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Linearisation of Second-Order Differential Equations

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Given a second order differential equation on a manifold we find necessary and sufficient conditions for the existence of a coordinate system in which the system is linear. The main tool to be used is a linear connection defined by the system of differential equations.

1. Introduction

We will consider the following problem: Given a system of second order differential equations

$$\ddot{x}^i = f^i(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n), \qquad i = 1, \dots, n,$$

we look for necessary and sufficient conditions for the existence of a coordinate transformation

$$\bar{x}^i = \xi^i(x^1, \dots, x^n), \qquad i = 1, \dots, n,$$

such that, in the new coordinates $\bar{x}^i$, the system has one of the following forms:

- Linear in velocities

$$\ddot{\bar{x}}^i = \alpha^i_j(\bar{x})\dot{\bar{x}}^j + \beta^i(\bar{x})\,;$$

- Linear in all coordinates

$$\ddot{\bar{x}}^i = A^i_j\dot{\bar{x}}^j + B^i_j\bar{x}^j + C^i, \qquad i = 1, \dots, n,$$

for some contants A^i_j, B^i_j and C^i.

As a guiding example we can consider the wellknown case of the equation for geodesics on a manifold: Given the differential equation

$$\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0,$$

can we find a local coordinate system $(\bar{x}^i)$ such that in the new coordinates the differential equation has the trivial form

$$\ddot{\bar{x}}^i = 0.$$

The answer to this problem is also wellknown [5]. If we consider the connection D whose Christoffell symbols are the functions Γ^i_{jk}, then the answer to the problem is YES if and only if the connection D is locally flat, i. e. if the curvature tensor $R^i_{j\,kl}$ vanishes.

In this way we reduce the problem of the existence of some special coordinate system to the problem of the integrability of a connection. In the general case we will follow a similar idea.

2. The Non-Linear Connection

In geometric terms, a system of second order differential equations (hereafter shortened to SODE) on a manifold M is interpreted as a vector field Γ on the tangent bundle to M, such that $T\tau(\Gamma(v)) = v$ for every $v \in TM$, where $\tau\colon TM \to M$ is the tangent bundle projection. The local coordinate expression of such vector field is

$$\Gamma = v^i \frac{\partial}{\partial x^i} + f^i(x,v)\frac{\partial}{\partial v^i}\,,$$

and the actual differential equation is the equation for the integral curves of Γ:

$$\frac{dx^i}{dt} = v^i \qquad\qquad \frac{dv^i}{dt} = f^i(x,v).$$

Every SODE defines an Ehresman connection on the tangent bundle (see [3, 1]), that is, a decomposition

$$T(TM) = \mathrm{Hor}(\tau) \oplus \mathrm{Ver}(\tau)$$

In local coordinates, the vertical distribution is generated by the vector fields

$$V_i = \frac{\partial}{\partial v^i}\,,$$

and the horizontal one by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma^j_i \frac{\partial}{\partial v^j} ,$$

where the coeficients of the connection are defined by

$$\Gamma^i_j = -\frac{1}{2} \frac{\partial f^i}{\partial v^j} .$$

In more intrinsec terms, the Lie derivative with respect to Γ of the vertical endommorphism S satisfies $[\mathcal{L}_\Gamma S]^2 = I$, so that it has eigenvalues 1 and -1. The eigenspace corresponding to the eigenvalue 1 is the vertical subbundle, and thus the eigenspace corresponding to the eigenvalue -1 defines a complementary distribution, i.e., a horizontal distribution. The projectors onto these subbundles are

$$P_H = \frac{1}{2}(I - \mathcal{L}_\Gamma S) \qquad \text{and} \qquad P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S).$$

Since the coefficientes of the connection are (minus one half of) the derivatives of the forces with respect to the velocities, we can think that the integrability of the non-linear connection implies that the forces are independent of the velocities in some coordinates. This is NOT the case. The integrability of the horizontal distribution implies the existence of coordinates (y^i, z^i) on TM such that the horizontal distribution is spanned by $\partial/\partial y^i$ while the vertical one is spanned by $\partial/\partial z^i$. Nevertheless, these coordinates are not natural coordinates in the tangent bundle (i.e., $\dot{y}^i \neq z^i$), and the system of differential equations is expressed in these coordinates as a system of $2n$ first order differential equations. Therefore, it has no sense to speak about forces in such systems of coordinates.

