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# The Fractional Langevin Equation: Brownian Motion Revisited

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## INTRODUCTION

It is well known that the concept of diffusion is associated with random motion of particles in space, usually denoted as Brownian motion, see e.g. [1-3]. Diffusion is considered normal when the mean squared displacement of the particle during a time interval becomes, for sufficiently long intervals, a linear function of it. When this linearity breaks down, degenerating in a power law with exponent $\alpha > 0$ different from one, diffusion is referred to be anomalous: slow if $0 < \alpha < 1$, fast if $\alpha > 1$; see e.g. [4].

According to the classical approach started by Langevin and known as the Einstein-Ornstein-Uhlenbeck theory of Brownian motion, normal diffusion and Brownian motion are associated with Langevin equation. More specifically, the classical Langevin equation addresses the dynamics of a Brownian particle through Newton's law by incorporating the effect of the Stokes fluid friction and that of thermal fluctuations in the vicinity of the particle into a random force with suitably assigned properties. These properties are derived from the requirement that the particle velocity asymptotically attains a stationary Maxwellian distribution. Over the period of the diffusing particle, the random force arising from molecular collisions undergoes such a rapid fluctuations that it is approximated well by a white noise. For large time intervals $t$, it emerges that the mean squared displacement becomes proportional to $t$ with the diffusion coefficient being a half of the proportionality constant, in the one dimensional case.

In Section 1 we summarize the salient mathematical aspects of the classical Langevin equation, showing the exponential time decay of the velocity correlation function and the linear long-time behaviour of the mean squared

displacement. We also consider the generalized version of the Langevin equation introduced by Kubo [2] to account for a general retarded effect of the frictional force and the two fluctuation-dissipation theorems by Kubo. In Section 2, following the approach originally started by Widom [5] and Case [6] and resumed in [3-4], we shall consider the modification of the Langevin equation on the basis of hydrodynamics, which takes into account the added mass and the Basset-Boussinesq retarding force. We shall improve the analysis of the previous authors, interpreting the retarding force in the framework of *Fractional Calculus* and providing the analytical expressions of the autocorrelation functions (both for velocity and random force) and of the mean squared displacement. We shall conclude noting that, for "not heavy" Brownian particles, there is the possibility for anomalous diffusion, with $\alpha > 1$, in a long time interval, before the normal diffusion is established.

## 1. The Classical and Genelarized Langevin Equations

According to the classical Langevin approach the dynamics in one dimension for a Brownian particle is described by

$$\frac{dX}{dt} = V \tag{1.1}$$

$$m\,\frac{dV}{dt} = F \tag{1.2}$$

where $m$ is the particle mass, $X = X(t)$, $V = V(t)$ are the particle position and velocity, and $F$ is the force acting on the particle from molecules of the fluid surrounding the Brownian particle.

The force $F$ may be divided into two parts. The first part is the *frictional force* and is taken to be proportional to the particle velocity, i.e.

$$F_v = -\frac{m}{\sigma}\,V\,, \tag{1.3}$$

where $1/\sigma$ is the friction coefficient for unit mass. One usually introduces the *mobility coefficient* as

$$\mu := \sigma/m\,. \tag{1.4}$$

If the Stokes law is assumed for a spherical particle of radius $a$, see e.g. [7], we have

$$1/\mu = 6\pi\,a\,\rho_f\,\nu\,, \tag{1.5}$$

where $\rho_f$ and $\nu$ are the density and the kinematic viscosity of the fluid, respectively. Introducing the characteristic parameters

$$T_* := a^2/\nu\,, \quad \gamma := \rho_p/\rho_f\,, \tag{1.6}$$

where $\rho_p$ is the particle density, we obtain

$$\frac{1}{\mu} = \frac{9}{2\gamma}\frac{m}{T_*} \Longrightarrow \sigma := \frac{2}{9}\gamma T_*\,. \tag{1.7}$$

The second part of the force, arising from rapid thermal fluctuations, is regarded as random, independent of the motion of the particle. This part is called the *random force* and is hereafter denoted by $R(t)$.

