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Geometrical Modeling of Material Aging†

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I. INTRODUCTION

Material aging is understood as changes of material properties with time. The aging is usually observed as an improvement of some properties and a deterioration of others. For example an increase of rigidity and strength and reduction in toughness with time are commonly observed in engineering materials ([1],[6]). In an attempt to model aging phenomena on a continuum (macroscopical) level one faces three major tasks. The first is to identify an adequate **age parameter** that represents, on a macroscopic scale, the micro and submicroscopical features, underlying the aging phenomena such as nucleation, growth and coalescence of microdefects, physico-chemical transformations etc. The age parameter should be considered as a parameter of state, in addition to the conventional parameters such as stress tensor and temperature.

The second task consists of formulation of a constitutive equation of aging, i.e., equations of age parameter evolution expressed in terms of controlling factors, e.g., load and temperature. It is expected that at common circumstenses a small variation of controlling factors results in a small variation of age parameter. However, at certain conditions, a sudden large variation of age parameter may result from a small perturbation of controlling factors. Experimental examination, classification and analysis of the condition that lead to such a catastrophic behavior, constitute the third task of the modeling. Formulation of local failure criteria within the scope of continuum mechanics is an example of this task.

In many engeneering materials the aging is manifested in variations of mass density as well as in the spectrum of relaxation time. Thus in a macroscopic

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test the aging can be detected in variations of **intrinsic** (material) **length** and **time scales**. Following this notion, in the present paper we employ the **material metric tensor** G as an **age parameter**. An evolution of G in **4D -material space-time** determines in our approach an inelastic behaviour and time dependent material properties recorded by an **external** observer.

The objective of the present work is to derive the constitutive equations of aging based on Extremal Action Principle. The variational approach seems to be most promising in view of complexity of the problem and lack of experimental data. It provides with a guide line for the experimental examination of the basic assumptions and modifications, if necessary.

The major task in implementation of extremal action principle is the construction of an appropriate Lagrangian. In variational formulations of Elasticity theory the Lagrangian is usually constructed in terms of invariants of the gradient of deformation. In Classical Field Theories invariants of metric, connection and the corresponding curvatures together with the gradients of "material fields" are emploied in the Lagrangian ([9],[11],[14]). In the present paper we combine the above approaches and revisit the classical continuum mechanics from the point of view of an intrinsic (material) geometry that includes an inner time.

In Sec. II we discuss the kinematics of an aging media emploing a 4-dimentional material space-time $P = \mathbb{R} \times B$ endowed with the 4D-metric G of Lorentz type (intrinsic metric) embedded into 4-dimentional Absolute (Newton's) space-time M^4. We define the mass form and formulate the mass conservation law that, in the context, gives a non-trivial relation between the "reference density" ρ_0 and the time evolution of the material metric G. A strain tensor E^{el} and a ground "state" are introduced as a measure of deformation and a natural analog of the "unstrained state" respectively. The central part of the work is the Sec. III where we propose a variational formulation of aging theory. The equation of Elasticity together with the generalized Hooke's equation are conventionaly derived considering the variation of the action integral with respect to the deformations ϕ^i. Similarly, new equations of evolution of the age parameter (and, therefore, of elastic moduli, mass density and inelastic deformation) result from the variation of the action integral with respect to the material metric tensor. The balance equations (conservation laws) resulting from the symmetries of the Absolute (Newton's) space-time and material (intrinsic) space-time respectively and the relations between them are discussed in the Sec. IV. Considerations of the paper are illustrated in Sec. V by the example - linearized model of aging of a rod whose time dependent elastic properties and irreversible deformation are associated with an evolving metric in 2D material space-time. More detailed exposition of the results presented here will be published in the article [2].

II. Kinematics of Aging Media

Material body is considered here, in a conventional way, as a 3D manifold B, i.e. a set of "idealized" material points (with the coordinates $X^I, I = 1,2,3$). Cylinder $P = \mathbb{R} \times B$ (with the coordinates $(X^0 = T, X^I, I = 1,2,3)$) equiped with the Lorentz type "intrinsic" ("material") metric tensor G with components G_{IJ} is refered to a material "space-time" (P, G). We require that all the sections $B_T = \{T = const\}$ are space-like, while the material "world lines" $\{\mathbb{R} \times (X^I, I = 1,2,3)\}$ are time-like with respect to the metric G.

Metric G defines the 4D volume element $dV = \sqrt{-|G|}d^4X$, we denote the determinant of the matrix (G_{IJ}) by the symbol $|G|$.

