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Quasinormable Spaces of Polynomials *

José M. Ansemil

Dpto. de Análisis Matemático, Fac. de C. Matemáticas,
Univ. Complutense, 28040-*Madrid, Spain*

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Let us start by considering a real or complex Banach space X and the space of n-homogeneous continuous polynomials on X. This space, which will be represented by $\mathcal{P}(^nX)$, is clearly a Banach space when we endowed it with the norm

$$\|P\| = \sup\{|P(x)|: \|x\| \leq 1\}, \quad P \in \mathcal{P}(^nX).$$

The topology on $\mathcal{P}(^n\mathrm{X})$ induced by this norm is, of course, a natural topology, but there are others, as the compact open topology τ_o. With this topology $\mathcal{P}(^nX)$ is not normable if X has infinite dimension ($(\mathcal{P}(^nX), \tau_o)$ is an infinite dimensional Montel space (see [7])), but at least it is quasinormable [12, 8].

The class of quasinormable spaces includes the class of normable spaces. It has been introduced by Grothendieck in 1954 [10]. It consists of the locally convex spaces E with the property that for every neighborhood U of 0 in E there exists another neighborhood V of 0 in E such that for every $\alpha > 0$ there is a bounded subset B in E with $V \subset \alpha U + B$.

It is clear that every normable space is quasinormable, but not every metrizable space is quasinormable. There is a classical example by Grothendieck-Köthe of a non quasinormable Fréchet-Montel space. The quasinormability of $(\mathcal{P}(^nX), \tau_o)$ is not based on the normability of the space X. It is also present when the space is a real or complex Fréchet space [12, 8].

In the context of Fréchet spaces E there are, apart from the compact open topology τ_o, other natural topologies on $\mathcal{P}(^nE)$ as the topology τ_b of uniform convergence on the bounded subsets of E, the strong topology β as a dual of the space of n-symmetric tensors ($n \geq 2$) endowed with the projective topology [9] and finally the Nachbin ported topology. This topology is generated by the

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family of seminorms p on $\mathcal{P}(^nE)$ that satisfy that for every open neighbourhood V of 0 in E there exists a positive constant C such that

$$p(P) \leq C \sup\{|P(x)| : x \in V\} \text{ for all } P \text{ in } \mathcal{P}(^nE).$$

We have $\tau_o \leq \tau_b \leq \beta \leq \tau_\omega$ on $\mathcal{P}(^nE)$. If E is a normed space, $\tau_o < \tau_b = \beta = \tau_\omega$ and this topology agrees with the norm on $\mathcal{P}(^nE)$ [7].

As we have already mentioned $(\mathcal{P}(^nE), \tau_o)$ is a quasinormable space. With τ_ω, $\mathcal{P}(^nE)$ is also quasinormable, because it can be seen directly from the definition of τ_ω that $(\mathcal{P}(^nE), \tau_\omega)$ is an inductive limit of a sequence of normed spaces and these spaces are quasinormable [10]. What happens with τ_b and β? Concerning β we can say that $(\mathcal{P}(^nE), \beta)$ is always quasinormable. This follows from the fact that this space is the strong dual of a Fréchet space (the completion of the symmetric projective tensor product of E by itself n times), and hence it is a DF space, so it is quasinormable.

For τ_b the question is more complicated. There are examples by Peris [13] of Fréchet spaces E such that $(\mathcal{P}(^nE), \tau_b)$ is not quasinormable. Nevertheless, it is quasinormable in several cases. Let us mention some examples:

-The Fréchet-Montel spaces (because for them $\tau_b = \tau_o$ on $\mathcal{P}(^nE)$).

-The spaces with the property $(BB)_{n,s}$ [1] (which means that every bounded subset of the projective symmetric n-tensor product of E is contained in the absolutely convex closed hull of a symmetric tensor product of n copies of a bounded subset of E). With this property $\tau_b = \beta$ on $\mathcal{P}(^nE)$.

-The spaces with a T-Schauder decomposition and the density condition. In this situation it happens that $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$ [9].

In spite of what we have seen, the quasinormability of $(\mathcal{P}(^nE), \tau_b)$ is not conditioned to the fact that the topology of uniform convergence on bounded subsets agrees with the compact-open, the strong or the Nachbin ported topologies. In what follows we are going to present an example of a Fréchet space E such that $(\mathcal{P}(^nE), \tau_b)$ is quasinormable but all the topologies we are considering on $\mathcal{P}(^nE)$ are different. The result is in a joint paper with Fernando Blasco and Socorro Ponte [2].