Then (1.2) is written as a stochastic equation as

$$\frac{dV}{dt} = -\frac{1}{\sigma}V(t) + \frac{1}{m}R(t)\,, \tag{1.8}$$

and it is referred to as the *classical Langevin equation.*

It is assumed that the stochastic processes $V(t)$ and $R(t)$ be *stationary.* This means that the respective *autocorrelation functions* $C_V$ and $C_R$,

$$C_V(t_0,t) := \langle\, V(t_0)\, V(t_0+t)\,\rangle = C_V(t)\,, \quad t>0\,, \tag{1.9}$$

and

$$C_R(t_0,t) := \langle\, R(t_0)\, R(t_0+t)\,\rangle = C_R(t)\,, \quad t>0\,, \tag{1.10}$$

do not depend on $t_0$. Hereafter we will assume $t_0 = 0$.

As a consequence, because of the Wiener-Khintchine theorem [3], the *power spectra* or *power spectral densities* $I_V(\omega)$ and $I_R(\omega)$ ($\omega \in \mathbb{R}$) are provided by the Fourier transforms of the respective autocorrelation functions. We write

$$\begin{aligned} I_V(\omega) = \widehat{C}_V(\omega) &:= \int_{-\infty}^{+\infty} C_V(t)\,\mathrm{e}^{-i\omega t}\,dt\,, \\ C_V(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} I_V(\omega)\,\mathrm{e}^{+i\omega t}\,d\omega\,, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} I_R(\omega) = \widehat{C}_R(\omega) &:= \int_{-\infty}^{+\infty} C_R(t)\,\mathrm{e}^{-i\omega t}\,dt\,, \\ C_R(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} I_R(\omega)\,\mathrm{e}^{+i\omega t}\,d\omega\,. \end{aligned} \tag{1.12}$$

We assume that any process $f(t)$ be causal, i.e. vanishing for $t < 0$, so that the Fourier transform $\widehat{f}(\omega)$ of $f(t)$ is related to the Laplace transform by the identity

$$\widehat{f}(\omega) = \overline{f}(s)|_{s=i\omega}\,, \quad \overline{f}(s) := \int_0^\infty \mathrm{e}^{-st}\, f(t)\, dt\,, \quad s \in \mathbb{C}\,. \tag{1.13}$$

It is assumed that the random force has zero mean and is uncorrelated to the particle velocity at initial time $t = 0$; in other words,

$$\langle\, R(t)\,\rangle = 0\,, \quad \langle\, V(0)\, R(t)\,\rangle = 0\,, \quad t > 0\,. \tag{1.14}$$

Furthermore, if the Brownian particle has been kept for a sufficiently long time in the fluid at (absolute) temperature $T$, the equipartition law

$$m\,\langle\, V^2(0)\,\rangle = k\,T\,, \tag{1.15}$$

where $k$ is the Boltzmann constant, is assumed for the energy distribution.

It can be shown (see below) that the previous assumptions lead to the following relevant results

$$C_V(t) = \langle\, V^2(0)\,\rangle\, \mathrm{e}^{-t/\sigma} = \frac{k\,T}{m}\, \mathrm{e}^{-t/\sigma}\,, \tag{1.16}$$

and

$$C_R(t) = \frac{m^2}{\sigma}\, \langle\, V^2(0)\,\rangle\, \delta(t) = \frac{m\,k\,T}{\sigma}\, \delta(t)\,. \tag{1.17}$$

The result (1.16) shows that the velocity autocorrelation function decays exponentially in time with the decay constant $\sigma$, while (1.17) means that the power spectrum of $R(t)$ is to be *white*, i.e. independent on frequency, resulting

$$I_R(\omega) \equiv I_R = \frac{m\,k\,T}{\sigma}\,. \tag{1.18}$$

The two results can be generalized for the so-called *generalized Langevin equation* introduced by Kubo [2],

$$\frac{dV}{dt} = -\int_0^t \gamma(t-\tau)\, V(\tau)\, d\tau + \frac{1}{m}\, R(t)\,, \tag{1.19}$$

where the function $\gamma(t)$ represents a retarded effect of the frictional force. For this case Kubo introduced two *fluctuation-dissipation theorems* that, using the Laplace transforms, read respectively

$$\overline{C}_V(s) = \frac{\langle V^2(0) \rangle}{s + \overline{\gamma}(s)}, \tag{1.20}$$

and

$$\overline{C}_R(s) = m^2 \langle V^2(0) \rangle \overline{\gamma}(s). \tag{1.21}$$