History of deformation of the body B is represented by a diffeomorphic embedding $\phi : P \to M$ of the material space-time P into the Minkowski space $M = \mathbb{R}^4$ (with the coordinates $(t = x^0, x^i, i = 1,2,3)$), equiped with the 3D Eucledian space metric h with components δ_{ij}. Later we restrict ϕ by requiring $t = \phi^0(X) = T$. Such deformations are called "sinchronized".

Using the deformation ϕ, we define the slicing of P by the level surfaces of the zeroth component of ϕ

$$B_{\phi,t} = \phi^{0\ -1}(t) = \{(T,X) \in P | \phi^0(T,X) = t\}. \tag{2.1}$$

For the sinchronized deformation $B_{\phi,t} = B_T$, therefore these surfaces are **spacelike** (see above). We assume the same to be true in the general case.

There is a "flow vector field" u_ϕ in P associated with the slicing $B_{\phi,t}$. It is the only time directed vector field orthogonal to the slices $B_{\phi,t}$ for all values of t and $< u_\phi, u_\phi >= -1$ (see [5],[14]). $u_\phi = [-G_{00}]^{-\frac{1}{2}}\frac{\partial}{\partial T}$ for sinchronized deformation ϕ and the block-diagonal metric G in coordinates $T, X^I, I = 1,2,3$.

In addition to the volume element, the **mass form** $dM = \rho_0 dV$ is defined in P. The reference mass density ρ_0, defined by this representation satisfies the mass conservation law $\mathcal{L}_{u_\phi} dM = 0$, where $\mathcal{L}_{u_\phi}$ is the substantial (Lie) derivative in the direction of the field u. In the sinchronized case the mass conservation law is equivalent to the following representation of the reference density:

$$\rho_0(T, X^I) = \rho_0(0, X)\sqrt{\frac{G_{00}}{|G|}}, \tag{2.2}$$

where $\rho_0(0, X)$ is the initial values of ρ_0 (we assume that $G(0, X)$ is the Minkowski metric). Space density ρ is defined, as usual, by the condition $\phi^*(\rho dv) = \rho_0 dV$, that gives $\rho \circ \phi = \frac{\rho_0\sqrt{-G}}{J(\phi)}$, where $J(\phi)$ is the Jacobian of the deformation ϕ.

Slicing $B_{\phi\ ,t}$ defines the covariant tensor $\gamma = G + u_\phi \otimes u_\phi$ (see [9],[14]). Denote by Π the orthonormal projector $\Pi = G^{-1}\gamma$ to the planes tangent to the slices $B_{\phi\,t}$. Tensor γ induces the time dependent 3D-metric g_t on the slices $B_{\phi\ ,t}$ (see [5]). In the synchronized case and the block-diagonal metric G, g_t is just the restriction of 4D-metric G to the slices B_T. We do not put any futher condition on the metric g_t. In particular, it may have non-zero curvature (i.e. incompatibility of deformation). Apparently there are residue stresses associated with this curvature.

We also introduce the 4D tensor $K_4 = G^{-1}C_4(\phi) - u_{\dot\phi} \otimes u_{.\ \phi}$, where $C_4(\phi) = \phi^* h$, and define 3D-elastic covariant strain tensor E^{el} as follows

$$E(\phi)^{el} = \frac{1}{2}\Pi ln(K)\Pi = \frac{1}{2}\Pi ln(G^{-1}\phi^* h - u_{\dot\phi} \otimes u_{.\ \phi})\Pi. \tag{2.3}$$

Then, 3D elastic strain tensor $E(\phi)^{el}$ results from the restrictions of tensors G^{-1} and $\phi^* h$ to the slices $B_{\phi,t}$. It is a natural measure of a deviation of the actual state from the "ground state" $\bar\phi$, $\bar\phi^* h = g_t$. For the synchronized deformation ϕ and the block-diagonal metric G, $E^{el} = \frac{1}{2}ln(g_t^{-1}C(\phi))$ i.e conventional logarithmic measure of deformation.

The total deformation E^{tot} of the body at each given moment T that measures the deviation of the deformed Eucledian metric $\phi^* h|_{B_{\phi,t}}$ from the initial (Eucledian) 3D-metric h on $B_{\phi,t}$ translated there from B, $E^{tot} = \frac{1}{2}ln(h^{-1}C(\phi))$ in important practical cases can be represented as the sum of the elastic deformation $E^{el} = \frac{1}{2}ln(g_t^{-1}C(\phi))$ and an irreversible deformation $E^{ir} = \frac{1}{2}ln(h^{-1}g_t)$ (the logarithm of (1,1)-tensors is taken on the slices $B_{\phi\ ,t}$):

$$E^{tot} = E^{el} + E^{ir}. \tag{2.4}$$

The following diagram presents the above decomposition: The actual state under the load at any given moment T results from both elastic (with the variable elastic moduli) and inelastic (irreversible) deformations. The "ground state" of the body is characterized by the 3D-metric g_t. This state is the background to which the elastic deformation is added to reach the actual state.

