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On Non-Holonomic Second-Order Connections with Applications to Continua with Microstructure

Marek Elżanowski and Serge Preston

Department of Mathematical Sciences, Portland State University, P.O.Box 751, *Portland, Oregon* 97207-0751, *U.S.A., e-mail: marek@mth.pdx.edu, serge@mth.pdx.edu*

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1. Introduction

Motivated by the theory of uniform elastic structures [2] we try to determine the conditions for the local flatness of locally integrable connections on non-holonomic frame bundles of order 2. Utilizing the results of Yuen [8] as well as our results for the holonomic case [3], we show that the locally integrable non-holonomic 2-connection is locally flat if, and only if, its projection to the bundle of linear frames is symmetric and the so-called inhomogeneity tensor (cf., Elżanowski and Prishepionok [4]) vanishes.

In the last section of this short paper we show how these results can be interpreted in the framework of the theory of uniformity of simple elastic materials with microstructure.

2. Non-Holonomic Second-Order Frame Bundles

Let $\mathcal{B}$ be an n-dimensional C^∞-manifold. Denote by $\hat{H}^2(\mathcal{B})$ the space of all 2-frames of $\mathcal{B}$. Respectively, let $H^2(\mathcal{B})$ be the space of all *holonomic* 2-frames of $\mathcal{B}$ (cf., Yuen [8]). Given a holonomic frame $p^2 \in H^2(\mathcal{B})$ there exists always a local (about the origin of $\mathbb{R}^n$) diffeomeorphism $f : U(0) \subset \mathbb{R}^n \to \mathcal{B}$ such that $\pi^2(p^2) = f(0)$ and its second jet at $0 \in \mathbb{R}^n$ can be identified with p^2. π^2, respectively $\hat{\pi}^2$, denote here the standard projections onto the base manifold $\mathcal{B}$.

Choosing a coordinate chart $\{y^1, \cdots y^n\}$ on $\mathcal{B}$ about $y = f(0)$ and a cartesian coordinate system $\{x^I, \cdots, x^N\}$ on $\mathbb{R}^n$ we have $p^2 = (y^i, y^i_A, y^i_{AB})$ where $i = 1, \cdots, n$, $A, B = 1, \cdots, n$, $det\ y^i_A \neq 0$ and $y^i_{AB} = y^i_{BA}$. In contrast, an arbitrary (*non-holonomic*) frame $\hat{p}^2 \in \hat{H}^2(\mathcal{B})$ is usually identified with a first jet at the origin of a local differentiable map $\hat{f} : U(0) \subset I\!R^n \to H^1(\mathcal{B})$. $H^1(\mathcal{B})$ denotes here the space of all linear frames of $\mathcal{B}$. Given a local coordinate system on $H^1(\mathcal{B})$, say $\{y^i, y^i_A\}$, $\hat{p}^2 = (y^i, y^1_A, y^i_{,B}, y^i_{A,B})$ where, in general, $y^i_A \neq y^i_{,A}$ and $y^i_{A,B} \neq y^i_{B,A}$. If, however, the map $\hat{f}$ is such that $\hat{f}(0) = j(\pi^1 \circ \hat{f})(0)$ i.e., $(y^i, y^i_A) = (y^i, y^i_{,A})$ for some choice of a coordinate chart on $\mathcal{B}$, then the frame, the function $\hat{f}$ induces, is called *semi-holonomic*. The space of all semi-holonomic 2-frames of $\mathcal{B}$ will be denoted by $\tilde{H}^2(\mathcal{B})$. It is now obvious from the later construction that $\hat{H}^2(\mathcal{B}) \supset \tilde{H}^2(\mathcal{B}) \supset H^2(\mathcal{B})$.