From the comparison between (1.8) and (1.19), we recognize that the classical case can be obtained from the generalized one interpreting the convolution in (1.19) in the generalized sense (see e.g. [8]) and putting

$$\gamma(t) = \frac{1}{\sigma}\,\delta(t) \iff \overline{\gamma}(s) = \frac{1}{\sigma}, \tag{1.22}$$

where $\delta(t)$ denotes the delta Dirac distribution.

In Appendix A we prove the statements (1.20) and (1.21), which thus reduce to the classical results (1.16) and (1.17) accounting for (1.22).

It can be readily shown that the mean squared displacement of a particle, starting at the origin at $t_0 = 0$, is given by

$$\langle X^2(t) \rangle = 2 \int_0^t (t - \tau)\, C_V(\tau)\, d\tau = 2 \int_0^t d\tau_1 \int_0^{\tau_1} C_V(\tau)\, d\tau. \tag{1.23}$$

For this it is sufficient to recall that $X(t) = \int_0^t V(t')\, dt'$, and to use the definition (1.9) of $C_V(t)$.

For the classical case, $C_V(t)$ is provided by (1.16), so that we obtain from (1.23)

$$\langle X^2(t) \rangle = 2 \langle V^2(0) \rangle \sigma \left[ t - \sigma \left( 1 - \mathrm{e}^{-t/\sigma} \right) \right]. \tag{1.24}$$

Introducing the *diffusion coefficient* $D$ as

$$D = \sigma \langle V^2(0) \rangle = \mu\, k\, T, \tag{1.25}$$

where we have used (1.4) and (1.15), we can deduce for large times the well-known property

$$\langle X^2(t) \rangle = 2D\, t \left\{ 1 + O\left[ (t/\sigma)^{-1} \right] \right\}, \quad \text{as} \quad t \to \infty. \tag{1.26}$$

The relationship stated in (1.25), which is called the *Einstein relation*, provides us with a very good basis of experimental verification that Brownian motion is in fact related to the thermal motion of molecules. We also note the following relevant results for the *diffusion coefficient*

$$D = \lim_{t\to\infty} \frac{\langle X^2(t)\rangle}{2\,t} = \int_0^\infty C_V(t)\,dt = \overline{C}_V(0)\,. \tag{1.27}$$

## 2. The Fractional Langevin Equation

On the basis of hydrodynamics the equation of motion (1.8) is not all correct since it ignores the effects of the added mass and of the retarded viscous force, which are due to the acceleration of the particle, see e.g. [5-7].

The added mass effect introduces a modification in the L.H.S. of (1.2) in that it requires to substitute the mass of the particle with the so-called effective mass, namely

$$m \to m_e := m + \frac{1}{2}\,m_f = m\left(1 + \frac{1}{2\gamma}\right), \tag{2.1}$$

where $\gamma = \rho_p/\rho_f$ according to (1.6). As a consequence, in order to do not change the mobility coefficient in the Stokes drag, we have to introduce $\sigma_e$ such that

$$\mu := \sigma/m = \sigma_e/m_e\,, \tag{2.2}$$

namely, recalling (1.6-7),

$$\sigma_e := \sigma\left(1 + \frac{1}{2\gamma}\right) = \frac{2\gamma+1}{9}\,T_*\,, \tag{2.3}$$

where $T_* = a^2/\nu$. With respect to the classical analysis, it turns out that the added mass effect, if it were present alone, would be only to lengthen the time scale ($\sigma \to \sigma_e > \sigma$), slowing down the exponential decay for the velocity correlation function (1.16) and for the mean square displacement (1.24), but without modifying the value of the diffusion coefficient $D$.

