EXTRACTA MATHEMATICAE Vol. 11, Núm. 1, 65 – 79 (1996)

Noether (and Gauge) Transformations for Higher Order Singular Lagrangians

Xavier Gràcia and Josep M. Pons

Departament de Matemàtica Aplicada i Telemàtica Universitat Politècnica de Catalunya, Campus Nord, edifici C3, 08071 *Barcelona, Catalonia, Spain*

Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona, Av. Diagonal 647, 08028 *Barcelona, Catalonia, Spain*

AMS *Subject Class.* (1991): 70H33, 70H35, 58F05, 34A09

1. Introduction

Noether's symmetries play an important role in the study of dynamical systems described by a variational principle. In this talk an analysis of this symmetries is presented; this is mainly based on the articles [GP 92] [GP 95]. The characterization of Noether's symmetries as presented herein is especially relevant when dealing with a singular lagrangian, since it allows to compute its gauge transformations.

Here we will consider a time-independent mechanical lagrangian, of first or higher order. However, the results may be extended to time-dependent lagrangians with minor changes, and to field theory through the usual methods of Theoretical Physics.

The case of higher order lagrangians appears as an extension of the first order case; however, they have several relevant differences, and for the sake of clearness they will be presented separately.

For this development several geometric structures for first order and higher order lagrangians are needed [GP 89] [GPR 91]. However, we shall frequently use a coordinate language. Other references with geometric aspects related to this paper are [CF 93] [CLM 89] [CLM 91].

The organization of the paper is as follows. Section 2 contains some considerations on symmetry transformations. Then we consider the case of first order lagrangians: in section 3 some geometric structures and our characterization of Noether's transformations are explained. This is also done for higher order lagrangians in section 4. Finally, section 5 is devoted to an example.

2. Symmetry transformations, gauge transformations

Let us consider, for instance, a differential equation[1] $F(t, q, \dot{q}) = 0$ for a path $q(t)$. A transformation $q(t) \rightsquigarrow \tilde{q}(t)$ is a *symmetry transformation* of the differential equation if it transforms solutions into solutions. A *gauge transformation* is a family of symmetry transformations depending on an arbitrary function of time.

It is usually easier to study infinitesimal symmetry transformations δq (and at the end integrate them). Let us explain the meaning of this statement. For instance one may have $\delta q = A(t, q, \dot{q}, \ddot{q}, \dots)$. Then the (finite) transformation of a path q_0 generated by this infinitesimal transformation is the one-parameter family (q_ε) which is the solution of the partial differential equation

$$\frac{\partial q_\varepsilon(t)}{\partial \varepsilon} = A(t, q_\varepsilon(t), \dot{q}_\varepsilon(t), \ddot{q}_\varepsilon(t), \dots)$$

with initial condition q_0, provided that this solution exists. This is a generalization of the case of point transformations, those where δq depends only on (t, q).

For a gauge transformation the dependence of δq on the arbitrary function $\varphi(t)$ is usually as a local functional:

$$\delta q = A(t, q, \dot{q}, \ddot{q}, \dots ; \varphi, \dot{\varphi}, \ddot{\varphi}, \dots).$$

Of course, these concepts may be given a geometric expression. For a point transformation, the paths are transformed by the flow of a certain vector field in a manifold Q (or $\mathbf{R} \times Q$). For a generalized transformation, if we denote by $\pi\colon \mathbf{R} \times Q \to \mathbf{R}$ the trivial fibration, then δq is represented by a π-vertical vector field along the projection $\mathbf{R} \times \mathrm{T}^l Q \to \mathbf{R} \times Q$ for a certain $l \in \mathbf{N}$. However, in this paper we will not insist in these geometric questions.

There are some interesting cases of symmetry transformations when dealing with lagrangian dynamics. In this paper we will be mainly concerned with Noether's transformations, but comparison with hamiltonian symmetry transformations is always interesting.

An infinitesimal *Noether's transformation* of a lagrangian L is an infinitesimal transformation δq such that the variation of L under this transformation is a total derivative:

$$\delta L = \mathrm{D}_t F$$

[1] Indices of coordinates will be omitted

for a certain F [Noe 18] [Bes 21] [Olv 86]. Here $\mathrm{D}_t = \dfrac{\partial}{\partial t} + \sum_i q_{i+1} \dfrac{\partial}{\partial q_i}$ is the total time-derivative, and the variation of L is $\delta L = \dfrac{\partial L}{\partial q}\delta q + \dfrac{\partial L}{\partial \dot q}\delta \dot q + \dots$, where for instance $\delta \dot q = \mathrm{D}_t \delta q$ and so on. It can be shown that a Noether's symmetry is a symmetry transformation of the Euler-Lagrange equations of L.