As it is well known (cf., Saunders [7]) these spaces are the principal bundles over $\mathcal{B}$ with the structure groups $\hat{\mathcal{G}}^2$, $\tilde{\mathcal{G}}^2$ and $\mathcal{G}^2$, respectively. The group $\mathcal{G}^2$, which is the set of all 2- jets of the origin preserving local diffeomorphisms of $I\!R^n$, is the semidirect product of the general linear group $GL(n,I\!R)$ and the algebra of all $I\!R^n$-valued symmetric bi-linear forms $\mathcal{N}^2_1(I\!R^n, I\!R^n)$. On the other hand, $\hat{\mathcal{G}}^2$ is the set of all first jets of the local automorphisms of $H^1(\mathcal{B})$ preserving the zero fibre and is isomorphic to the zero fibre of $\hat{H}^2(I\!R^n)$. As a group it can be viewed as a semidirect product of two copies of $GL(n,I\!R)$ and the algebra $\hat{\mathcal{N}}^2_1(I\!R^n, I\!R^n)$ of all $I\!R^n$-valued bi-linear forms. It acts on the non-holonomic frame bundle $\hat{H}^2(\mathcal{B})$ on the right as follows: take a 2-frame $\hat{p}^2 = (y^i, y^i_A, y^i_{,B}, y^i_{A,B})$ and let $(p^A_C, g^B_D, \alpha^A_{CD})$ represent an element of $\hat{\mathcal{G}}^2$ then, using the standard shorthand,

$$(y^i, y^i_A, y^i_{,B}, y^i_{A,B})(p^A_C, g^B_D, \alpha^A_{CD}) = (y^i, y^i_A p^A_C, y^i_{,B} g^B_D, y^i_{A,B} g^B_D p^A_C + y^i_A \alpha^A_{CD}).$$

Obviously, this action is consistent with the action of $\tilde{\mathcal{G}}^2$ on $\tilde{H}^2(\mathcal{B})$ and the action of $\mathcal{G}^1 = GL(n,I\!R)$ on the bundle of linear frames $H^1(\mathcal{B})$. Indeed, if the frame $\hat{p}^2$ is semi-holonomic and $p^A_C = g^A_C$, for any $A, C = 1, \cdots, n$, then, the resulting frame is also semi- holonomic. If, moreover, $y^i_{A,B}$ and α^A_{CD} are symmetric the action produces a holonomic frame.

In addition to being the principal bundles over the manifold $\mathcal{B}$ these spaces are affine bundles over the bundle of linear frames $H^1(\mathcal{B})$ with the standard fibers $\hat{\mathcal{N}}^2_1, \tilde{\mathcal{N}}^2_1$ and $\mathcal{N}^2_1$, respectively.

3. Non-Holonomic 2-Connections

Suppose now that a linear connection on the frame bundle of non-holonomic 2-frames is given. Such a connection is represented on $\hat{H}^2(\mathcal{B})$ by an equivariant 1-form ω^2 with values in the Lie algebra $\hat{\mathfrak{g}}^2$ of its structure group $\hat{\mathcal{G}}^2$, i.e., for every vector χ from the tangent space $T\hat{H}^2(\mathcal{B})$ $\omega^2(\chi) \in \hat{\mathfrak{g}}^2$. At any 2-frame $\hat{p}^2$ the kernel of the connection form ω^2, as a linear map on the tangent space $T_{\hat{p}^2}\hat{H}^2(\mathcal{B})$, is a n-dimensional vector subspace $\mathcal{H}_{\omega^2}(\hat{p}^2)$ called the horizontal space of the connection ω^2 at $\hat{p}^2$. This gives an n-dimensional equivariant differentiable (horizontal) distribution $\mathcal{H}_{\omega^2}$ on $\hat{H}^2(\mathcal{B})$.