The retarded viscous force effect is due to an additional term to the Stokes drag, which is related to the history of the particle acceleration. This additional drag force, proposed independently by Boussinesq [9] and Basset [10] in earlier times, is nowadays referred to as the *Basset force*. As a consequence, the frictional force (1.3-5) is to be substituted as follows

$$F_v = -6\pi\,a\,\rho_f\,\nu\left\{V(t) + \frac{a}{\sqrt{\pi\nu}}\int_{-\infty}^{t}\frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}\,d\tau\right\}. \tag{2.4}$$

Using (1.6-7) and requiring the *causality* of the processes, we can re-write (2.4) as

$$F_v = -\frac{9}{2\gamma} m \left[ \frac{1}{T_*} V(t) + \frac{1}{\sqrt{T_*}} B(t) \right] , \tag{2.5}$$

where

$$B(t) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}} d\tau = \frac{1}{\Gamma(1/2)} \int_{0^-}^{t} \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}} d\tau . \tag{2.6}$$

The lower limit of the integral has been written as $0^-$ to account for the possible discontinuity in the velocity particle at $t = 0$. Basing on the *Fractional Calculus* recalled in Appendix B, we can write, see (B.11-12) and (B.6),

$$B(t) = D_0^{1/2} V(t) = \Phi_{-1/2}(t) * V(t) , \tag{2.7}$$

where $D_0^{1/2}$ denotes the fractional derivative of order 1/2 and

$$\Phi_{-1/2}(t) := \frac{t^{-3/2}}{\Gamma(-1/2)} = -\frac{t^{-3/2}}{2\sqrt{\pi}} . \tag{2.8}$$

Then, adding the random force $R(t)$, the complete Langevin equation (1.2) turns ou to be

$$\frac{dV}{dt} = -\frac{1}{\sigma_e} \left[ 1 + \sqrt{T_*} D_0^{1/2} \right] V(t) + \frac{1}{m_e} R(t) . \tag{2.9}$$

We agree to refer to (2.9) as to the *fractional Langevin equation.*

We recognize that our fractional Langevin equation is a particular case of the generalized Langevin equation (1.19) with

$$\gamma(t) = \frac{1}{\sigma_e} \left[ \delta(t) - \sqrt{T_*} \frac{t^{-3/2}}{2\sqrt{\pi}} \right] \Longleftrightarrow \overline{\gamma}(s) = \frac{1}{\sigma_e} \left[ 1 + \sqrt{T_*} s^{1/2} \right] . \tag{2.10}$$

Consequently, we can use (2.10) to compute the correlations functions $C_V(t)$, $C_R(t)$ starting from their Laplace transforms (1.20-21), respectively. Then, the mean squared displacement can be derived from $C_V(t)$ according to (1.23).

Let us first consider the random force. The inversion of the Laplace transform $\overline{C}_R(s)$ yields

$$C_R(t) = m_e^2 \langle V^2(0) \rangle \gamma(t) , \tag{2.11}$$

where $\gamma(t)$ is provided by (2.10). We thus recognize that for our fractional Langevin equation the random force cannot be longer represented uniquely by a white noise; an additional "fractional" noise is present due to the term $t^{-3/2}$ which, as formerly noted by Case [6], is to be interpreted in the generalized sense of tempered distributions [8].

Let us now consider the velocity correlation. Using (1.20) and (2.10) it turns out

$$\overline{C}_V(s) = \frac{\langle V^2(0) \rangle}{s + \left[1 + \sqrt{T_*}\, s^{1/2}\right]/\sigma_e} = \frac{\langle V^2(0) \rangle}{s + \sqrt{\beta/\sigma_e}\, s^{1/2} + 1/\sigma_e}, \tag{2.12}$$

where, because of (2.3) and (1.6),

$$\beta := \frac{T_*}{\sigma_e} = \frac{9}{2\gamma + 1} = \frac{9\rho_f}{2\rho_p + \rho_f}. \tag{2.13}$$

We note from (2.13) that $0 < \beta < 9$, the limiting cases occurring for $\gamma = \infty$ and $\gamma = 0$, respectively. We also recognize that the effect of the Basset force is expected to be negligible for $\beta \to 0$, i.e. for particles which are sufficiently heavy with respect to the fluid ($\rho_p \gg \rho_f$).

