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Local Algebras and the Largest Spectrum Finite Ideal †

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1. Introduction

M.R.F. Smyth proved in [9, Theorem 3.2] that the socle of a semiprimitive associative Banach complex algebra coincides with the largest algebraic ideal. Later M. Benslimane, A. Kaidi and O. Jaa showed [3] the equality between the socle and the largest spectrum finite ideal in semiprimitive alternative Banach complex algebras. In fact, they showed that every spectrum finite one-sided ideal of a semiprimitive alternative Banach complex algebras is contained in the socle.

In this note it is given a new proof of this last result by using the notion of local algebra attached to an element of an (associative, alternative or Jordan) algebra. Only the associative case will be considered here since there is no essential difference between the associative and alternative cases.

This local approach is inspired by the method followed by O. Loos [7] to solve the corresponding problem for Jordan Banach pairs, although this author deals with the more general notion of subquotient.

2. Preliminaries

Throughout this section and unless otherwise stated, A will denote an associative algebra (not necessarily with a unit element) over an arbitrary ring of scalars Φ.

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Every associative algebra A gives rise to a *quadratic Jordan algebra* A^+ over the same linear structure as A and with quadratic operations $x \mapsto x^2$ and $x \mapsto U_x$, where $U_x y = xyx$ is the usual Jordan operator. This allows us to apply techniques of the theory of Jordan algebras to the study of associative algebras, which proves to be very useful when dealing with symmetrical notions, that is, notions which remain invariant under passing to the opposite algebra, and therefore can be stated in Jordan terms. Invertibility, semiprimeness, primeness, von Neumann regularity and the Jacobson radical are examples of symmetrical notions, as well as the notion of socle of a semiprime associative algebra.

For $a \in A$, the *a-homotope* of A, denoted by $A^{(a)}$, is the associative algebra over the same linear structure as A with the new product $x \cdot_a y = xay$.

It is well known that the Jacobson radical, $Rad(A)$, of an associative algebra A consists precisely of the *properly quasi-invertible* (p.q.i.) elements, those $z \in A$ for which all az (equivalently all za) are quasi-invertible (q.i.) in the sense that $1 - az$ is invertible in the unital hull A' of A, obtained by adjoining a unit element to A if A is not unital. McCrimmon proved [8, Proposition 1] the following result whose proof is here included for completeness.

LEMMA. *The element az (za) is q.i. in the associative algebra A if and only if z is q.i. in the homotope $A^{(a)}$. Thus z is p.q.i. if and only if it is q.i in all homotopes $A^{(a)}$.*

Proof. First we note (see [4, Proposition 6, p.16]) that az is q.i. if and only if so is za. Now, if z is q.i. in $A^{(a)}$, there exists $v \in A$ such that $z + v = zav = vaz$. Multiply these equalities on the left by a to obtain $az + av = (az)(av) = (av)(az)$, which proves that az is q.i. in A. Suppose now that za and therefore az as well are q.i. in A. Then the left multiplication operator $L_{1-za} : A' \to A'$ and the right multiplication operator $R_{1-az} : A' \to A'$ are invertible. Since A is a two-sided ideal of A', their restrictions to A are also invertible. In particular, there exist v and w in A such that $L_{1-za}v = v - zav = -z$ and $R_{1-az}w = w - waz = -z$. Hence $v = w$ and z is q.i. in $A^{(a)}$. ∎

Recall that for a semiprime associative algebra A, the left socle of A coincides with the right socle. This ideal is simply called the *socle* of A and will be denoted by $Soc(A)$. Looking at the Jordan structure of A, a Φ-submodule I of A is said to be an *inner ideal* of A if $xA'x \subset I$ for every $x \in I$. By [5, 2.6(i)], $Soc(A)$ is the sum of all minimal inner ideals of A, where minimal is taken with respect to the set of all nonzero inner ideals of A. It is easily seen.

(1) *If I is a minimal inner ideal of A and $x \in A$, then either $xIx = 0$ or xIx is a minimal inner ideal of A.*

3. THE LOCAL ALGEBRAS OF AN ASSOCIATIVE ALGEBRA

Let A be an associative algebra and a and element of A. It is readily seen that the set $Ker(a) = Ker(U_a)$, where $U_a \colon A \to A$ is the linear operator defined by $x \mapsto axa$, is an ideal of the a-homotope $A^{(a)}$, so we can consider the quotient algebra $A^{(a)}/Ker(a)$, called the *local algebra* of A at a, and which will be denoted by A_a. We will write $\overline{x} \in A_a$ to denote the coset of an element $x \in A$.