Transition from the reference state to the "ground state" that manifests in the evolution of the (initial) Eucledian metric h to the metric g_t can not

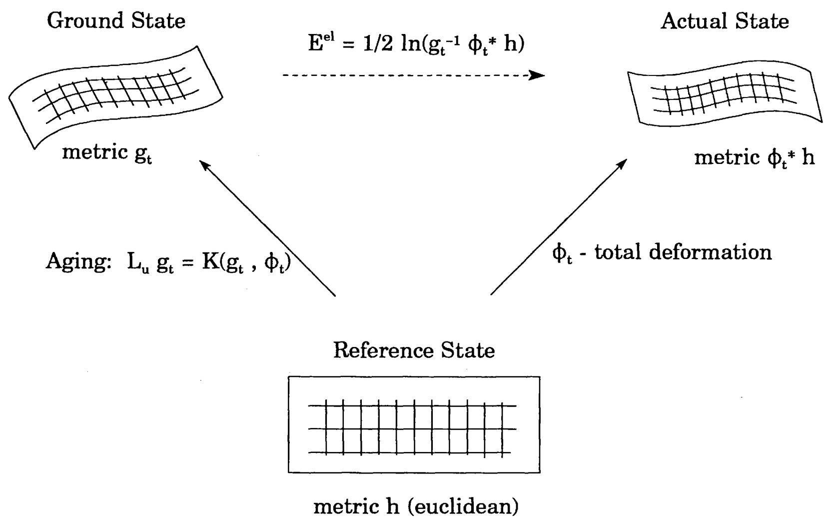

Figure 1. Geometrical modeling of physical systems

be described, in general, by any point transformation. Transition from the "ground state" to the actual state at the moment t also is not compatible in this sense. Yet the transition from the reference state to the actual state is represented by the diffeomorphism ϕ_t. Here we consider the **material 4D-metric** G and the **deformation** ϕ (or elastic strain tensor $E^{el}(\phi)$) to be the dynamical variables of the theory. Reference density ρ_0 is found by the formula given above if its initial value $\rho_0(T=0)$ is known.

III. Variational Formulation of Equations of Aging

Variational principle Following the framework of the classical field theory we take Lagrangian density $\mathcal{L}(G,E)$ refered to the **volume form** $dV = \sqrt{-|G|}d^4X$ as a density that depends on the dynamic variables of our model i.e. 4D-material metric G and the deformation history ϕ: $\mathcal{L}(G,E^{el}) = L(G,E^{el})dV$, with $L(G,E^{el})$ being the Lagrangian.

Deformation ϕ is 3-dimensional in the sense that it reflects only spacial part h of the metric in M and, the 4D-tensor $C_4(\phi) = \phi^*h$ is the degenerate metric in P. We compare it with the tensor $\gamma = G + u_\phi \otimes u_\phi$. Elastic deformation E^{el} measures the deviation of $C(\phi)$ from γ on the slices $B_{\phi,t}$.

Coincidence of $\gamma|_{B_{\phi,t}}$ and $C(\phi)$ is possible if there is no elastic deformation and if deformation ϕ describes just an evolution of the metric g_t.

Based on these arguments we present the Lagrangian $L(G,E)$ as the sum of the "ground state" Lagrangian $L_m(G)$ that depends on the metric G only and the elastic part $L_e(G,E)$ that is a perturbation of the metric part due to the elastic deformation:

$$L = L_m(G) + L_e(E^{el}) \tag{3.1}$$

In a quasistatic theory we ignore the kinetic energy and the last term is simply related to the elastic strain energy f that is assumed to be a function of two first invariants of the (1,1)-strain tensor E^{el}, $Tr(E^{el})$ and $Tr((E^{el})^2)$.

$$L_e(E^{el}) = \bar{\rho}_0(f(E^{el}) + \rho_0(0)F), \tag{3.2}$$

where $\bar{\rho}_0 = \rho_0/\rho_0(0) = \sqrt{\frac{G_{00}}{|G|}}$ is the reference density normalized to its initial value and the strain energy has the form $f(E^{el}) = \mu Tr(E^{el\ 2}) + \lambda(Tr(E^{el}))^2$, μ and λ are initial values of elastic constants. F is the potential of the body forces.