As shown by Kobayashi [5] (see also Elżanowski and Prishepionok [3]), every linear connection on the second order frame bundle is uniquely induced by the so-called $\mathcal{E}$-connection of order 3, i.e., a $\mathcal{G}^1$-invariant section $\varepsilon^3 : H^1(\mathcal{B}) \to \hat{H}^2(B)$. Indeed, in general, every $\mathcal{E}$-connection of order $k+1$ induces a GL-reduction $N_{\omega^k} \equiv \pi_k^{k+1}(\varepsilon^{k+1}(H^1(\mathcal{B})))$ of $\hat{H}^{k+1}(\mathcal{B})$, called the *characteristic manifold* of the connection ω^k and a GL-reduction $M_{\omega^k} \equiv \varepsilon^{k+1}(H^1(\mathcal{B}))$. Equivalently, the mapping ε^{k+1} induces the characteristic manifold N_{ω^k} and a GL-invariant partial section $q^k : N_{\omega^k} \to \hat{H}^{k+1}(\mathcal{B})$ (onto the manifold M_{ω^k}) called the *characteristic section*.

In particular, given a 2-frame $\hat{p}^2 \in N_{\omega^2}$ a vector $\chi \in T_{\hat{p}^2}\hat{H}^2(\mathcal{B})$ is a horizontal vector of ω^2 if, and only if, $\theta_3(q^2_*(\chi)) = 0$ where θ_3 is the $\mathfrak{g}^2$-component of the *fundamental form* θ^3 on $\hat{H}^3(\mathcal{B})$.[1] The kernel of the form θ_3 is called the *standard horizontal space* of the frame $q^2(\hat{p}^2)$. The horizontal distribution of the connection ω^2 on the submanifold N_{ω^2} is therefore the GL-equivariant distribution of the standard horizontal spaces of the corresponding (through the section q^2) frames from the image of the $\mathcal{E}$- connection ε^3 [2]. The extension of this distribution to the whole of $\hat{H}^2(\mathcal{B})$ is done uniquely by the associated action of the structure group $\hat{\mathcal{G}}^2$ on the tangent space.

The 2-connection $\hat{\omega}^2$ is called a *holonomic connection*, respectively a *semi-holonomic connection*, if it is a reduction of a non-holonomic 2-connection to the corresponding reduction of the bundle of non-holonomic 2-frames. Note that in either of these two cases the corresponding $\mathcal{E}$-connection ε^3 is not necessarily a section into a reduction of $\hat{H}^3(\mathcal{B})$ to the holonomic (respectively semi-holonomic) structure group $\mathcal{G}^3$ (respectively $\tilde{\mathcal{G}}^3$).

In what follows, we consider only 2-connections $\hat{\omega}^2$ which are locally integrable i.e., their horizontal distributions are locally integrable differential distributions. Therefore, by the Frobenious theorem, for every base point $x \in \mathcal{B}$ there exists a local section $\hat{\mathfrak{l}}^2$ of the non-holonomic second order frame bundle $\hat{H}^2(\mathcal{B})$ such that the horizontal distribution $\mathcal{H}_{\hat{\omega}^2}$ is a lift of the tangent space $T\mathcal{B}$. Namely, $\mathcal{H}_{\hat{\omega}^2} = \hat{\mathfrak{l}}^2_*(T\mathcal{B})$.

[1]For the definition of the fundamental for on a frame bundle consult Kobayashi [5] or Elżanowski and Prishepionok [3].

[2]Note that, in general, such a distribution is not tangent to the characteristic manifold $N_{\hat{\omega}^2}$.

Given a locally integrable non-holonomic second order connection $\hat{\omega}^2$ it locally induces two, in general different, linear connections. Indeed, if $\hat{\omega}^2$ is locally integrable then for every point $x \in \mathcal{B}$ there exists always a local section $\hat{\mathfrak{l}}^2 : U(x) \to \hat{H}^2(\mathcal{B})$ generating its horizontal distribution. There is also a section $\mathfrak{p}^1 : U(x) \to H^1(\mathcal{B})$ such that $\mathfrak{p}^1 \equiv \hat{\pi}^2_1 \circ \hat{\mathfrak{l}}^2$. The section $\hat{\mathfrak{p}}^2$ generates a horizontal distribution on $H^1(\mathcal{B})$. We therefore have a locally integrable linear connection on $H^1(\mathcal{B})$, denoted $proj\omega^1$. It is easy to see that the distribution $\mathcal{H}_{proj\omega^1}$ is a projection, by $\hat{\pi}^2_{1*}$, of the horizontal distribution of the 2-connection $\hat{\omega}^2$. It can also be shown (cf., Elżanowski and Prishepionok [3]) that $N_{proj\omega^1}$, the characteristic manifold of $proj\omega^1$, is the $\hat{\pi}^2_1$-projection of the characteristic manifold of $\hat{\omega}^2$.