An associative algebra A is called *semiprimitive* if the Jacobson radical $Rad(A) = 0$

PROPOSITION 1. *Given $a \in A$, the mapping $x \mapsto axa$ from $A^{(a)}$ to A induces a monomorphism of Φ-modules $\varphi : A_a \to A$ such that:*

(i) *$\varphi(Rad(A_a)) = Rad(A) \cap aAa$. Hence, if A is semiprimitive, then all its local algebras are also semiprimitive.*

(ii) *The mapping $\overline{I} \mapsto \varphi(\overline{I})$ is an isomorphism from the lattice of inner ideals of A_a onto the lattice of inner ideals of A contained in aAa.*

Proof. (i) By the lemma, we just need to show that $\overline{x}$ is q.i. in $(A_a)^{\overline{y}}$ if and only if axa is q.i. in $A^{(y)}$, for all $y \in A$; but this follows from the following of equivalence:

$$\overline{x} \cdot_a \overline{y} \cdot_a \overline{z} = \overline{x} + \overline{z} = \overline{z} \cdot_a \overline{y} \cdot_a \overline{x} \Longleftrightarrow (axa)y(aza) = axa + aza = (aza)y(axa)$$

where $z \in A$.

(ii) Inner ideals of A_a are of the form $\overline{I}$ for an inner ideal I of $A^{(a)}$ such that $I \supset Ker(a)$, and clearly, for $x \in I$,

$$(axa)A(axa) = a(xaAax)a = a(U_x^{(a)} A^{(a)})a \subset aIa,$$

where $U_x^{(a)}$ denotes the U_x-operator in the a-homotope $A^{(a)}$. This proves that $aIa = \varphi(\overline{I})$ is an inner ideal of A, clearly contained in aAa.

Conversely, let K be an inner ideal of A contained in aAa. Then the set $I = \{x \in A : axa \in K\}$ is an inner ideal of $A^{(a)}$ containing $Ker(a)$. Indeed, $a(U_x^{(a)} A^{(a)})a = a(xaAax)a = (axa)A(axa) \subset K$ because K is an inner ideal of A, which completes the proof. ∎

Remark. Note that if A is semiprime the converse of (i) is also true: If all the local algebras of A are semiprimitive, then A is semiprimitive.

Recall that $a \in A$ is said to be von Neumann regular if $a = aba$ for some $b \in A$. Note that if a is von Neumann regular with $a = aba$, then the mapping $x \mapsto axa$ defines an homomorphism of $A^{(a)}$ into $A^{(b)}$ whose kernel is precisely $Ker(a)$, so this mapping induces an isomorphism of A_a onto aAa regarded as a subalgebra of $A^{(b)}$. In particular, if $e = e^2$ is an idempotent, then A_e is isomorphic to eAe. (Note that eAe is a subalgebra of A as well as of $A^{(e)}$).

PROPOSITION 2. *Let A be an associative algebra and let $a \in A$ be von Neumann regular with $a = aba$. Then $\overline{b}$ is a unit element of A_a.*

If A is semiprime, the converse is also true: If A_a is unital with a unit element $\overline{b}$, then $a = aba$.

Proof. Suppose first $a = aba$. Then, for every $x \in A$, we have $a(xab-x)a = ax(aba)-axa = axa-axa = 0$, so $\overline{x}\cdot_a\overline{b} = \overline{x}$ for any $\overline{x} \in A_a$. Similarly, $\overline{b}\cdot_a\overline{x} = \overline{x}$, which proves that A_a is unital with a unit element $\overline{b}$.

Assume now that A is semiprime and that A_a is unital, and let $b \in A$ be such that $\overline{b}$ is a unit element of A_a. We claim that $a = aba$. Indeed, for any $x \in A$, we have

$$(a - aba)x(a - aba) = a(x - bax)(a - aba) = a(x - bax - xab + baxab)a = 0$$

since $\overline{b}$ is a unit element of A_a. Now $(a - aba)A(a - aba) = 0$ implies $a = aba$ by semiprimeness of A. ∎

The following result is a local characterization of the elements of the socle.

PROPOSITION 3. *Let A be a semiprime associative algebra. Then all the local algebras A_a of A are also semiprime. Moreover, $a \in A$ is in the socle if and only if A_a is semisimple in the classical sense, equivalently, A_a is unital and coincides with its socle.*

Proof. If $\overline{x}\cdot_a A_a \cdot_a \overline{x} = 0$ then $(axa)A(axa) = a(xaAax)a = 0$, which implies $axa = 0$ by semiprimeness of A, so $\overline{x} = 0$ which proves that A_a is semiprime.