In more general consideration, one can take the strain energy L_e as a function of joint invariants of tensor E^{el} with the tensor $\mathcal{K}$ and the Ricci tensor $Ric(g_t)$ of the metric g_t.

Notice that when the intrinsic metric G coincides with the Minkowski metric (with c=1), tensor E^{el} is the usual strain tensor of the classical elasticity theory ([10],[11]) and the expression (3.2) is the conventional quadratic form of the strain energy of linear elasticity.

Term $L_m(G)$ in (3.1) can be interpreted as the "cohesive energy" of the solid. We assume that the ground state Lagrangian $L_m(G)$ depends on the invariants of the tensor of extrinistic curvature $\mathcal{K} = \mathcal{L}_{u_\phi}\gamma$ of slices $B_{\phi,t}$ in the material space-time P (see [5,14]) and on the Ricci tensor $Ric(g_t)$ of the metric g_t. In the case of a block-diagonal metric G, $\gamma = \left(\begin{smallmatrix} 0 & 0 \\ 0 & g_t \end{smallmatrix}\right)$ and, therefore, $\mathcal{K}$ is, essentially, the time derivative of the 3D projection g_t of material metric G: $\mathcal{K}^I_J = \frac{1}{\sqrt{-G_{00}}} G^{IA} G_{AJ,0}$. Tensor $\mathcal{K}$ is interpreted as the rate of change of effective intrinsic spacial scales in the media due to different inelastic processes together with the influence of elastic deformation on these processes.

The ground state Lagrangian L_m is constructed as a linear combination of quadratic invariants of tensors $Ric(g_t)$ and $\mathcal{K}$:

$$L_m(G) = \gamma_0 + \xi Tr(\mathcal{K}) + \alpha Tr(\mathcal{K}^2) + \beta Tr(\mathcal{K})^2 + \tau R(g_t). \tag{3.3}$$

Here γ_0 is the initial energy density (per unit mass) that is considered to be constant and serves as the parameter of the theory, $R(g_t)$ is the scalar curvature of the 3D metric g. Coeficients α, β, ξ, τ are also parameters to be choosen later.

In simpliest cases (homogeneous case, 1D case) scalar curvature R of the metric g_t is zero and the last term in (3.3) vanishes. More general case where g has nonzero curvature localized on some surfaces or lines (situation studied in gravity by A. Taub [15]) will be considered elsewhere.

Notice that the 4D-scalar curvature $R(G)$ of the metric G can be expressed as $-(tr(\mathcal{K}^2) - (tr\mathcal{K})^2) + R(g_t)$, up to a divergence term (see [5], [14]). As a result, Hilbert-Einstein action $R(G)\sqrt{-|G|}$ is the special case of (3.3). Following the standard procedure for the Lagrangian formulations of the Elasticity (see [11]) we add the surface term $\int \dot{W}(\phi, G)d^3\Sigma$ with $\dot{W}$ representing the power of surface truction; to formulate the action integral on a tube domain $U = [0,t] \times V$, with $(V, \partial V)$ being an arbitrary subdomain of B with the boundary ∂V:

$$A_U(G,\phi) = \int_V (L_m(\mathcal{K}) + L_e(E))dV + \int_{\partial V} \dot{W}(\phi, G)d^3\Sigma. \tag{3.4}$$

Euler-Lagrange Equations Variation principle of extremal action $\delta A = 0$ taken with respect to the dynamic variables ϕ and G gives a system of Euler-Lagrange equations that can be interpreted as the coupled elasticity and aging equations

$$\frac{\partial \mathcal{L}_e}{\partial \phi^m} - \frac{\partial}{\partial X^I}\left(\frac{\partial \mathcal{L}_e}{\partial \phi^m_{,I}}\right) + \rho_0\sqrt{-|G|}(\nabla F)_m = 0, \qquad I = 0,1,2,3. \tag{3.5}$$

$$\frac{\delta \mathcal{L}_m}{\delta G^{IJ}} = -\frac{\sqrt{-|G|}}{2}T_{IJ}, \qquad I = 0,1,2,3. \tag{3.6}$$