3. The Pull-Back Bundle

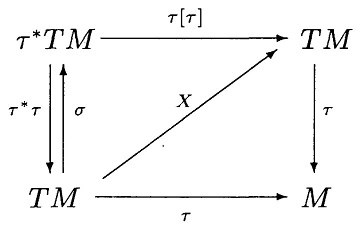
As we saw in the last section, the integrability of the non-linear connection does not solve our problem. This should be expected, since the problem of linearisability is a linear problem and the integrability of a non-linear connection is a non-linear condition. In the next section we will define a linear connection associated to the SODE, which can be thought as the linearisation of the non-linear connection. In the present section we will consider the differential geometric structures that we will need there.

Consider the tangent bundle $\tau\colon TM \to M$ and the pull-back bundle (see [5]) $\tau^*\tau\colon \tau^*TM \to TM$, where

$$\tau^*TM = \{\, (w_1, w_2) \in TM \times TM \mid \tau(w_1) = \tau(w_2)\,\}$$

and $\tau^*\tau$ is the projection onto the the first factor. The linear connection will be defined on this bundle.

A section σ of this bundle is said to be a vector field along τ. Alternatively a vector field along τ can be considered as a map $X\colon TM \to TM$ such that $X(v)$ is a tangent vector a $\tau(v)$. We have the following commutative diagram:



where $\tau[\tau]$ is the projection onto the second factor. The $C^\infty(TM)$-module of vector fields along τ will be denoted $\mathcal{X}(\tau)$. The local expression af a vector field along τ is

$$X = X^i(x, v)\frac{\partial}{\partial x^i}\,.$$

We will say that a vector field X along τ is basic if it is induced by a vector field on M, i.e., the coefficients X^i do not deppends on the coordinates v^i.

The projection $T\tau\colon T(TM) \to TM$ induces a map $\tau_\#\colon \mathcal{X}(TM) \to \mathcal{X}(\tau)$. In local coordinates

$$\tau_\# \left(X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial v^i}\right) = X^i\frac{\partial}{\partial x^i}\,.$$

Similarly the vertical lift $\xi^V\colon TM \to TTM$ induces a map ${}^V\colon \mathcal{X}(\tau) \to \mathcal{X}(TM)$, which in coordinates is given by

$$\left(X^i\frac{\partial}{\partial x^i}\right)^V = X^i\frac{\partial}{\partial v^i}\,.$$

The sequence

$$0 \longrightarrow \mathcal{X}(\tau) \overset{V}{\longrightarrow} \mathcal{X}(TM) \overset{\tau_\#}{\longrightarrow} \mathcal{X}(\tau) \longrightarrow 0$$

is exact. A non-linear connection can be seen as a splitting of this sequence, that is, a map ${}^{H}\colon \mathcal{X}(\tau) \to \mathcal{X}(TM)$, called the horizontal lift, such that $\tau_{\#}(X^{H}) = X$ for all $X \in \mathcal{X}(\tau)$. In coordinates

$$\left(X^i \frac{\partial}{\partial x^i}\right)^{H} = X^i H_i .$$

The connection also defines a map $\kappa\colon \mathcal{X}(TM) \to \mathcal{X}(\tau)$, called the connection map, given by $\kappa = \xi^{V-1} \circ P_V$. In local coordinates

$$\kappa\left(X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial v^i}\right)^{V} = \left(Y^i + \Gamma^i_j X^j\right) \frac{\partial}{\partial v^i} .$$

The connection map is characterized by $\kappa(X^{H}) = 0$ and $\kappa(X^{V}) = X$, for all vector field X along τ.

4. The Linear Connection

We remind that a linear connection on a bundle $\pi\colon E \to N$ is a linear map that associates to every vector field U on N a derivation D_U of the module of sections of the bundle. Thus, D satisfies

$$\begin{aligned} D_U(\sigma + \lambda) &= D_U\sigma + D_U\lambda \\ D_U(f\sigma) &= U(f)\,\sigma + f D_U\sigma \\ D_{U+V}\sigma &= D_U\sigma + D_V\sigma \end{aligned}$$

for σ and λ section of E, f a function on N, and U, V vector fields on N.

We will consider the case $E = \tau^*TM$, $N = TM$ and $\pi = \tau^*\tau$. The sections of this bundle are the vector fields along τ, so that a linear connection associates a derivation D_U of $\mathcal{X}(\tau)$ to every vector field U on TM.