The relation above can be equivalently written

$$[L]\delta q + \mathrm{D}_t G = 0,$$

where $[L] = 0$ are the Euler-Lagrange equations of L,

$$[L] = \frac{\partial L}{\partial q} - \mathrm{D}_t \frac{\partial L}{\partial \dot q} + \dots ,$$

and G is related to F through a certain expression. Notice that this second definition of a Noether's transformation is a conservation law: the function G is a constant of motion.

On the other hand, for the hamiltonian formalism it is interesting to study symmetry transformations generated by a function, *i.e.*, $\delta q = \{q, G\}$ and similarly for δp. In the case of a singular lagrangian there is a correspondence between first-class primary hamiltonian constraints and generators of hamiltonian gauge symmetries —see [GP 88] for instance— but this correspondence may fail.

In [GP 92] it is shown that, at least in certain examples, our characterization of Noether's transformations can be used to construct lagrangian gauge transformations for systems not possessing hamiltonian gauge generators; this observation can be extended to higher order lagrangians [GP 95].

3. The case of first order lagrangians

Let us consider an autonomous first order lagrangian, perhaps singular, $L\colon \mathrm{T}Q \to \mathbf{R}$. The well-known Legendre's map $\mathcal{F}L$ of L connects the velocity space $V = \mathrm{T}Q$ with the phase space $P = \mathrm{T}^*Q$. Its local expression (in natural coordinates) is

$$\mathcal{F}L(q, v) = (q, \hat p),$$

where $\hat p = \partial L / \partial v$ are the momenta.

The time-evolution operator K connecting velocity and phase spaces [BGPR 86] is less known. Following [GP 89], it is the vector field along the

Legendre's map that satisfies certain conditions (namely, it satisfies a "second order condition" and the pull-back of its contraction with the canonical symplectic form of P is the differential of the lagrangian energy). Its local expression is

$$K(q,v) = v\frac{\partial}{\partial q} + \frac{\partial L}{\partial q}\frac{\partial}{\partial p}.$$

This operator can be used to:

- write the Euler-Lagrange equation intrinsically,
- construct all the lagrangian constraints from the hamiltonian ones,
- characterize the generators of hamiltonian symmetry transformations, and
- characterize the conserved quantities of Noether's transformations.

Our main interest in this paper is the last of these applications. However, let us comment briefly on the other ones.

3.1. Equations of motion Let ξ be a path in V. Let us consider the differential equation [GP 89]

$$\mathrm{T}(\mathcal{F}L) \circ \dot{\xi} = K \circ \xi.$$

In other words, the following diagram must be commutative:

$$\begin{tikzcd}
 & \mathrm{T}(V) \arrow[r, "\mathrm{T}(\mathcal{F}L)"] \arrow[d] & \mathrm{T}(P) \arrow[d] \\
I \arrow[ur, "\dot{\xi}"] \arrow[r, "\xi"'] & V \arrow[ur, "K"] \arrow[r, "\mathcal{F}L"'] & P
\end{tikzcd}$$

In coordinates, if ξ is represented by $(q(t), v(t))$, this reads

$$\dot{q} = v, \qquad \frac{\partial^2 L}{\partial v\, \partial q}\dot{q} + \frac{\partial^2 L}{\partial v\, \partial v}\dot{v} = \frac{\partial L}{\partial q},$$

that is to say, the Euler-Lagrange equation for ξ.

This equation can be also expressed in terms of vector fields[2]:

$$\mathrm{T}(\mathcal{F}L) \circ X \underset{N}{\simeq} K.$$

[2] $f \underset{N}{\simeq} 0$ means $f = 0$ on N.

This means that, if X is a vector field on V tangent to the submanifold N, then X satisfies the equation above if and only if its integral curves satisfy the Euler-Lagrange equation.

3.2. CONSTRAINTS From now on let us assume that L is a singular lagrangian, that is to say, the Legendre's map is not a local diffeomorphism. In coordinates, this means that the hessian matrix $W = \dfrac{\partial^2 L}{\partial v\,\partial v}$ is singular.

Then the Euler-Lagrange equation is singular, and existence and uniqueness of solutions may fail: the solutions cover only a subset $V^{(f)}$ of V, and many solutions may pass through a point.

On the other hand, the definition of the hamiltonian formalism is more difficult. Then, assuming certain regularity conditions, one may use Dirac's theory, which in more geometric terms amounts to consider the hamiltonian formalism as a presymplectic system on the primary hamiltonian constraint submanifold $P^{(1)} = \mathcal{F}L(V) \subset P$. Then also the solutions cover only a subset $P^{(f)}$ of P, and many solutions may pass through a point.