Moreover, given the locally integrable connection $\hat{\omega}^2$, the generating section $\hat{\mathfrak{l}}^2$ and its projection $\mathfrak{p}^1$ there exists also a partial section $\hat{q}^2 : \mathfrak{p}^1(U(x) \to \hat{H}^2(\mathcal{B})$. This section when extended equivariantly, by the action of the general linear group[3], to the whole manifold $H^1(U(x))$ becomes a local $\mathcal{E}$-connection of order 2. As we have stated earlier, its existence is equivalent to the availability of a new linear connection $i\hat{\omega}^2$, called the *induced connection*, the characteristic manifold of which is the entire bundle of linear frames over $U(x)$[4] while the characteristic section is the equivariant extension of the partial section q^2. Let us also add that the manifold $M_{i\hat{\omega}^2}$ is the GL-reduction of $\hat{H}^2(\mathcal{B})$ based on the image $q^2(\mathfrak{p}^1(U(x)))$.

4. Locally Flat Non-Holonomic 2-Connections

We are now in the position to address the main question posed in the Introduction i.e., what are the necessary and sufficient conditions an arbitrary 2-connection must satisfy to be locally flat ? We may start by saying that it was shown by Yuen [8] that a semi-holonomic connection $\tilde{\omega}^2$ is locally flat if, and only if, it has no torsion, it is curvature free and is the prolongation of a linear connection. Here by the *torsion* we understand the $\mathbb{R}^n \oplus \tilde{\mathfrak{g}}^1$-valued 2-form $d\theta^2|_{\mathcal{H}_{\tilde{\omega}^2}}$ while the *curvature* is the $\tilde{\mathfrak{g}}^2$-valued 2-form $d\tilde{\omega}^2|_{\mathcal{H}_{\tilde{\omega}^2}}$ (see e.g., Cordero at al.[1]). The 2-connection (possibly non-holonomic) $\hat{\omega}^2$ is the *prolongation* of some linear connection, say ω^1, if its horizontal distribution is a differential lift of the horizontal distribution $\mathcal{H}_{\omega^1}$. More precisely, the prolongation of the

[3]Realize that the general linear group can be embedded canonically into the group $\hat{\mathcal{G}}^2$.

[4]Characteristic manifolds of all linear connections are identical and all equal to the bundle of linear frames $H^1(\mathcal{B})$, as evident from the definition of the characteristic manifold (cf., Elżanowski and Prishepionok [3]).

connection ω^1, with its characteristic section q^2, is the 2-connection, denoted by $\mathcal{P}(\omega^1)$, such that at every $p^1 \in H^1(\mathcal{B})$

$$\mathcal{H}_{\mathcal{P}(\omega^1)}(q^2(p^1)) = q^2_*(\mathcal{H}_{\omega^1}(p^1)).$$

This immediately implies that:

PROPOSITION 1. (Elżanowski and Prishepionok [3]) *A 2-connection is the prolongation of some linear connection if, and only if, its horizontal distribution is tangent to its characteristic manifold.*

It can be shown (see Elżanowski and Prishepionok [3] and Elżanowski [2]) that the prolongation of a linear connection is unique. It can also be proved that for any linear connection ω^1 its prolongation is such that $proj\mathcal{P}(\omega^1) = \omega^1$ and that $\hat{\omega}^2 = \mathcal{P}(\omega^1)$ if, and only if, $N_{\hat{\omega}^2} = q^2(N_{\omega^1})$. Moreover,