The linear connection defined by a SODE Γ is given by

$$D_U X = \kappa[P_H U, X^{V}] + \tau_{\#}[P_V U, X^{H}] .$$

In local coordinates, if we define the functions Γ^i_{jk} on TM by

$$\Gamma^i_{jk} = \frac{\partial \Gamma^i_j}{\partial v^k} = -\frac{1}{2}\frac{\partial^2 f^i}{\partial v^j \partial v^k} ,$$

then the linear connection is determined by

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} , \qquad D_{\frac{\partial}{\partial v^i}} \frac{\partial}{\partial x^j} = 0 .$$

Thus, if X and Y are vector fields along τ we have

$$\begin{aligned} D_{X^H}Y &= X^i[H_i(Y^j) + \Gamma^j_{ik}Y^k]\frac{\partial}{\partial x^j}, \\ D_{X^V}Y &= X^i\,\frac{\partial Y^j}{\partial v^i}. \end{aligned}$$

Two properties that will be use in the next section are:

- A vector field Y along τ is basic if and only if $D_V Y = 0$ for every vertical vector field V on TM.
- If X and Y are basic vector fields, then

  $$D_{X^H}Y - D_{Y^H}X = [X, Y],$$

  where $[X, Y]$ denotes the Lie braket as vector fields on M.

Further properties of the linear connection can be found in [4].

5. Linearisation of Sodes

Since the coefficients of the linear connection are the second derivatives of the forces with respect to the velocities, the vanishing of these coefficients in some coordinates imply that the forces are linear in velocities.

THEOREM 1. *A second order diferential equation is linearisable in velocities if and only if the linear connection is flat.*

Proof. The 'only if' part is trivial. For the 'if' part, assume that the linear connection is flat. Then there exists a local base $\{X_i\}$ of parallel vector fields along τ. It follows that $D_V X_i = 0$ for every vertical vector field V on TM, so that X_i are basic vector fields. Moreover they pairwise commute:

$$[X_i, X_j] = D_{X_i^H}X_j - D_{X_i^H}X_j = 0.$$

Therefore, by Frobenius theorem, there exists a coordinate system (x^i) such that $X_i = \partial/\partial x^i$. In these coordinates, the coefficients of the linear connection vanish, and hence $\partial^2 f^i/\partial v^j \partial v^k = 0$. It follows that the forces are affine functions in velocities. ∎

Another geometric object associated to the SODE is the Jacobi endomorfism. It is the linear endomorfism Φ of $\mathcal{X}(\tau)$ defined by

$$\Phi(X) = \kappa[\Gamma, X^H].$$

In local coordinates the components of Φ are

$$\Phi^i_j = -\frac{\partial f^i}{\partial x^j} - \Gamma^i_k\Gamma^k_j - \Gamma(\Gamma^i_j).$$

In terms of the Jacobi endomorphism, we can charaterize the linear SODES as follows.

THEOREM 2. *A second order differential equation is linearisable if and only if the linear connection is flat and the Jacobi endomorphism is parallel.*

Proof. If the linear connection is flat, we can consider 'affine' coordinates (x^i), in which the forces are linear in velocities:

$$f^i = \alpha^i_j(x)v^j + \beta^i(x).$$

The expression of the Jacobi endormorphism is then

$$\Phi^i_j = -\frac{\partial \beta^i}{\partial x^j} - \frac{1}{4}\alpha^i_k\alpha^k_j + \frac{1}{2}\left(\frac{\partial \alpha^i_j}{\partial x^k} - 2\,\frac{\partial \alpha^i_k}{\partial x^j}\right)v^k.$$

Since Φ is parallel iff its components in affine coordinates are constant, it follows that the coefficients of v^k vanish and the other terms are constant. But the vanishing of $\frac{\partial \alpha^i_j}{\partial x^k} - 2\,\frac{\partial \alpha^i_k}{\partial x^j}$ implies $\frac{\partial \alpha^i_j}{\partial x^k} = 0$, and thus α^i_j are constant: $\alpha^i_j = A^i_j$. On the other hand $\Phi^i_j = -\frac{\partial \beta^i}{\partial x^j} - \frac{1}{4}\alpha^i_k\alpha^k_j$ is constant iff $\frac{\partial \beta^i}{\partial x^j}$ is constant: $\frac{\partial \beta^i}{\partial x^j} = B^i_j$. It follows that the forces have the expression

$$f^i = A^i_j x^j + B^i_j + C^i,$$

for some constants C^i.

The 'only if' part is a matter of straightforward calculation. ∎

For time-dependent systems of second order differential equations see [2].

References

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