The subsets $V^{(f)} \subset V$ and $P^{(f)} \subset P$ can be described by constraints. An important result is that the lagrangian constraints χ can be obtained by applying the operator K to the hamiltonian constraints ϕ [Pon 88]:

$$\chi = K \cdot \phi.$$

Here K, as any vector field along a map, acts as a differential operator on functions. In coordinates this reads $\chi(q,v) = v\,\mathcal{F}L^*\left(\dfrac{\partial \phi}{\partial q}\right) + \dfrac{\partial L}{\partial q}\,\mathcal{F}L^*\left(\dfrac{\partial \phi}{\partial p}\right)$.

Let us consider in particular the primary hamiltonian constraints ϕ_μ, which define the primary hamiltonian constraint submanifold $P^{(1)} \subset P$. The vectors $\gamma_\mu = \mathcal{F}L^*\left(\dfrac{\partial \phi_\mu}{\partial p}\right)$ constitute a basis of the kernel of the hessian W, and the primary lagrangian constraints —the first compatibility conditions arising from the Euler-Lagrange equations— are

$$\chi_\mu = K \cdot \phi_\mu = \left(\frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q\,\partial v}\right)\gamma_\mu;$$

these constraints define the primary lagrangian constraint submanifold $V^{(1)} \subset V$.

It is also worth noticing that the primary hamiltonian constraints allow to construct a frame (Γ_μ) for $\operatorname{Ker} \mathrm{T}(\mathcal{F}L)$; in coordinates these read

$$\Gamma_\mu = \gamma_\mu \frac{\partial}{\partial v}.$$

These vector fields are useful to test the (local) projectability of a function in velocity space to a function in phase space.

3.3. GENERATORS OF HAMILTONIAN SYMMETRY TRANSFORMATIONS Let us only state the main result [GP 88]: a function $G_H(t,q,p)$ in phase space generates (through Poisson bracket) a hamiltonian symmetry transformation if and only if[3]

$$K \cdot G_H \underset{V^{(f)}}{\cong} 0,$$

where $V^{(f)}$ is the final lagrangian constraint submanifold. This condition can be expressed in another equivalent condition; both are useful to find hamiltonian gauge transformations.

3.4. NOETHER'S TRANSFORMATIONS Let us consider an infinitesimal Noether's transformation for a first order autonomous lagrangian, for which $F + G = \hat{p}\delta q$. We have the following result [GP 92]:

THEOREM 1. *Let $\delta q(t,q,v)$ be a Noether's transformation with conserved quantity $G_L(t,q,v)$. Then G_L is projectable to a function $G_H(t,q,p)$ such that*

$$K \cdot G_H \underset{V^{(1)}}{\simeq} 0.$$

Conversely, given a function $G_H(t,q,p)$ such that

$$K \cdot G_H = -\sum_{\mu} r^{\mu} \chi_{\mu}$$

(a combination of primary lagrangian constraints) then

$$\delta q = \mathcal{F}L^* \left(\frac{\partial G_H}{\partial p} \right) + \sum_{\mu} r^{\mu} \gamma_{\mu}$$

is a Noether's transformation with conserved quantity $G_L = \mathcal{F}L^*(G_H)$.

Before discussing the meaning of this result and sketching its proof, let us remark that here the operator K acts on a time-dependent function and so it carries an extra term $\partial/\partial t$; of course, in this context K can be regarded as a vector field along the time-dependent Legendre's map $\mathcal{F}L \colon \mathbf{R} \times \mathrm{T}Q \to \mathbf{R} \times \mathrm{T}^*Q$ —see [CFM 94] for some geometric aspects of this extension.

[3] $f \underset{N}{\cong} 0$ means $f \underset{N}{\simeq} 0$ and $df \underset{N}{\simeq} 0$.

Notice that, whatever be the lagrangian —assuming the usual mild regularity conditions that allow the construction of the hamiltonian formalism— the function G_L is projectable through $\mathcal{F}L$. However, δq is not necessarily projectable, and indeed its projectability is equivalent to the projectability of the functions r^μ; if this is satisfied then it can be proved that, after redefinition of G_H, this function generates a hamiltonian symmetry transformation.

On the other hand, there is some indetermination in the conserved quantity G_L of δq; see [GP 94] for a discussion about this question.

The proof of the theorem is quite simple. First, one must take the conservation law deduced from δq and look at the coefficient of the acceleration to obtain $\Gamma_\mu \cdot G_L = 0$ and conclude the projectability of G_L, $G_L = \mathcal{F}L^*(G_H)$; this shows that $\delta q - \mathcal{F}L^*\,(\partial G_H/\partial p)$ is a null vector $\sum_\mu r^\mu \gamma_\mu$ of the hessian W. Second, one must introduce this expression into the conservation law and use the basic relation

$$K \cdot g = [L]\,\mathcal{F}L^*\left(\frac{\partial g}{\partial p}\right) + \mathrm{D}_t\,\mathcal{F}L^*(g)$$

(g may be time-dependent) and the expression of the primary lagrangian constraints.