Suppose now that A_a is semisimple, equivalently, A_a is unital and coincides with its socle. If $\overline{b}$ is a unit element of A_a, then $\overline{b} = \overline{x}_1 + \ldots + \overline{x}_n$ where, for each $1 \leq i \leq n$, $\overline{x}_i \in \overline{I}_i$ with $\overline{I}_i$ a minimal inner ideal of A_a. Then, by Proposition 2,

$$a = aba = ax_1a + \ldots + ax_na \in \varphi(\overline{I}_1) + \ldots + \varphi(\overline{I}_n),$$

where for each $1 \leq i \leq n$, $\varphi(\overline{I}_i)$ is a minimal inner ideal of A by Proposition 1(ii). Thus, $a \in Soc(A)$.

To the converse, let $a \in Soc(A)$. By using the fact that every element in the socle generates a principal right ideal determined by an idempotent [2, F.1.7], it is easily seen that every element in the socle is von Neumann regular. Hence $a = aba$, where without loss of generality, we may assume that $b \in Soc(A)$. Then we can write $a = aba = ax_1a + \ldots + ax_na$, where $x_i \in I_i$ for some minimal inner ideal I_i of A. Hence, by Proposition 1(ii) and (1), $\overline{b} = \overline{x}_1 + \ldots + \overline{x}_n \in Soc(A_a)$, but by Proposition 2, $\overline{b}$ is a unit element of A_a. Therefore, A_a is unital and coincides with its socle, equivalently, A_a is semisimple, as required. ∎

4. The largest spectrum finite ideal in Banach algebras

Let A be an associative algebra over a field F. The *spectrum* of an element $a \in A$ is the set $Sp(a, A) \subset F$ defined as follows:

$$Sp(a, A) \cup \{0\} = \{\lambda \in F^* \colon \lambda^{-1}a \text{ is not q.i}\} \cup \{0\}$$

with $0 \notin Sp(a, A)$ if and only if A is unital and a is invertible in A.

A direct consequence of the lemma is the following result concerning the spectrum.

$$Sp(x, A^{(a)}) \cup \{0\} = Sp(xa, A) \cup \{0\} = Sp(ax, A) \cup \{0\}. \tag{2}$$

By a normed algebra we will mean a (real or complex) associative algebra A endowed with a norm making continuous the product of A. If this norm is complete, then we will say that A is a Banach algebra.

PROPOSITION 4. *Let A be a normed (Banach) algebra. Then every local algebra A_a of A is also a normed (Banach) algebra.*

Proof. It is clear that the homotope $A^{(a)}$, with the same norm as A, is a normed (Banach) algebra. Moreover, $Ker(a)$ is a closed ideal of $A^{(a)}$. Hence, for the quotient norm, the local algebra A_a is also a normed (Banach) algebra. ∎

In the case of a normed algebra, it is possible to refine the characterization of socle elements given in Proposition 3. This is a well known result in the complex case (see, for instance, [9, 3.1(i)]), which is here included for completeness.

PROPOSITION 5. *Let A be a semiprime normed algebra. Then $x \in A$ is in the socle if and only if xAx is finite dimensional.*

Proof. By Proposition 3, $x \in Soc(A)$ if and only if A_x is classically semisimple, but A_x is a normed algebra (Proposition 4), and a normed algebra is classically semisimple if and only if it is finite dimensional (apply Wedderburn-Artin theorem [6, p.203] together with the structure of normed division algebras [4, 14. Theorems 2 and 7]). Finally, by Proposition 1, xAx is finite dimensional if and only if so is A_x. ∎

As a direct consequence of the above proposition, we have that the socle of a semiprime normed algebra A is *algebraic* in the sense that, for every $a \in A$, there exists a nonzero polynomial $p(x)$ such that $p(a) = 0$. Hence $Soc(A)$ has finite spectrum [4, Prop. 5, p. 21].

THEOREM. *Let A be a semiprimitive (complex) Banach algebra and let L be a left ideal of A whose elements have finite spectrum. Then L is contained in the socle of A.*

Proof. Given $a \in L$, it follows from (2) that $Sp(x, A^{(a)})$ is finite for every $x \in A$, and hence it is clear that each element of A_a has also finite spectrum, but A_a is a semiprimitive Banach algebra (Propositions 1(i) and 4). Then, by [1, 5.4.2], A_a has finite dimension and hence aAa is finite dimensional by Proposition 1. Therefore, $a \in Soc(A)$ by Proposition 5. ∎

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