Elasticity Equations (3.5) are obtained by taking the variation δA with respect to the components ϕ^i inside the domain U. These equations (except of one with $I = 0$) coincide with the conventional equations of equilibrium of the Nonlinear Elasticity. However their special features are associated with the different definition of the elastic strain tensor E^{el} and with the dependence of the elastic Lagrangian $\mathcal{L}_e$ on time through the metric G and in general through the Ricci tensor and the extrinsic curvature tensor. As a result, tensor of elastic constants is a function of these parameters and, therefore, of time. Evolution of these parameters is defined by the equations (3.6) (refered as Aging Equations). Zeroth equation is an identity for sinchronized deformations. The Hooke's law (obtained by the equating zero of surface variation of deformation history ϕ^i) takes the form

$$\frac{\partial \mathcal{L}_e}{\partial \phi^m_{,I}} = P^I_m, \quad I, m = 0, 1, 2, 3. \tag{3.7}$$

Here $P^I_m = -\frac{\partial P^I}{\partial \phi^m}$ is the first Piola-Kirchoff tensor, with P^I being the components of the traction surface density ($\dot{W} d^3\Sigma = P^I d\Sigma_I$). Using Hooke's law and assuming the absence of body forces ($\nabla F = 0$) one can rewrite the elasticity Equations in the well known form

$$\frac{\partial \mathcal{L}_e}{\partial \phi^m} - \frac{\partial}{\partial X^I} P^I_m = 0, \quad m = 1, 2, 3 \tag{3.8}$$

(If $\mathcal{L}_e$ is traslationally invariant in space, the first term in the left side vanishes).

AGING EQUATIONS Variation of the action with respect to the metric G give us the equations of the material metric G evolution i.e. the aging equations (3.6) where $\frac{1}{2}\sqrt{-|G|}\, T_{IJ} = \frac{\delta(\mathcal{L}_e)}{\delta G^{IJ}}$ defines the "canonical" Energy-Momentum tensor (EMT). This tensor is symmetric and has a close relation with the Eshelby Energy-Momentum tensor b_{IJ} ([4],[7]). Indeed, (see [2]) components of these tensors in a case of a block-diagonal metric G are related as follows

$$\sqrt{-G}\ T_{IJ} = b_{(IJ)} + \mathcal{L}_e G_{IJ}(1 - \delta^0_I \delta^0_J) \tag{3.9}$$

Notice also that the spacial part of the tensor T coincide with the symmetrized second Piola-Kirchoff tensor S: $\sqrt{-|G|} T_{IJ} = S_{(IJ)},\ I, J = 1, 2, 3.$

Equations (3.5-3.6) together with the expression (2.2) for the reference density form the closed system of equations for dynamical variables (G_{IJ}, E^{el}). They complemented with the initial and boundary conditions, provides one with a closed non-linear boundary value problem for deformation and material properties evolution.

In general system (3.5-6) seems rather complex especially if the L_e dependence on $Ric(g_t)$ or $\mathcal{K}$ is included. Yet some problems can be readily analysed.

BLOCK-DIAGONAL METRIC G, SINCHRONIZED ϕ. In this case

$$(K_{IJ}) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{-G_{00}}} G_{IJ,0} \end{pmatrix}.$$

From this it follows that only derivatives in X^I, $I = 1, 2, 3$ that appears in $L_m = \gamma_0 + p(\mathcal{K}) + \tau R(g)$ (p is a sum of term linear by $\mathcal{K}$ and homogeneous function of invariants of tensor $\mathcal{K}$ of degree 2) are those in $R(g_t)$ and that g_t is equal to the restriction of G to the slices B_T for each t. No derivatives in G_{00} appears anywhere in $\mathcal{L}$. In particular, (00)-equation is not a dynamical equation but rather the condition, similar to the "energy condition" in the gravity, see [5].

This equation has the form

$$G_{00}(\sqrt{g_t}(\gamma + \tau R(g_t)) + f(E^{el})) + \sqrt{g_t}p_2(g^{IA}\dot{g}_{AJ}) = 0, \tag{3.10}$$

where $\sqrt{g_t}$ is the volume element of the 3D metric g, p_2 is the quadratic part of L_m and f is the strain energy. This equation can be used to exclude G_{00} from the other six equations (3.6). Alternatively, it can be used as the additional equation to select convinient variables (see [2], Example 2).

Spacial part of equation (3.6) takes the form

$$-\frac{\sqrt{g}}{2\sqrt{-G_{00}}}(\alpha g_{IJ}g_{AB} + \beta g_{IA}g_{JB})\,\ddot{g}^{AB} + q_{IJ}(g, \dot{g}, G_{00}, \dot{G}_{00}) + \tau\tilde{\mathcal{E}}_{IJ}(g_t) = S_{(IJ)}. \tag{3.11}$$

Here $\tilde{\mathcal{E}}_{IJ}(g)$ is the analog of the Einstein tensor of the 3D-metric g_t. The difference with the usual Einstein tensor is due to the presence of the factor $\sqrt{-G_{00}}$ in the term $\sqrt{-|G|}R_3(g)$ of the Lagrangian density ($\sqrt{-|G|} = \sqrt{-G_{00}}\sqrt{g_t}$). Term q on the left side depends on the metric coefficients and their first derivatives in time.