4. The case of higher order lagrangians

Here some results and notation from [GPR 91] will be presented. See also [LR 85] [BGPR 88].

We consider a kth order lagrangian $L\colon \mathrm{T}^kQ \to \mathbf{R}$. The Euler-Lagrange equations $[L] = 0$ can be written as a first order equation in $\mathrm{T}^{2k-1}Q$ (this is what will be called the lagrangian formalism of L), and there is a hamiltonian formulation of the theory in $\mathrm{T}^*(\mathrm{T}^{k-1}Q)$. Both spaces are connected through the Legendre-Ostrogradskiĭ's map

$$\mathcal{F}L(q_0, \dots, q_{2k-1}) = (q_0, \dots, q_{k-1}; \hat{p}^0, \dots, \hat{p}^{k-1}),$$

where the Ostrogradskiĭ's momenta are defined by

$$\hat{p}^i = \sum_{j=0}^{k-i-1} (-1)^j \mathrm{D}_t^j \left(\frac{\partial L}{\partial q_{i+j+1}}\right).$$

Equivalently,

$$\hat{p}^{k-1} = \frac{\partial L}{\partial q_k}, \qquad \hat{p}^{i-1} = \frac{\partial L}{\partial q_i} - \mathrm{D}_t \hat{p}^i.$$

Notice that $\hat{p}^i$ depends only on $q_0, \ldots, q_{2k-1-i}$.

Introducing the momenta step-by-step, for $0 \leq r \leq k$ an intermediate space P_r can be defined, with coordinates $(q_0, \ldots, q_{2k-1-r}; p^0, \ldots, p^{r-1})$. In particular, the lagrangian and hamiltonian spaces are $P_0 = \mathrm{T}^{2k-1}Q$ and $P_k = \mathrm{T}^*(\mathrm{T}^{k-1}Q)$.

Then the Legendre-Ostrogradskiĭ's map is decomposed into partial Ostrogradskiĭ's maps $\mathcal{F}_r \colon P_r \to P_{r+1}$, with local expression

$$\mathcal{F}_r(q_0, \ldots, q_{2k-1-r}; p^0, \ldots, p^{r-1}) = (q_0, \ldots, q_{2k-2-r}; p^0, \ldots, p^{r-1}, \hat{p}^r) :$$

$$\begin{array}{ccccccccccc}
\mathrm{T}^{2k-1}Q & & & & \xrightarrow{\qquad\qquad\qquad\mathcal{F}L\qquad\qquad\qquad} & & & & & & \mathrm{T}^*(\mathrm{T}^{k-1}Q) \\
\| & & & & & & & & & & \| \\
P_0 & \xrightarrow{\mathcal{F}_0} & P_1 & \longrightarrow & \cdots & \longrightarrow & P_r & \xrightarrow{\mathcal{F}_r} & P_{r+1} \longrightarrow \cdots & \xrightarrow{\mathcal{F}_{k-1}} & P_k
\end{array}$$

Now let us introduce the intermediate time-evolution operator K_r, which is a vector field along $\mathcal{F}_r$ satisfying certain conditions similar to those satisfyed by K in the first order case. In coordinates it reads

$$\begin{aligned}
K_r \;=\;& q_1 \frac{\partial}{\partial q_0} + \ldots + q_{2k-1-r} \frac{\partial}{\partial q_{2k-2-r}} + \\
& + \left(\frac{\partial L}{\partial q_0}\right) \frac{\partial}{\partial p^0} + \left(\frac{\partial L}{\partial q_1} - p^0\right) \frac{\partial}{\partial p^1} + \ldots + \left(\frac{\partial L}{\partial q_r} - p^{r-1}\right) \frac{\partial}{\partial p^r}.
\end{aligned}$$

All these operators are connected by $\mathrm{T}(\mathcal{F}_r) \circ K_{r-1} = K_r \circ \mathcal{F}_{r-1}$.

These operators are so as useful as in the first order case: they define dynamics in the intermediate spaces P_r, they relate the corresponding constraints, and are also useful to obtain hamiltonian symmetry generators and Noether's transformations. As before, let us commment on these points, and especially on the last one.

