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When is any Continuous Function Lipschitzian?

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It is well known that every continuous function on a compact metric space X is uniformly continuous on the space. However, this property does not characterize the compactness, but a larger class of spaces usually called UC spaces or Atsuji spaces. The first list of descriptions of these spaces was compiled by Monteiro and Peixoto [9] and a more extensive list of internal descriptions was furnished by Atsuji [1]. Some of such characterizations are these:

(i) Each open cover of the metric space X has a Lebesgue number [9].

(ii) If $\{x_n\}$ is a sequence of points in X without accumulation points, then all but finitely many numbers of x_n are isolated and $\inf I(x_n)$ for all the isolated points of X is positive, where $I(x) \doteq \sup\{\alpha > 0 : B(x,\alpha) = \{x\}\} > 0$ [1].

(iii) Whenever $\{A_n\}$ is a decreasing sequence of closed nonempty subsets of X with $\lim_n \beta(A_n) = 0$ then $\cap A_n \neq \emptyset$ (here $\beta(A) \doteq \inf\{d(a, X \setminus \{a\}) : a \in A\}$) [3].

Further several characterization of Atsuji spaces can be found in [1]-[13]. In particular, on this argument, Levine [7] shown the following

THEOREM 1. *Let $f_n : X_1 \to X_2$ be continuous for $n = 1, 2, \ldots$ where X_1 and X_2 are metric spaces with metric d_1 and d_2 respectively. Then, there exist metrics d_1^* and d_2^* for X_1 and X_2 respectively which are equivalent to d_1 and d_2 such that every f_n is uniformly continuous relative to d_1^* and d_2^*.*

Actually, the proof of the above theorem yields much more: the functions f_n are lipschitzian relative to d_1^* and d_2^*. So, it seems natural to ask if all continuous functions from X_1 to X_2 are lipschitzian relative to two suitable equivalent metrics of X_1 and X_2.

The following example shows that the answer is negative, also under the strong hypothesis of compactness of X_1 and X_2.

EXAMPLE 2. Consider $X_1 = X_2 = [0,1]$ and let d_1 and d_2 be two metrics equivalent to euclidean metric on $[0,1]$. We go to construct a continuous function
$f : ([0,1], d_1) \to ([0,1], d_2)$ not lipschitzian.

Let $x_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that

$$d_1\left(x_n, \frac{1}{n+1}\right) < \frac{1}{n} - \frac{1}{n+1}.$$

A such x_n certainly exists since $\frac{1}{n+1} + \frac{1}{k}$ approach $\frac{1}{n+1}$ for $k \to \infty$ with respect to the euclidean metric and so with respect to the equivalent metric d_1 also. Now, for any $n \in \mathbb{N}$, let $y_n \in [0,1]$ such that

$$\frac{1}{2}x_n < y_n < x_n, \quad \frac{1}{2}x_n < d_2(y_n, 0) < x_n.$$

A such y_n certainly exists since $\frac{3}{4}x_n + \frac{1}{k} \in (\frac{1}{2}x_n, x_n)$ for sufficiently large k and $\frac{3}{4}x_n + \frac{1}{k}$ approach $\frac{3}{4}x_n$ for $k \to \infty$ with respect to the euclidean metric and so with respect to the equivalent metric d_2 also. We define $f : [0,1] \to [0,1]$ by

$$f(x) \doteq \begin{cases} 0 & \text{if } \ x = 0 \\ y_n \dfrac{(n+1)x - 1}{(n+1)x_n - 1} & \text{if } \ \dfrac{1}{n+1} \le x \le x_n \\ y_n \dfrac{nx - 1}{nx_n - 1} & \text{if } \ x_n \le x \le \frac{1}{n} \end{cases}$$

Of course the function f is continuous relative to d_1 and d_2 on $[0,1]$. But

$$\frac{d_2\left(f(x_n), f\left(\frac{1}{n+1}\right)\right)}{d_1\left(x_n, \frac{1}{n+1}\right)} > \frac{d_2(y_n, 0)}{\frac{1}{n} - \frac{1}{n+1}} > \frac{n(n+1)}{2}x_n > \frac{n}{2}$$

in such a way that the function f is not lipschitzian.

The behavior explained in the above example is not unexpected. Indeed, now we prove that each metric space X fails to have the property that all real valued (continuous or uniformly continuous) functions on X are lipschitzian, unless X has a finite number of points only.

THEOREM 3. *Let (X, d) be a metric space. Are equivalent:*

(a) *Each real valued continuous function on X is lipschitzian*

(b) *Each real valued uniformly continuous function on X is lipschitzian*

(c) *Each real valued function on X is lipschitzian*

(d) *X is a finite set.*

Proof. We need to prove only (b) $\Rightarrow$(c)$\Rightarrow$(d).

Proof of (b) $\Rightarrow$(c)

STEP 1. The set X' of all accumulation points of X is empty.

Suppose, on the contrary, that there exists an $x_0 \in X'$. Then the function

$$\phi(x) \doteq \sqrt{d(x, x_0)}$$

is uniformly continuous but not lipschitzian.

STEP 2. Each real valued function on X is uniformly continuous.

Suppose, on the contrary, that there is a function $f : X \to \mathbb{R}$ that is not uniformly continuous. Then we construct a function $g : X \to \mathbb{R}$ that is uniformly continuous but not lipschitzian. Since f is not uniformly continuous, there exist an $\varepsilon_0 > 0$ and two sequences $\{x_n\}$, $\{y_n\}$ in X for which $d(x_n, y_n) < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon_0$ (so $x_n \neq y_n$).

Consider the function $g : X \to \mathbb{R}$ defined by

$$g(x) \doteq \begin{cases} 0 & \text{if } \ x \neq x_n \\ \sqrt{d(x_n, y_n)} & \text{if } \ x = x_n \end{cases}$$

Then g is not lipschitzian, of course, but it is uniformly continuous. Indeed, let $\varepsilon > 0$ fixed. We choose $n_\varepsilon > \frac{1}{\varepsilon^2}$ and $\delta_\varepsilon > 0$ in such a way that

$$\delta_\varepsilon < \min_{1 \leq j \leq n_\varepsilon} I(x_j).$$

A such δ_ε certainly exists by STEP 1. One can verify easily that $d(x, y) < \delta_\varepsilon \Rightarrow |g(x) - g(y)| < \varepsilon$.

Proof of (c) $\Rightarrow$(d)

Suppose, on the contrary, that X is an infinite set and let $\{x_n\} \subset X$ be a sequence of distinct elements of X. Then the function $h : X \to \mathbb{R}$ defined by

$$h(x) \doteq \begin{cases} 0 & \text{if } \ x \neq x_n \\ nd(x_n, x_{n+1}) & \text{if } \ x = x_n \ (n \text{ even}) \\ 0 & \text{if } \ x = x_n \ (n \text{ odd}) \end{cases}$$

is not lipschitzian, of course. ∎

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