Right side of (3.7) contains no derivatives of metric coefficients. Third term in the right side contains space derivatives of G_{IJ}, $I, J = 1, 2, 3$ but does not contain time derivatives. The first two terms in the left side on the contrary, contain only derivatives in time but no space derivatives. This equation is of the second order in time. In the case where the term with the constant α dominate one with the constant β (for example, if $\beta = 0$) this equation can be easily transformed to the normal form

$$\frac{\partial^2 G_{IJ}}{\partial t^2} + F(G, \frac{\partial G}{\partial t}, \frac{\partial G}{\partial X}, \frac{\partial^2 G}{\partial X^2}, \nabla\phi) = 0.$$

Below we consider 3 special cases.

1) Homogeneous media In a case of a homogeneous media tensor $\mathcal{E}_{IJ}(g)$ is identically zero. As a result, (3.11) becomes a system of quasilinear **ordinary** differential equations of the second order for G_{IJ}. Cauchy problem for this system is correct if $\alpha >> \beta$.

In a case, where $Ric(g_t) \approx 0$, a good approximation of the general system (3.5-6) can be proposed. If the total deformation ϕ is approximated by the ground deformation $\bar{\phi}$ in evaluation of EMT T_{IJ} in the right side of (3.6), the latter becomes decoupled from equilibrium equation (3.5). This allows to study aging equations separately and, after obtaining solution G of these equations, substitute them into elastic equilibrium equation (3.5) and solve it as the usual elasticity equation **with variable elastic moduli.**

2) STATIC CASE If G does not depend on time. $\mathcal{K} = 0$, $p(\mathcal{K}) = 0$ and (3.10) reduces to the "energy balance equation" (with scalar curvature $R_3(g)$ plaing the role of metric energy) while (3.11) becomes the second equilibrium equation describing the stress produced by the curvature of the metric g and "frozen" into the media (comp. [9],[11]).

3) HOMOGENEOUS ROD (1-D CASE) In a case of a 1D media (rod) the curvature of g is identically zero. Then the equations (3.6) reduces to a nonlinear dynamical system for G_{00} and G_{11} (see [2], Example 2).

IV. BALANCE EQUATIONS

As it is usual for a Lagrangian field theory, action of any one-parameter group of transformations of the space $P \times M$ commuting with the projecton to the first factor leads to the corresponding balance law (see [11]). In particularly, translations in the "laboratory space-time" M lead to the equations of motion (3.5) (including zeroth one that is trivially valid here), rotations in the "laboratory space" lead to the angular momentum balance law (conservation law in isotropic case). Respectivelly, translations in the "material space-time" P lead to the **energy balance law** (translations along T axis) and to the **material momentum balance law** (called also "pseudomomentum") , rotations in the material space B lead to the "material angular momentum" balance law (see [7],[8],[13]). In terms of the 4D Eshelby tensor

$$b = -\mathcal{L}_e \delta^I_J + \frac{\partial \mathcal{L}_e}{\partial \phi^i_{,I}} \phi^i_{,J},$$

material energy-momentum conservation law has the form (see [2])

$$div_4 b = b^I_{J,I} = \frac{\partial}{\partial X^I}\left(\mathcal{L}_m \delta^I_J - \frac{\partial \mathcal{L}_m}{\partial G^{AB}_{,I}} G^{AB}_{,J} - \frac{\partial \mathcal{L}_e}{\partial G^{AB}_{,I}} G^{AB}_{,J}\right). \tag{4.1}$$

Relations between the space and material balance (conservation) laws are given by the deformation gradient:

$$\begin{pmatrix} \eta_0 \\ \dots \\ \eta_4 \end{pmatrix} = \begin{pmatrix} 1 & \phi^1_{,0} & \dots & \phi^3_{,0} \\ 0 & \phi^1_{,1} & \dots & \phi^1_{,3} \\ \dots & \dots & \dots & \\ 0 & \dots & \dots & \phi^3_3 \end{pmatrix} \begin{pmatrix} \nu_0 \\ \dots \\ \nu_4 \end{pmatrix}.$$

Similar to the relativistic elasticity ([9]) system of material momentum balance laws $\eta_I = 0, I = 1,2,3$ is equivalent to the elasticity equations $\nu_m = 0$ while energy balance law $\eta_0 = 0$ (which is the material law) follows from any of these two systems: $\eta_0 = \sum_{i=1}^{i=3} V^i \nu_i$. This reflects the fact that the deformations we consider are not really 4-dimensional, and that we restrict the class of deformations to sinchronized ones.