4.1. Equations of motion Given a path ξ_r in P_r $(0 \leq r < k)$, we can consider the differential equation

$$\mathrm{T}(\mathcal{F}_r) \circ \dot{\xi}_r = K_r \circ \xi_r,$$

that is to say, the following diagram must be commutative:

$$\begin{tikzcd}
 & \mathrm{T}(P_r) \arrow[r, "\mathrm{T}(\mathcal{F}_r)"] \arrow[d] & \mathrm{T}(P_{r+1}) \arrow[d] \\
I \arrow[ur, "\dot{\xi}_r"] \arrow[r, "\xi_r"'] & P_r \arrow[r, "\mathcal{F}_r"'] \arrow[ur, "K_r"] & P_{r+1}
\end{tikzcd}$$

Alternatively, one can consider the equation of motion in terms of a vector field X_r in P_r:

$$\mathrm{T}(\mathcal{F}_r) \circ X_r \simeq K_r.$$

All these dynamics are equivalent to the Euler-Lagrange equations.

4.2. CONSTRAINTS Now let us assume that L is a singular lagrangian: $\mathcal{F}L$ (or any of the $\mathcal{F}_r$) is not a local diffeomorphism. In coordinates, this means that the hessian matrix with respect to the highest order derivatives, $W = \dfrac{\partial^2 L}{\partial q_k \, \partial q_k}$, is not invertible. In this case, assuming as usual certain mild regularity conditions, it is still possible to construct the hamiltonian formulation in P_k. Then the equations of motion in each space P_r yield constraints, and it can be proved that a basis for the constraints of P_r can be obtained by applying K_r to the constraints of P_{r+1}.

For instance, if $P_k^{(1)} = \mathcal{F}_{k-1}(P_{k-1}) \subset P_k$ is the primary hamiltonian constraint submanifold, it can be defined through some primary hamiltonian constraints ϕ_k^μ. Then application of K_{k-1} to them yields the primary constraints of P_{k-1}, and this proceeds in the same way until the primary lagrangian constraints ϕ_0^μ, which define the primary lagrangian constraint submanifold $P_0^{(1)} \subset P_0$. This is also true for the secondary constraints and so on.

As for first order lagrangians, the primary constraints yield a basis for $\operatorname{Ker} W$:

$$\gamma_\mu = \mathcal{F}_{k-1}^* \left(\frac{\partial \phi_k^\mu}{\partial p^{k-1}} \right).$$

(Notice that γ_μ depends only on $(q_0, \ldots, q_k)$.) Then, a basis for $\operatorname{Ker} \mathrm{T}(\mathcal{F}_r)$ is constituted by the vector fiels

$$\Gamma_\mu^r = \gamma_\mu \frac{\partial}{\partial q_{2k-1-r}},$$

which can be used to test the projectability of a function in P_r to P_{r+1}.

It is interesting to notice that the primary lagrangian constraints can be expressed also as

$$\phi_0^\mu = K_0 \cdot \phi_1^\mu = (-1)^{k-1}[L]\gamma_\mu = (-1)^{k-1}\alpha\gamma_\mu,$$

where we use the notation provided by separing the highest order derivative in the Euler-Lagrange equations:

$$[L] = \sum_{r=0}^{k}(-1)^r \mathrm{D}_t^r \left(\frac{\partial L}{\partial q_r}\right) = \frac{\partial L}{\partial q_0} - \mathrm{D}_t \hat{p}_0 = \alpha - (-1)^{k-1} q_{2k} W.$$

4.3. GENERATORS OF HAMILTONIAN SYMMETRY TRANSFORMATIONS Given a function $G_H(t, q_0, \ldots, q_{k-1}, p^0, \ldots, p^{k-1})$ in phase space, the necessary and sufficient condition for it to generate a hamiltonian symmetry transformation is

$$K_{k-1} \cdot G_H \underset{P_{k-1}^{(f)}}{\cong} 0.$$

This result is similar to that for first order lagrangians, but it uses K_{k-1} instead of K.

4.4. NOETHER'S TRANSFORMATIONS Let us consider again the two definitions of a Noether's transformation; for a kth order lagrangian, the relation between the functions F and G is $F + G = \sum_{r=0}^{k-1} \hat{p}^r \delta q_r$. Now the main result is [GP 95]:

THEOREM 2. *Let $\delta q(t, q_0, \ldots, q_{2k-1})$ be a Noether's transformation with conserved quantity G_L. Then G_L is projectable to a function G_I in P_1 such that*

$$K_0 \cdot G_I \underset{P_0^{(1)}}{\simeq} 0,$$

where $P_0^{(1)}$ is the primary lagrangian constraint submanifold. Conversely, given a function $G_I(t, q_0, \ldots, q_{2k-2}, p^0)$ such that

$$K_0 \cdot G_I = -\sum_\mu r^\mu (\alpha\gamma_\mu)$$

(a combination of primary lagrangian constraints) then

$$\delta q = \mathcal{F}_0^* \left(\frac{\partial G_I}{\partial p^0}\right) + \sum_\mu r^\mu \gamma_\mu$$

is a Noether's transformation with conserved quantity $G_L = \mathcal{F}_0^*(G_I)$.