As far as we know, at least in this context, an explicit inversion of the Laplace transform (2.12) in terms of elementary functions has not yet been carried out. Widom [5] and Case [6] have only provided integral representations of the velocity correlation function, from which they have derived the long-time asymptotic behaviour ($\propto t^{-3/2}$). In our notation, applying the asymptotic theorem for $s \to 0$ to (2.12), we get as $t \to \infty$

$$C_V(t) \simeq \langle V^2(0) \rangle \frac{\sqrt{\beta}}{2\sqrt{\pi}} \left(\frac{t}{\sigma_e}\right)^{-3/2} = \frac{\langle V^2(0) \rangle}{2\beta\sqrt{\pi}} \left(\frac{t}{T_*}\right)^{-3/2}. \tag{2.14}$$

The presence of such a long-time tail, pointed out also in [3-4], was first observed by Alder and Wainwright [11] in a computer simulation of velocity correlation functions.

The explicit inversion of (2.12) is hereafter carried out, basing on our previous analysis of the original and generalized Basset problems, in the framework of the *Fractional Calculus* and *Mittag-Leffler functions* [12-14]. For this aim let us recall the following Laplace transform pairs

$$\begin{aligned} &\frac{1}{(s^{1/2} - a_+)(s^{1/2} - a_-)} \div \\ &\qquad \frac{1}{a_+ - a_-} \left[a_+ E_{1/2}(a_+ \sqrt{t}) - a_- E_{1/2}(a_- \sqrt{t})\right], \end{aligned} \tag{2.15}$$

$$\frac{1}{(s^{1/2}-a)^2} \div E_{1/2}(a\sqrt{t})\ [1+2\,a^2\,t]+2\,a\sqrt{t/\pi}\,, \tag{2.16}$$

where

$$E_{1/2}(a\sqrt{t}) := \sum_{n=0}^{\infty} \frac{a^n\,t^{n/2}}{\Gamma(n/2+1)} = \mathrm{e}^{a^2\,t}\,\mathrm{erfc}(-a\sqrt{t}) \tag{2.17}$$

denotes the *Mittag-Leffler function* of order 1/2. In fact, re-writing (2.12) as

$$\overline{C}_V(s) = \frac{\langle\, V^2(0)\,\rangle}{(s^{1/2}-a_+)\,(s^{1/2}-a_-)} \tag{2.18}$$

where

$$\begin{aligned} a_\pm &= \frac{-\sqrt{\beta}\pm(\beta-4)^{1/2}}{2\sqrt{\sigma_e}} \quad \text{if} \quad \beta\neq 4\,, \\ a_\pm = a &= -\frac{1}{\sqrt{\sigma_e}} \quad \text{if} \quad \beta = 4\,, \end{aligned} \tag{2.19}$$

we easily obtain the required $C_V(t)$ in terms of *Mittag-Leffler functions*, as pointed out in (2.15-16).

Furthermore, it can be proved [14] that $C_V(t)$ results for $t>0$ a decreasing function, completely monotonic, i.e. $(-1)^n\,C_V^{(n)}(t)>0$, with the asymptotic behaviour given by (2.14), for any physical value of $\beta$.

In order to compute $\langle\, X^2(t)\,\rangle$, according to (1.23) we have to consider the 2-fold primitives of the functions in the R.H.S. of (2.15-16), vanishing at $t=0$. In particular, the repeated integral for the *Mittag-Leffler function* turns out

$$\begin{aligned} I_0^2\,E_{1/2}(a\sqrt{t}) &= \sum_{n=0}^{\infty} \frac{a^n\,t^{n/2+2}}{\Gamma(n/2+3)} \\ &= \frac{1}{a^4}\left[E_{1/2}(a\sqrt{t})-1-2a\frac{t^{1/2}}{\sqrt{\pi}}-a^2t-\frac{4}{3}a^3\frac{t^{3/2}}{\sqrt{\pi}}\right]. \end{aligned} \tag{2.20}$$