Energy balance law plays special role in our considerations. If the elastic Lagrangian $L_e = L_e(E^{el})$ depends only on the strain tensor E^{el} and if L_m is function of $\mathcal{K}$ only, energy balance law takes the following form

$$\frac{\partial}{\partial T}\left(\mathcal{E}^{tot} = -\mathcal{L}_e - \mathcal{L}_m + \frac{\partial \mathcal{L}_m}{\partial G^{AB}_{,0}} G^{AB}_{,0}\right) = -\frac{\partial}{\partial X^I}\left(P^I_i \phi^i_{,0}\right). \tag{4.2}$$

Here $\mathcal{E}^{tot}$ is the total inner energy density, P^I_i is the first Piola-Kirchoff stress tensor. That equation has the standard form "rate of change of inner energy equals to the 3D flow of energy". The total inner energy $\mathcal{E}^{tot}$ is composed of the usual elastic strain energy (first term) and the "metric energy" (sum of the second and third terms). For the external "observer" all three terms represent the total energy while from the point of view of "internal observer" this total interior energy comprises three terms corresponding to the different processes going in the media. In the next section this balance will be presented in the more specific terms in the case of an aging rod. The sum in the right side represents the flow of the traction forces power. In more general case an additional flows related to the change of the inner metric G ("material forces power") appear (comp. [3],[12],[13]). In the case of the conventional elasticity intrinsic metric G does not depend on time and the energy balance takes its classical form.

First term in the left side of (4.2) is the elastic strain energy, second and third together represent a "cohesive" energy i.e. a part of the total energy density associated with the integrity of the media. Reduction of the cohesive energy due to the aging can be related to an increase of brittleness. This relation is a subject of different article.

V. Linearized Problem

Minkowski metric $G^0_{IJ} = \nu_I \delta_{IJ}$, where $\nu_I = -1$ for $I = 0$ and $= +1$ for $I = 1, 2, 3$ and the equilibrium embedding $\phi^i(T, X) = X^i$ are solutions of system (3.5)-(3.6) for $\gamma = 0$. Here we consider the linearization of the system (3.5-3.6) for small total deformation, that implies a small decline of a material metric from the Minkowski metric as well as small components of elastic strain.

We decompose $\phi^i(X^I) = X^i + u^i$, $u^0 = 0$ and consider the gradients of deviations u^i to be small.

We also decompose the block-diagonal metric G in a similar fasion: $G_{IJ} = G^0_{IJ} + \epsilon^p_{IJ}$. Here ϵ^p is the small deviation of matherial metric G from the Minkowski metric G^0. Denote by ϵ^p_3 the spacial part of tensor ϵ^p. We consider ϵ^p_{IJ} to be of the same magnitude as $u^i_{,I}$.

We have (using eucledian metric to rise and lower the indices) $\phi^i_{,I} = \delta^i_I + u^i_{,I}$, $C(\phi)_{IJ} = \delta_{IJ} + (u^I_{,J} + u^J_{,I})$. For the elastic strain tensor we have

$$E^{el} = \frac{1}{2}\left[(u^I_{,J} + u^J_{,I}) - \epsilon^p_{IJ}\right] = E^{tot} - \epsilon^p_3. \tag{5.1}$$

where $E^{tot} = \frac{1}{2}(u^I_J + u^J_I)$ is the total strain.

Notice that in this approximation space components of strain tensors coincide with the coresponding components of material strain tensors. We also have $E^{ir} = \frac{1}{2} ln(h^{-1} g_t) \approx \epsilon^p_3$, so ϵ^p_3 is the linearized tensor of inelastic deformation.