As for first order lagrangians, notice that here K_0 acts on a time-dependent function, and so it contains the term $\partial/\partial t$. Notice also that there is some indetermination in the functions r^μ, and this allows also the existence of Noether's transformations with vanishing conserved quantity.

However, now there is no guarantee that the conserved quantity G_L be projectable to the hamiltonian space P_k. Moreover, even in the case that both the conserved quantity G_L and the coefficients of the Noether's transformation δq are projectable to P_k, it could happen that any of the possible functions G_H which are projection of G_L generates a hamiltonian symmetry transformation.

In conclusion, in contrast to the first order case, where there are two possibilities for a Noether's transformation according to its projectability to phase space, for a higher order lagrangian there are several different possibilities according to the degree of projectability of the coefficients of a Noether's transformation and its conserved quantity.

Finally, let us sketch the proof of the theorem. The highest order derivative, q_{2k}, in

$$[L]\delta q + \mathrm{D}_t G_L = 0$$

appears linearly, and its coefficient is

$$(-1)^k W \delta q - \frac{\partial G_L}{\partial q_{2k-1}} = 0;$$

so, contraction with the null vectors γ_μ shows that $G_L = \mathcal{F}_0^*(G_I)$ for a certain function G_I in P_1. Next we use the general relation

$$K_0 \cdot g = [L]\,\mathcal{F}_0^*\left(\frac{\partial g}{\partial p^0}\right) + \mathrm{D}_t\,\mathcal{F}_0^*(g)$$

and after these substitutions and looking again at the coefficient of q_{2k} in the resulting expression we conclude that $\delta q - \mathcal{F}_0^*\left(\frac{\partial G_I}{\partial p^0}\right)$ is a null vector of W:

$$\delta q - \mathcal{F}_0^*\left(\frac{\partial G_I}{\partial p^0}\right) = \sum_\mu r^\mu \gamma_\mu$$

for some $r^\mu(t, q_0, \dots, q_{2k-1})$. Using this expression we obtain

$$K_0 \cdot G_I + \sum_\mu r^\mu(\alpha\gamma_\mu) = 0.$$

All this reasoning can be inverted to obtain the converse.

5. An example

For a relativistic particle we consider a lagrangian proportional to the curvature of its world line [Pis 86] [BGPR 88] [GP 95],

$$L = \alpha \frac{\sqrt{\Delta_2}}{\Delta_1} = \alpha \frac{\sqrt{(\boldsymbol{x}_1\boldsymbol{x}_1)(\boldsymbol{x}_2\boldsymbol{x}_2) - (\boldsymbol{x}_1\boldsymbol{x}_2)^2}}{(\boldsymbol{x}_1\boldsymbol{x}_1)},$$

where α is a constant parameter. Here $\boldsymbol{x}(t)$ is in Minkowski space $\mathbf{R}^d$, we write $\boldsymbol{x}_n$ for its nth time-derivative, and

$$\Delta_n = \det((\boldsymbol{x}_i\boldsymbol{x}_j))_{1\le i,j\le n}.$$

We also put

$$\boldsymbol{e}_1 = \boldsymbol{x}_1, \qquad \boldsymbol{e}_2 = \boldsymbol{x}_2 - \frac{(\boldsymbol{x}_2\boldsymbol{e}_1)}{(\boldsymbol{e}_1\boldsymbol{e}_1)}\boldsymbol{e}_1, \qquad \boldsymbol{e}_3 = \boldsymbol{x}_3 - \frac{(\boldsymbol{x}_3\boldsymbol{e}_2)}{(\boldsymbol{e}_2\boldsymbol{e}_2)}\boldsymbol{e}_2 - \frac{(\boldsymbol{x}_3\boldsymbol{e}_1)}{(\boldsymbol{e}_1\boldsymbol{e}_1)}\boldsymbol{e}_1.$$

The partial Ostrogradskiĭ's transformations are

$$\begin{array}{ccccc} P_0 = \mathrm{T}^3(\mathbf{R}^d) & \xrightarrow{\mathcal{F}_0} & P_1 & \xrightarrow{\mathcal{F}_1} & P_2 = \mathrm{T}^*(\mathrm{T}(\mathbf{R}^d)) \\ (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) & \mapsto & (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \hat{\boldsymbol{p}}^0) & & \\ & & (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{p}^0) & \mapsto & (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{p}^0, \hat{\boldsymbol{p}}^1), \end{array}$$

where the momenta are defined by

$$\begin{aligned} \hat{\boldsymbol{p}}^1 &:= \frac{\partial L}{\partial \boldsymbol{x}_2} = \frac{\alpha}{\sqrt{\Delta_2}}\boldsymbol{e}_2, \\ \hat{\boldsymbol{p}}^0 &:= \frac{\partial L}{\partial \boldsymbol{x}_1} - \mathrm{D}_t\hat{\boldsymbol{p}}^1 = -\frac{\alpha}{\sqrt{\Delta_2}}\boldsymbol{e}_3. \end{aligned}$$

To be precise, P_0 is not all $\mathrm{T}^3(\mathbf{R}^d)$, but the open subset defined by $\Delta_1 > 0$, $\Delta_2 > 0$. Similarly for P_1 and P_2.