To give the example we will consider the Banach space ℓ_1 and a particular Fréchet space, we take the projective tensor product of both spaces and then we consider its completion. This will be our example. Let me explain which Fréchet space we are going to consider.

From Grothendieck thesis (1955) [11], it was open the question of knowing if the completion of the projective tensor product of two Fréchet-Montel spaces is also a Montel space. Now we know that this is not true and the counterexample was given by Taskinen in 1986 [14] who built a Fréchet-Montel space, which we are going to denote by T, such that $T\hat{\otimes}_\pi T$ (the completion or the proyective tensor product) contains a complemented copy of ℓ_1 and hence it is not a Montel space. This counterexample has been modified by Taskinen [15] to get, from a given Köthe sequence space, λ_1, a Fréchet-Montel space (which we will also denote by T) such that $T\hat{\otimes}_\pi T$ contains a complemented copy of the given λ_1. Even this complemented copy of λ_1 is inside $T\hat{\otimes}_{s,\pi}T$ (the completion of the symmetric proyective tensor product). Our counterexample will be $E = \ell_1\hat{\otimes}_\pi T$.

THEOREM. *For every $n \geq 2$ the space $(\mathcal{P}(^nE), \tau)$ (with $E = \ell_1\hat{\otimes}_\pi T$), is quasinormable for $\tau = \tau_o, \tau_b, \beta$ and τ_ω, and $\tau_o < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$.*

Sketch of the proof: We already know that $(\mathcal{P}(^nE), \tau)$ is quasinormable for $\tau = \tau_o, \beta$ and τ_ω. Let us see how it is also quasinormable for τ_b. The proof will be a consequence of the following,

LEMMA. *For $E = \ell_1\hat{\otimes}_\pi T$ we have $(\mathcal{L}(^nE), \tau_b) \cong \ell_\infty([T\hat{\otimes}_\pi \overset{n}{\overbrace{\cdots}} \hat{\otimes}_\pi T]', \tau_o)$.*

($\mathcal{L}(^nE)$ stands for the space of n-linear continuous functions on E and τ_b for the topology of uniform convergence on the bounded subsets of $E\times\cdots\times E$). The result is also true with any Fréchet-Montel space F instead of T.

From the Lemma the quasinormability of $(\mathcal{P}(^nE), \tau_b)$ follows in this way:

The space $(\mathcal{L}(^nE), \tau_b)$ is isomorphic to $\ell_\infty(G', \tau_o)$ where G is a Fréchet space and this space is quasinormable [6]. Then, since $(\mathcal{P}(^nE), \tau_b)$ is a complemented subspace of $(\mathcal{L}(^nE), \tau_b)$ [7], it is quasinormable.

Let us see how $\tau_o < \tau_b < \beta < \tau_\omega$ on $\mathcal{P}(^nE)$ for $n \geq 2$. Indeed, if τ_o agrees with τ_b, then E has to be a Montel space [7], what is not possible because $E = \ell_1\hat{\otimes}_\pi T$ has ℓ_1 as a complemented subspace.

If τ_b agrees with β then the space E has the property $(BB)_{n,s}$ [9] and hence $(BB)_{2,s}$ [4], but then T would also have this property [4] (T is complemented in E) but, as we have mentioned, Taskinen's space fails to have this property (if it had it $T\hat{\otimes}_{s,\pi}T$ would be a Montel space what is impossible because it contains, as a complemented subspace, a non distinguished λ_1 space).

Finally, if β agrees with τ_ω then $E\hat{\otimes}_{s,\pi}E$ is distinguished ([7] and [4]). This would imply that $T\hat{\otimes}_{s,\pi}T$ is a also distinguished what is impossible because we have chosen T such that $T\hat{\otimes}_{s,\pi}T$ contains, as a complemented subspace, a non distinguished λ_1.

We note that the results in the above theorem are obtained for $n \geq 2$. What happens for $n = 1$? Of course $(\mathcal{P}(^1E), \tau_b)$, that is exactly the strong dual of E, is quasinormable, $\tau_o \neq \tau_b$ (since E is not a Montel space), nothing to say about the possibility of the equality between τ_b and β (β only has sense for $n \geq 2$), but $\tau_b = \tau_\omega$ because $E = \ell_1 \hat{\otimes}_\pi T$ is a distinguished space [5] (τ_ω is always the barrelled topology associated with τ_b [7]).

Nevertheless, our example can be modified to get the above theorem also for $n = 1$. The details of this can be seen in Fernando Blasco's thesis [3].

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