For the determinant of the material metric G and for the tensor of extrinistic curvature we have the following approximate expressions: $|G| \approx -1 + \epsilon^p_{00} - tr(\epsilon^p_3), \sqrt{-|G|} \approx 1 - \frac{1}{2}(\epsilon^p_{00} - tr(\epsilon^p_3))$, trace is taken with respect to the eucledian 3D metric, $\mathcal{K} = \frac{\partial}{\partial t}\epsilon^p$. For the reference density we have

$$\rho_0 \approx \rho_0(0, X)(1 - \frac{tr(\epsilon^p_3)}{2}).$$

Lagrangian of this linearized problem is (in the absence of the body forces)

$$\mathcal{L}_{lin} = (\gamma + \kappa tr(\epsilon^p_3))\left(1 - \frac{\epsilon p_{00}}{2} + \frac{tr(\epsilon^p_3)}{2}\right) + \xi \frac{\epsilon^p_{00}}{2} tr(\epsilon^p_3)_{,0} +$$
$$\alpha tr((\epsilon^p_{3\ ,0})^2) + \beta(tr(\epsilon^p_{3,0}))^2 + \frac{\mu}{2} tr(E^{el})^2 + \frac{\lambda}{4}\left(tr(E^{el})\right)^2, \tag{5.2}$$

Hooke's law has the conventional form $P = \mu E^{el} + \frac{\lambda}{2} tr(E^{el}) I$, I being the 3D unit tensor, and, in this approximation, Cauchy tensor σ coinside with the Piola-Kirchoff tensor P.

Linearized equations have the form

Equilibrium equation:

$$div(\sigma) = 0. \tag{5.3}$$

This equation is similar to the usual linear equilibrium equation with the force $div(\epsilon_3^p)$.

Metric Equations (linearized equations (3.6))

$$\kappa tr(\epsilon_3^p) + \xi tr(\epsilon_3^p)_{,0} = \frac{\gamma}{2}, \tag{5.4}$$

and

$$\left(\frac{\gamma}{2} + \kappa(1 - \frac{\epsilon_{00}^p}{2}) + \kappa tr(\epsilon_3^p)\right) I - \frac{\xi}{2} tr(\epsilon_3^p)\epsilon_{00\ ,0}^p I - 2\left(\alpha\epsilon_{3\ ,00}^p + \beta I(tr(\epsilon_3^p))_{,00}\right) = \sigma. \tag{5.5}$$

Here γ is considered to be a small parameter.

Equation (5.4) gives (we take $\epsilon^p(t = 0) = 0$)

$$tr(\epsilon_3^p) = \frac{\gamma}{2\kappa}(1 - e^{-\frac{\kappa}{\xi}t}). \tag{5.6}$$

Equations (5.5) are equations of the second order in time for the five free components of the tensor ϵ_3^p and one of the first order in time for ϵ_{00}^p.

Consider a 1D case (rod). Here we have only components ϵ_{00}^p and ϵ_{11}^p. We put $\beta = 0$. Young module E is, in this approximation, constant. Then, (5.6) gives

$$\epsilon_{11}^p = \frac{\gamma}{2\kappa}(1 - e^{-\frac{\kappa}{\xi}t}).$$

Equation (5.6) for constant stress σ has the solution

$$\epsilon_{00}^p = ce^{-\frac{\kappa}{\xi}t} + (\frac{\kappa + 2\gamma - 2\sigma}{\kappa}) + \frac{2\alpha\kappa - \xi^2}{\xi^3} te^{-\frac{\kappa}{\xi}t}.$$

Adding the condition that at $t = 0$ $\epsilon_{00}^p = 0$ we get

$$\frac{\gamma}{\kappa} = -\frac{c+1}{2} + \frac{\sigma}{\kappa}.$$

If, in addition, we require that $\epsilon_{11}^p = 0$ if the load is zero ($\sigma = 0$), then $c = -1$ and we have

$$\epsilon_{00}^p = 1 - e^{-\frac{\kappa}{\xi}t} + \frac{2\alpha\kappa - \xi^2}{\xi^3} te^{-\frac{\kappa}{\xi}t}, \tag{5.7}$$

and for the irreversible deformation

$$\epsilon_{11}^{p} = \frac{\sigma}{\kappa}(1 - e^{-\frac{\kappa}{\xi}t}). \tag{5.8}$$

This represents the well known creep behavior of a material with fading memory ([6]).

VI. Conclusion

A variational approach is proposed to formulate constitutive equations for aging media. The approach is based on the assumption that the metric tensor of the inner (material) space-time geometry together with an elastic strain tensor constitute a complete set of parameters of state. This assumption combined with classical Hamilton's principle provides a framework for derivation the constitutive and balance equations modeling material behaviour. Selection of a particular form of the Lagrangian, as it is usual in a variational formulation, leads to a particular constitutive equations. Thus, for one of the simplest linearized case the approach leads to a model of well studied creep behavior of a material with fading memory. Analysis of various forms of Lagrangian, the resulting models of material behavior, comparison with the experimental data as well as with conventional thermodynamic restrictions is the subject of our next work.

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