The lagrangian is singular since the hessian matrix

$$W_{\mu\nu} := \frac{\partial^2 L}{\partial x_2^\mu\, \partial x_2^\nu} = \frac{\alpha}{\sqrt{\Delta_2}}\left(\eta_{\mu\nu} - \frac{e_{1\mu}e_{1\nu}}{(\boldsymbol{e}_1\boldsymbol{e}_1)} - \frac{e_{2\mu}e_{2\nu}}{(\boldsymbol{e}_2\boldsymbol{e}_2)}\right)$$

has rank $d-2$.

The intermediate evolution operators are

$$\begin{aligned}
K_1 &:= \boldsymbol{x}_1\frac{\partial}{\partial \boldsymbol{x}_0} + \boldsymbol{x}_2\frac{\partial}{\partial \boldsymbol{x}_1} + \frac{\partial L}{\partial \boldsymbol{x}_0}\frac{\partial}{\partial \boldsymbol{p}^0} + \left(\frac{\partial L}{\partial \boldsymbol{x}_1} - \boldsymbol{p}^0\right)\frac{\partial}{\partial \boldsymbol{p}^1} \\
&= \boldsymbol{x}_1\frac{\partial}{\partial \boldsymbol{x}_0} + \boldsymbol{x}_2\frac{\partial}{\partial \boldsymbol{x}_1} - \left(\boldsymbol{p}^0 + \frac{\alpha}{\sqrt{\Delta_2}}\left(\frac{\Delta_2}{\Delta_1^2}\boldsymbol{e}_1 + \frac{\dot{\Delta}_1}{2\Delta_1}\boldsymbol{e}_2\right)\right)\frac{\partial}{\partial \boldsymbol{p}^1}, \\
K_0 &:= \boldsymbol{x}_1\frac{\partial}{\partial \boldsymbol{x}_0} + \boldsymbol{x}_2\frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{x}_3\frac{\partial}{\partial \boldsymbol{x}_2} + \frac{\partial L}{\partial \boldsymbol{x}_0}\frac{\partial}{\partial \boldsymbol{p}^0} \\
&= \boldsymbol{x}_1\frac{\partial}{\partial \boldsymbol{x}_0} + \boldsymbol{x}_2\frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{x}_3\frac{\partial}{\partial \boldsymbol{x}_2},
\end{aligned}$$

and the Euler-Lagrange equations are

$$[L] := \frac{\partial L}{\partial \boldsymbol{x}_0} - \mathrm{D}_t\hat{\boldsymbol{p}}^0 = -\mathrm{D}_t\hat{\boldsymbol{p}}^0 = 0.$$

As a hamiltonian we can take

$$H = (\boldsymbol{p}^0\boldsymbol{x}_1),$$

and there are two primary constraints in the hamiltonian space P_2:

$$\phi_2^1 = (\boldsymbol{p}^1\boldsymbol{x}_1), \qquad \psi_2^1 = \frac{1}{2}\left((\boldsymbol{p}^1\boldsymbol{p}^1) - \frac{\alpha^2}{(\boldsymbol{x}_1\boldsymbol{x}_1)}\right).$$

Proceeding with the hamiltonian stabilization we obtain secondary constraints

$$\phi_2^2 = \{\phi_2^1, H\} = -(\boldsymbol{p}^0\boldsymbol{x}_1), \qquad \psi_2^2 = \{\psi_2^1, H\} = -(\boldsymbol{p}^0\boldsymbol{p}^1)$$

and a tertiary constraint

$$\psi_2^3 = \{\psi_2^2, H\} = (\boldsymbol{p}^0\boldsymbol{p}^0).$$

All them are first-class.

The constraints in P_1 are obtained by applying the operator K_1 to the hamiltonian constraints. There are only three independent constraints remaining: ϕ_1^1, ψ_1^1 and ψ_1^2.

Similarly from the intermediate constraints we obtain the lagrangian constraints by application of K_0. Then we have only $\psi_0^1 = K_0 \cdot \psi_1^1$.