PROPOSITION 2. *Any locally integrable 2-connection $\hat{\omega}^2$ (holonomic or not) is the prolongation of some linear connection if, and only if,*

$$i\hat{\omega}^2 = proj\hat{\omega}^2.$$

Proof. If the locally integrable connection $\hat{\omega}^2$ is the prolongation of some linear connection - and it can only be the prolongation of its own projection $proj\hat{\omega}^2$ - then its characteristic distribution is tangent to its characteristic manifold. Consequently, $N_{\hat{\omega}^2} = \hat{\mathfrak{l}}^2 GL \equiv M_{i\hat{\omega}^2}$ by the definition of the induced connection. Also, as stated earlier, $q^2(N_{proj\hat{\omega}^2}) = N_{\hat{\omega}^2}$. This proves that the characteristic sections of the induced and the projected connection are identical. The projected connection and the induced one are, therefore, equal as they have the same characteristic manifolds and the same characteristic sections. The converse is now obvious. ∎

For a locally integrable 2-connection $\hat{\omega}^2$ to be the prolongation is, therefore, equivalent to the vanishing of the $\hat{\mathfrak{g}}^1$-valued tensorial form

$$\mathcal{D}_{\hat{\omega}^2} \equiv i\hat{\omega}^2 - proj\hat{\omega}^2.$$

Obviously, the vanishing of the tensor $\mathcal{D}_{\hat{\omega}^2}$ does not yet guarantee the local flatness of the non-holonomic connection $\hat{\omega}^2$. Its vanishing is, however, both the necessary and sufficient condition for the local flatness of any locally integrable holonomic connection ω^2. Indeed, if the locally integrable connection ω^2 is holonomic then the corresponding local section $\mathfrak{l}^2$ goes into the holonomic 2-frame bundle $H^2(\mathcal{B})$. Consequently, its induced connection $i\omega^2$ has, as we showed in [3], vanishing torsion. In fact, it is symmetric only if its $\mathcal{E}$-connection is a section into the holonomic frame bundle (cf., Yuen [8]). The horizontalistribution of $i\omega^2$ may, however, be nonintegrable. Inontrast, $proj\omega^2$ is locally integrable but it probablyas some non-vanishing torsion. The vanishing of the tensor $\mathcal{D}_{\omega^2}$ makes these two linear connections equal and so both locally flat. The vanishing of $\mathcal{D}_{\omega^2}$, according to Proposition 2, proves also that the 2-connection ω^2 is the prolongation. As the prolongation of the locally flat linear connection is also locally flat [ ] the connection ω^2 is locally flat too.

If we now turn our attention to the semi-holonomic case, say $\tilde{\omega}^2$, the vanishing of the corresponding tensor $\mathcal{D}_{\tilde{\omega}^2}$ makes the induced connection equal to the projected one. Both connections are also curvature free as the 2-connection $\tilde{\omega}^2$ is curvature-free. There is no guarantee, however, that they are locally flat as there is no indication that they have zero torsions. In contrast with the holonomic case, the induced connection is not necessarily symmetric as its corresponding $\mathcal{E}$-connection (the invariantly extended partial section $\tilde{q}^2$) is not a section into the holonomic frame bundle $H^2(\mathcal{B})$. To make this happen, however, it is enough to demand that the torsion of the induced connection, which is now identical with the projected connection, vanish. We therefore have:

PROPOSITION 3. *Any locally integrable semi-holonomic 2- connection $\tilde{\omega}^2$ is locally flat if, and only if, the tensor $\mathcal{D}_{\tilde{\omega}^2}$ vanishes and the projected connection $proj\omega^2$ is symmetric.*

