $L(E,F)$. ∎

We now consider compactness and weak compactness of operators in the band generated by homomorphisms.

PROPOSITION 12. *Let $T \in L(C(X), C(Y))$ be positive and satisfy $T^* : Y \to C_a'(X)$. Then T weakly compact implies T compact.*

Proof. For $y_0 \in Y$ and $y_\alpha \to y_0$, suppose that $T^*y_0 = \sum_{i=1}^\infty a_i\hat{x}_i$ ($a_i > 0, x_i$ distinct). For a given $N > 0$, write $T^*y = a_1(y)\hat{x}_1 + ... + a_N(y)\hat{x}_N + \sum_{i=1}^\infty b_i\hat{z}_i$ with $z_i \neq x_j$ for any $j = 1, ..., N$ and $i \in \mathbb{N}$. Since T is weakly compact, we have $T^*y_\alpha \to T^*y_0$ in $\sigma(C', C'')$, and we may conclude $a_i(y_\alpha) \to a_i = a_i(y_0)$ for $i = 1, ..., N$ by applying T^*y_α to 1_{x_i}, the projection of the unit $\mathbf{1}$ on the band of C'' dual to C'_{x_i}.

Given $\epsilon > 0$ find N so that $\sum_{N+1}^\infty a_i < \epsilon/5$ and write T^*y_α as above. Choose α_1 so that $|a_i(y_\alpha) - a_i| < \frac{\epsilon}{5}$ for $\alpha \geq \alpha_1$. For these α, we conclude

$$\begin{aligned}
||T^*y_\alpha &- T^*y_0|| \\
&\leq \sum_1^N |a_i - a_i(y_\alpha)| + ||T^*y_\alpha - \sum_1^N a_i(y_\alpha)\hat{x}_i|| + ||T^*y_0 - \sum_1^N a_i\hat{x}_i|| \\
&< \frac{\epsilon}{5} + ||T^*y_\alpha|| - ||\sum_1^N a_i(y_\alpha)\hat{x}_i|| + \frac{\epsilon}{5} \\
&< \frac{\epsilon}{5} + ||T^*y_0|| + \frac{\epsilon}{5} - \sum_1^N a_i + \frac{\epsilon}{5} + \frac{\epsilon}{5} \\
&= \frac{4\epsilon}{5} + ||T^*y_0 - \sum_1^N a_i\hat{x}_i|| < \epsilon.
\end{aligned}$$

Thus $T^*y_\alpha \to T^*y_0$ in norm, and we conclude that T is compact. ∎

THEOREM 13. *Let $T \in L(C(X), C(Y))$ be weakly compact. If $T_a \in L(C(X), C(Y))$ is the operator determined by $T_a^*y = P_a \circ T^*y$ where $P_a : C'(X) \to C_a'(X)$ is the projection onto $C_a'(X)$, then T_a is compact and is in the order closure of the set of sums of homomorphisms. If $C(Y)$ is Dedekind* *complete, then T_a is the projection of T onto the band generated by lattice homomorphisms.*

Proof. Note first that $T_a^*y = P_a \circ T^*y$ defines a weakly compact element of $L(C(X), C(Y))$, since $T^* : Y \to C'(X)$ maps convergent nets to $\sigma(C', C'')$ convergent nets and $P_a : C' \to C'_a$ preserves $\sigma(C', C'')$ convergence. The preceeding proposition implies that T_a is compact.

By [3, Theorem 5], T_a is in the norm closure (hence the order closure) of the set of local homomorphisms, which are in turn the suprema of finite sums of homomorphisms.

For $C(Y)$ Dedekind complete, let T_1 be the projection onto the band generated by lattice homomorphisms. For any $S \leq T$ which is a finite sum of lattice homomorphisms, we have $S \leq T_a$ (since $S^* \leq T_a^*$) and thus $T_1 \leq T_a$. Since $T_1 \geq T_a$ (T_a is in the band generated by finite sums of homomorphisms), we conclude $T_1 = T_a$. ∎

Finally, we state and prove a characterization of the norm closure of the set of finite sums of lattice homomorphisms.

THEOREM 14. *Let $T \in L(C(X), C(Y))$ be positive with $C(Y)$ Dedekind complete. T is in the norm closure of the set of sums of lattice homomorphisms if and only if $T = \sum_{i=1}^{\infty} R_i$ (norm) where $R_i \wedge R_j = 0$ if $i \neq j$ and each R_i is a lattice homomorphism.*

Proof. The implication ($\Leftarrow$) is clear. To prove ($\Rightarrow$) suppose that $||S_n - T|| \to 0$ where each S_n is a sum of lattice homomorphisms. We may assume $S_n \leq T$, replacing S_n with $S_n \wedge T$ if necessary. We may further assume that $S_n \leq S_{n+1}$, by replacing S_2 with $(S_2 - S_1) \vee 0 + S_1$, and repeating for $S_3, S_4, \ldots$ (recall that finite sums of lattice homomorphisms form an ideal).

Write $S_1 = R_{1,1} + \ldots + R_{1,k_1}$, where the $R_{i,j}$ are disjoint lattice homomorphisms. $S_2 - S_1$ is also a sum of lattice homomorphisms, hence, we may write $S_2 - S_1 = R_{2,1} + \ldots + R_{2,k_2}$, $k_2 \geq k_1$, where $R_{2,i} = (S_2 - S_1)_{R_{1,i}}$, $1 \leq i \leq k_1$, and $R_{2,j}$ are disjoint. Proceed inductively, writing $S_n - S_{n-1} = R_{n,1} + \ldots + R_{n,k_n}$, $k_n \geq k_{n-1}$ with

$$R_{n,i} = (S_n - S_{n-1})_{(\sum_{j=1}^{n-1} R_{j,i})}.$$

Define $R_i = \sum_{j=1}^{\infty} R_{j,i}$, letting $R_{j,i} = 0$ if not previously defined. Each R_i is a lattice homomorphism by lemma 1 and $R_i \wedge R_j = 0$ when $i \neq j$. Also,

$T - \sum_{i=1}^{k_n} R_i \geq 0$ and

$$||T - \sum_{i=1}^{k_n} R_i|| \leq ||T - \sum_{i=1}^{k_n} \sum_{j=1}^{n} R_{j,i}|| = ||T - S_n||$$

and we conclude $\sum_{i=1}^{k_n} R_i \to T$ in norm. ∎

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