Finally we obtain the null vectors of the hessian:

$$\gamma_\phi := \mathcal{F}_1^*\left(\frac{\partial \phi_2^1}{\partial \boldsymbol{p}^1}\right) = \boldsymbol{x}_1, \qquad \gamma_\psi := \mathcal{F}_1^*\left(\frac{\partial \psi_2^1}{\partial \boldsymbol{p}^1}\right) = \hat{\boldsymbol{p}}^1.$$

Once all these elements have been obtained, one may ask for the symmetry transformations. The results are the following [GP 95].

On one hand, the model does not have any hamiltonian gauge symmetry generator, in spite of having two primary first-class hamiltonian constraints.

On the other hand, the model has two independent gauge Noether's transformations. They can be obtained through our characterization, looking for a function G_I combination of secondary constraints of P_1, and they are

$$\begin{aligned} \delta \boldsymbol{x} &= \varphi(t)\boldsymbol{x}_1, \\ \delta \boldsymbol{x} &= 2\varphi(t)\hat{\boldsymbol{p}}^0 + \dot{\varphi}(t)\hat{\boldsymbol{p}}^1, \end{aligned}$$

where φ is an arbitrary function of time.

These transformations and their generating functions G_I are indeed projectable to the hamiltonian space; however, they do not yield hamiltonian gauge transformations.

Some other examples of computation of Noether's gauge transformations for first order lagrangians can be found in [GP 92].

References

[BGPR 86] Batlle, C., Gomis, J., Pons, J.M., Román-Roy, N., Equivalence between the lagrangian and hamiltonian formalism for constrained systems, *J. Math. Phys.,* **27** (1986), 2953–2962.

[BGPR 88] Batlle, C., Gomis, J., Pons, J.M., Román-Roy, N., Lagrangian and hamiltonian constraints for second-order singular lagrangians, *J. Phys. A: Math. Gen.,* **21** (1988), 2693–2703.

[Bes 21] Bessel-Hagen, E., Uber die Erhaltungssätze der elektrodynamik, *Math. Ann.,* **84** (1921), 258–276.

[CF 93] Cariñena, J.F., Fernández-Núñez. J., Geometric theory of time-dependent singular lagrangians, *Fortschrit. Phys.,* **41** (1993), 517–552.

[CFM 94] Cariñena, J.F., Fernández-Núñez, J., Martínez, E., Time-dependent K-operator for singular lagrangians, preprint Universidad de Zaragoza, 1994.

[CLM 89] Cariñena, J.F., López, C., Martínez, E., A new approach to the converse of Noether's theorem *J. Phys. A: Math. Gen.,* **22** (1989), 4777–4786.

[CLM 91] Cariñena, J.F., López, C., Martínez, E., Sections along a map applied to higher-order lagrangian mechanics. Noether's theorem, *Acta Appl. Math.,* **25** (1991), 127–151.

[GP 88] Gràcia, X., Pons, J.M., Gauge transformations, Dirac's conjecture and degrees of freedom for constrained systems, *Ann. Phys. (N.Y.),* **187** (1988), 355–368.

[GP 89] GRÀCIA, X., PONS, J.M., On an evolution operator connecting lagrangian and hamiltonian formalisms, *Lett. Math. Phys.,* **17** (1989), 175-180.

[GP 92] GRÀCIA, X., PONS, J.M., A hamiltonian approach to lagrangian Noether transformations, *J. Phys. A: Math. Gen.,* **25** (1992), 6357–6369.

[GP 94] GRÀCIA, X., PONS, J.M., Noether transformations with vanishing conserved quantity, *Ann. Inst. H. Poincaré A,* **61** (1994), 315-327.

[GP 95] GRÀCIA, X., PONS, J.M., Gauge transformations for higher order lagrangians, *J. Phys. A: Math. Gen.,* **28** (1995), 7181-7196.

[GPR 91] GRÀCIA, X., PONS, J.M., ROMÁN-ROY, N., Higher-order lagrangian systems: geometric structures, dynamics, and constraints, *J. Math. Phys.,* **32** (1991), 2744-2763.

[LR 85] DE LEÓN, M., RODRIGUES, P.R., "Generalized Classical Mechanics and Field Theory", North-Holland Math. Studies 112, Elsevier, Amsterdam, 1985.

[Noe 18] NOETHER, A.E., Invariante variationsprobleme *Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl.,* (1918), 235-257.

[Olv 86] OLVER, P.J., "Applications of Lie Groups to Differential Equations", 2nd ed., GTM 107, Springer, New York, 1993.

[Pis 86] PISARSKI, R.D., Field theory of paths with a curvature-dependent term *Phys. Rev. D,* **34** (1986), 670-673.

[Pon 88] PONS, J.M., New relations between hamiltonian and lagrangian constraints, *J. Phys. A: Math. Gen.,* **21** (1988), 2705-2715.