The asymptotic behaviour of $\langle\, X^2(t)\,\rangle$ as $t\to\infty$ can be easier obtained from its Laplace transform for $s\to 0$, and reads

$$\langle\, X^2(t)\,\rangle = 2D\,t\left\{1+O\left[(t/\sigma_e)^{-1/2}\right]\right\}, \quad \text{as} \quad t\to\infty\,, \tag{2.21}$$

where

$$D = \overline{C}_V(0) = \sigma_e\,\langle\, V^2(0)\,\rangle = \mu\,k\,T\,. \tag{2.22}$$

Note that in the RHS of (2.22) we have used the energy equipartition law (1.15) with the effective mass and (2.2).

## Conclusions

In this paper we have revisited the Brownian motion on the basis of the *fractional Langevin equation* (2.9), which turns out to be a particular case of the generalized Langevin equation (1.19) introduced by Kubo on 1966.

The importance of our approach is to model the Brownian motion more realistically than the usual one based on the classical Langevin equation (1.8), in that it takes into account also the retarding effects due to hydrodynamic backflow, i.e. the added mass and the Basset memory drag, as pointed out in (2.1) and (2.4), respectively.

On the basis of the two *fluctuation-dissipation theorems* (recalled in the Appendix A) and of the techniques of the *Fractional Calculus* (recalled in the Appendix B), we have provided the analytical expressions of the correlation functions (both for the random force and the particle velocity) and of the mean squared particle displacement. Consequently, the well-known results of the classical theory of the Brownian motion have been properly generalized.

The random force has been shown to be represented by a superposition of the usual white noise with a "fractional" noise, as pointed out in (2.10-11).

The velocity correlation function $C_V(t)$ exhibits a different behaviour from the classical case: it is no longer expressed by a simple exponential but by a combination of Mittag-Leffler functions of order 1/2, according to (2.15-19). As a consequence, one can derive for $C_V(t)$ a slower decay, proportional to $t^{-3/2}$ as $t \to \infty$, which indeed is more realistic than the usual exponential one, also in view of numerical simulations.

Finally, the mean squared displacement has been shown to maintain, for sufficiently long times, the linear behaviour which is typical of normal diffusion, with the same diffusion coefficient of the classical case, as seen in (2.21-22), i.e. $\langle X^2(t) \rangle \simeq 2\,D\,t$. However, the Basset memory force, which is responsible of the algebraic decay of the velocity correlation function, induces a retarding effect ($\propto t^{1/2}$) in the establishing of the linear behaviour, which is relevant when the parameter $\beta$ introduced in (2.13) is big enough. From numerical computations this effect is seen to be evident when $0 < \rho_p < 2\,\rho_f$, i.e. for "not heavy" Brownian particles; in these cases one can get a best fit in a long time interval with the law $\langle X^2(t) \rangle \simeq 2\,D_*\,t^\alpha$, with $0 < D_* < D$ and $1 < \alpha < 2$, which appears as a manifestation of fast anomalous diffusion [19].

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## Appendix A

Let us consider the *generalized Langevin equation* (1.19), that we write as

$$R(t) = m\left[\dot{V}(t) + \gamma(t) * V(t)\right], \tag{A.1}$$

where $\cdot$ denotes time differentiation and $*$ time convolution. The assumption of stationarity for the stochastic processes along with the following hypothesis

$$\langle R(t)\rangle = 0, \quad \langle V(0)\,R(t)\rangle = 0, \quad t > 0, \tag{A.2}$$

allows us to derive, by using the Laplace transforms, the two *fluctuation-dissipation theorems*

$$\overline{C}_V(s) := \overline{\langle V(0)\,V(t)\rangle} = \frac{\langle V^2(0)\rangle}{s + \overline{\gamma}(s)}, \tag{A.3}$$

and

$$\overline{C}_R(s) := \overline{\langle R(0)\,R(t)\rangle} = m^2\,\langle V^2(0)\rangle\,\overline{\gamma}(s). \tag{A.4}$$

Our derivation is alternative to the original one by Kubo who used Fourier transforms [2]; furthermore, it appears useful for the treatment of our *fractional Langevin equation.*

Multiplying both sides of (A.1) by $V(0)$ and averaging, we obtain

$$\langle V(0)\,\dot{V}(t)\rangle + \gamma(t) * \langle V(0)\,V(t)\rangle = 0. \tag{A.5}$$

The application of the Laplace transform to both sides of (A.5) yields

$$s\,\overline{\langle V(0)\,V(t)\rangle} - \langle V^2(0)\rangle + \overline{\gamma}(s)\,\overline{\langle V(0)\,V(t)\rangle} = 0, \tag{A.6}$$

from which we just obtain (A.3).