The completely non-holonomic case reduces, in fact, to the semi- holonomic situation. Namely, let us consider a locally integrable non-holonomic 2-connection $\hat{\omega}^2$. Let, as before, $\hat{\mathfrak{l}}^2$ be the integral local section of its horizontal distribution. There exist therefore both the projected connection $proj\hat{\omega}^2$ as well as the induced connection $i\hat{\omega}^2$. If the tensor $\mathcal{D}_{\hat{\omega}^2}$ vanishes and the torsion of the projected connection $proj\hat{\omega}^2$ is zero then $\hat{\omega}^2$ is not only the prolongation (of its own projection) but the induced connection $i\hat{\omega}^2$ is locally flat (its torsion and curvature vanish). Note also that as the induced connection $i\hat{\omega}^2$ is symmetric the local section $\mathfrak{l}^2$, which plays the role of the $\mathcal{E}$-connection of the induced one, must be the section into the holonomic second order frame bundle, [8], [3]. Thus, the 2-connection $\hat{\omega}^2$ is reducible to the holonomic connection and by Proposition 3 is locally flat. Note that in the completely non-holonomic case the 2- connection $\hat{\omega}^2$ can be projected onto the first or the second factor. The arguments presented above are simultaneously applicable in both situations.

5. Locally Homogeneous Deformable Continuous Bodies

Suppose now that the manifold $\mathcal{B}$ represents a continuous deformable material body. For simplicity let us assume that it can be covered by a global chart. We select one such chart $\psi_0 : \mathcal{B} \rightarrow \mathbb{R}^3$ as the *reference configuration* of the body $\mathcal{B}$ and identify the material body with is image $\psi_0(\mathcal{B}) \subset \mathbb{R}^3$. Given any other configuration, say $\phi : \mathcal{B} \rightarrow \mathbb{R}^3$, a 1-jet of $\psi_0 \times \phi^{-1}$ at $x \in \psi_0(\mathcal{B})$ is called *a local configuration* of the material point x. The mechanical properties of the so-called simple elastic body are completely characterized by the smooth real-valued energy function $\mathcal{W}$ on the space of all local configurations of $\mathcal{B}$. If some additional structure, given for example by a smooth distribution on $\mathcal{B}$ of deformable triads of vectors, is also available then the mechanical properties of such a simple material with microstructure are determine by a function $\mathcal{W}$ but now on the space of non-holonomic 2-frames of $\mathcal{B}$ (see e.g., de León and Epstein [6] or Elżanowski and Prishepionok [4]).

We say that such a material body is *smoothly uniform*, i.e., build of the same material points, it there exists a smooth section $\hat{\mathfrak{l}}^2 : \mathcal{B} \rightarrow \hat{H}^2(\mathcal{B})$, called the *material configuration*, such that the strain energy function $\mathcal{W}$ is constant on the image of $\hat{\mathfrak{l}}^2$.

The energy function $\mathcal{W}$ may have, over a given material point, a non-trivial isotropy group. If the material body $\mathcal{B}$ is uniform these isotropy groups are isomorphic. If the function $\mathcal{W}$ representing a uniform material body has a non-trivial isotropy group then the choice of the material configuration is not unique. Namely, any smooth action of the isotropy group will produce yet another material configuration.

The material configuration $\hat{\mathfrak{l}}^2$ induces on $\hat{H}^2(\mathcal{B})$ the so-called *material parallelism.* Such a parallelism is locally integrable but is not necessarily flat. However, if there is one which is locally flat, and so the inducing section $\hat{\mathfrak{l}}^2$ is locally generated by a coordinate system on the manifold $\mathcal{B}$ the body is called locally *homogeneous.* In other words, the simple material body with microstructure is locally homogeneous it admits a locally flat material parallelism. Adopting what we have said in the first part of this paper about the conditions for the local flatness of non-holonomic 2- connections, one can claim that:

Proposition 4. *A simple material body with the microstructure given by triads of deformable vectors is locally homogeneous if, and only if, there exists the material parallelism (the material 2-connection $\hat{\omega}^2$) such that its tensor $\mathcal{D}_{\hat{\omega}^2}$ vanishes and the projected linear connection $proj\hat{\omega}^2$ is symmetric.*

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