Multiplying both sides of (A.1) by $R(0)$ and averaging, we obtain

$$C_R(t) := \langle R(0)\,R(t)\rangle = m^2\left[\langle \dot{V}(0)\,\dot{V}(t)\rangle + \gamma(t) * \langle \dot{V}(0)\,V(t)\rangle\right]. \tag{A.7}$$

Noting that, by the stationary condition,

$$\langle \dot{V}(0)\,V(0)\rangle = 0, \quad \langle \dot{V}(0)\,V(t)\rangle = -\langle V(0)\,\dot{V}(t)\rangle, \tag{A.8}$$

the application of the Laplace transform to both sides of (A.7) yields

$$\overline{C}_R(s) = m^2\left\{s\,\overline{\langle \dot{V}(0)\,V(t)\rangle} - \overline{\gamma}(s)\left[s\,\overline{\langle V(0)\,V(t)\rangle} - \langle V^2(0)\rangle\right]\right\}. \tag{A.9}$$

Since

$$\overline{\langle \dot{V}(0)\, V(t) \rangle} = -\overline{\langle V(0)\, \dot{V}(t) \rangle} = -s\, \overline{\langle V(0)\, V(t) \rangle} + \langle V^2(0) \rangle\,, \quad \text{(A.10)}$$

we get

$$\begin{aligned} \overline{C}_R(s) = m^2 \Big\{ s \left[ -s\, \overline{C}_V(s) + \langle V^2(0) \rangle - \overline{\gamma}(s)\, \overline{C}_V(s) \right] \\ + \overline{\gamma}(s)\, \langle V^2(0) \rangle \Big\}\,, \end{aligned} \quad \text{(A.11)}$$

from which, accounting for (A.3), we just obtain (A.4).

## Appendix B

Here we recall the essentials of Riemann-Liouville *Fractional Calculus* basing on [14-18], and we interpret the Basset force in terms of a fractional derivative of order $1/2$.

Usually, the starting point to introduce the Riemann-Liouville fractional calculus is the well-known Cauchy's iterated formula, which provides the $n$-fold primitive of a given function $f(t)$ in terms of a single integral. If $t > c \in \mathbb{R}$, it reads

$$\begin{aligned} I_c^n\, f(t) &= \int_c^t \int_c^{\tau_{n-1}} \cdots \int_c^{\tau_2} \int_c^{\tau_1} f(\tau)\, d\tau\, d\tau_1 \ldots d\tau_{n-1} \\ &= \frac{1}{(n-1)!} \int_c^t (t-\tau)^{n-1}\, f(\tau)\, d\tau\,, \quad n = 1,\, 2,\, \ldots\,. \end{aligned} \quad \text{(B.1)}$$

Here $c$ denotes the point where the primitive is required to vanish along with its first $n-1$ derivatives. The passage from $n \in \mathbb{N}$ to $\alpha \in \mathbb{R}^+$ is now quite natural taking into account that $(n-1)! = \Gamma(n)$. Consequently we define

- *the fractional integral of $f(t)$ of order $\alpha$* (with starting point $c$)

$$I_c^\alpha\, f(t) := \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1}\, f(\tau)\, d\tau\,, \quad \alpha \in \mathbb{R}^+\,. \quad \text{(B.2)}$$

For $\alpha = 0$ we define $I_c^0\, f(t) = f(t)$ so that $I_c^0 \equiv \mathbb{I}$ where $\mathbb{I}$ is the identity operator. The choice with $c = -\infty$ is originally due to Liouville (1832), while with $c = 0$ to Riemann (1847).

In order to introduce the notion of fractional derivative of order $\alpha$, we need to consider the possibility to change $\alpha \to -\alpha$ in the r.h.s. of (B.2). While

the extension in (B.1) from $n$ to $\alpha > 0$ is quite legitimate, the actual proposal requires some care due to the convergence of the integral.

If $\alpha$ denotes any positive real number in the range $n-1 < \alpha < n$ with $n \in \mathbb{N}$ and $f(t)$ is a sufficiently well-behaved function, one usually defines

- *the fractional derivative of $f(t)$ of order $\alpha$* (with starting point $c$)

$$D_c^\alpha f(t) := \frac{d^n}{dt^n} I_c^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_c^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \,. \tag{B.3}$$

A different formula for the fractional derivative, alternative to (B.3), originally introduced by Caputo [14-15], is

$$\tilde{D}_c^\alpha f(t) := I_c^{n-\alpha} \frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_c^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \,. \tag{B.4}$$

We note in general that

$$\frac{d^n}{dt^n} I_c^{n-\alpha} f(t) \neq I_c^{n-\alpha} \frac{d^n}{dt^n} f(t) \,, \tag{B.5}$$

unless the function $f(t)$ along with its first $n-1$ derivatives vanishes at $t = c^+$.

For *causal* functions (i.e. vanishing for $t < 0$) the choice $c = 0$ is in order. In this case it is convenient to introduce the so-called Gel'fand-Shilov distribution [8]

$$\Phi_\lambda(t) := \frac{t^{\lambda-1}}{\Gamma(\lambda)} \Theta(t) \,, \qquad \lambda \in \mathbb{C} \,, \tag{B.6}$$

where $\Theta(t)$ is the unit step Heaviside function and $\Gamma(\lambda)$ is the Gamma function. For $\lambda = -n$ $(n = 0, 1, \dots)$, $\Phi_\lambda(t)$ reduces to the $n$-derivative (in the generalized sense) of the *Dirac delta distribution*,

$$\Phi_{-n}(t) := \frac{t^{-n-1}}{\Gamma(-n)} \Theta(t) = \delta^{(n)}(t) \,, \quad n = 0 \,, 1 \,, \dots \tag{B.7}$$

Assuming that the passage of the $n$-derivative in (B.3) under integral is legitimate, one recognizes that, for $n-1 < \alpha < n$,

$$D_0^\alpha f(t) = \tilde{D}_0^\alpha f(t) + \sum_{k=0}^{n-1} f^{(k)}(0^+) \, \Phi_{(k-\alpha+1)}(t) \,, \tag{B.8}$$

and, using the (generalized) technique of Laplace transforms,

$$\begin{aligned} \mathcal{L}\left\{\tilde{D}_0^\alpha f(t)\right\} &= s^\alpha \overline{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} f^{(k)}(0^+)\,, \\ \mathcal{L}\left\{D_0^\alpha f(t)\right\} &= s^\alpha \overline{f}(s)\,. \end{aligned} \tag{B.9}$$

Let us now consider the *causal* restriction of the Basset force. We easily recognize in (2.6) that

$$\begin{aligned} B(t) &= \frac{1}{\Gamma(1/2)} \int_{0-}^{t} \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}\, d\tau \\ &= \frac{1}{\Gamma(1/2)} \int_{0}^{t} \frac{dV(\tau)/d\tau}{\sqrt{t-\tau}}\, d\tau + V(0)\,\Phi_{1/2}(t)\,. \end{aligned} \tag{B.10}$$

Consequently, we can write the following equivalent expressions in terms of the derivatives of order 1/2,

$$B(t) = \tilde{D}_0^{1/2}\, V(t) + V(0)\,\Phi_{1/2}(t) = D_0^{1/2}\, V(t)\,. \tag{B.11}$$

Applying the property (B.9) to (B.11), we can also write

$$\overline{B}(s) = s^{1/2}\,\overline{V}(s) \iff B(t) = \Phi_{-1/2}(t) * V(t)\,, \tag{B.12}$$

where the convolution is to be intended in the generalized sense of Gel'fand-Shilov [8].

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