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Prolongations and Fields of Paths for Higher-Order O.D.E. Represented by Connections on a Fibered Manifold†

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1. Introduction

The prolongations of higher-order (semispray) connections on a fibered manifold over one-dimensional base are described. Within the framework of the adopted formalism, the order-reduction method of fields of paths for the corresponding ordinary differential equations is presented, generalizing the Hamilton-Jacobi integration method from the calculus of variations to the nonvariational situation.

As time goes on, the description of the geometry of time-dependent ordinary differential equations being motivated by the non-autonomous mechanics more or less hesitantly turns from that influenced by the tangent bundles geometry methods to that dealing with the theory of connections on fibered manifolds. In this respect, the main goal of the paper is to present a part of a recently developed formalism which allows a transparent discussion on certain phenomena related to the higher-order systems 'in normal form', i.e. solved with respect to the highest derivatives. For ordinary differential equations, such a system is represented by the so-called *higher-order (semispray) connection* $\Gamma^{(k+1)}$ on a fibered manifold $\pi\colon Y \to X$ over one-dimensional base X, which is a section of the affine bundle $\pi_{k+1,k}\colon J^{k+1}\pi \to J^k\pi$. Reaping the benefit of its compatibility with the underlying structures, such a system can be very naturally prolonged, composed with suitable morphisms within the appropriate 2-fibered manifolds and, last but not least, considered as the solution of some other *associated* connections. This leads to the presented indirect *method of fields of paths*, transferring the integration problem for $\Gamma^{(k+1)}$

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to that of looking for its *integrals* as the generators of (local) semispray connections representing the order-reduction of $\Gamma^{(k+1)}$. In fact, the method stands for a generalization of the *Hamilton-Jacobi* method from variational analysis, studied in this situation by Krupková [7], [8], to the nonvariational case – the situation of $k = 1$ was established in [9].

As regards the underlying structures and the notation, $\pi\colon Y \to X$ is an arbitrary fibered manifold over one-dimensional base X, $\pi_k\colon J^k\pi \to X$ its canonical k-th prolongation ($\pi_0 \equiv \pi$), $\pi_{k,r}\colon J^k\pi \to J^r\pi$ ($\pi_{k,k} \equiv \mathrm{id}_{J^k\pi}$) the induced projections. If the fibered coordinates on $J^k\pi$ are $t, q^\sigma, \dots, q^\sigma_{(k)}$, the additional induced coordinates on $J^1\pi_k$ are denoted by $q^\sigma_{;}, \dots, q^\sigma_{(k);}$. All mappings are supposed to be smooth, $\mathcal{F}(M)$ denotes the set of functions on a manifold M, $\mathcal{S}_U(\pi)$ means the set of sections γ of π defined on an open $U \subset X$, while $\mathcal{S}_{\mathrm{loc}}(\pi)$ means that the domains are not specified. In this respect, $(t, q^\sigma) \circ \gamma = (t, \gamma^\sigma)$. By $D^{k+1,k}$ we denote the total derivative along $\pi_{k+1,k}$, $V_{\pi_k}J^k\pi$ means the subbundle of π_k-vertical vector fields on $J^k\pi$, the canonical embedding $\iota_{1,k}\colon J^{k+1}\pi \to J^1\pi_k$ is defined by $\iota_{1,k}(j^{k+1}_x\gamma) j^1_x(j^k\gamma)$, and $J^r(\Phi, \mathrm{id}_x)$ denotes the r-th prolongation of a fibered morphism Φ over X. As usually, the summation convention is used far as possible.

2. Higher-Order (Semispray) Connections

Let $k \geq 1$. As a special type of connections on π_k, the so-called $(k+1)$-*holonomic connections* on π_k or more briefly $(k+1)$-*connections* on π are intrinsically related to the theory of higher-order equations. Due to the canonical embedding $\iota_{1,k}\colon J^{k+1}\pi \hookrightarrow J^1\pi_k$, a $(k+1)$-*connection* on π is a section (both global and local versions will appear) $\Gamma^{(k+1)}\colon J^k\pi \to J^{k+1}\pi$ of the affine bundle $\pi_{k+1,k}$. The *horizontal form* of $\Gamma^{(k+1)}$ is a π_k-projectable vector-valued 1-form $h_{\Gamma^{(k+1)}}\colon J^k\pi \to TJ^k\pi \otimes \pi_k^*(T^*X)$, which locally reads $h_{\Gamma^{(k+1)}} = D_{\Gamma^{(k+1)}} \otimes dt$, where the (local) absolute derivative $D_{\Gamma^{(k+1)}} = D^{k+1,k} \circ \Gamma^{(k+1)}$,

$$D_{\Gamma^{(k+1)}} = \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} q^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} + \Gamma^\sigma_{(k+1)} \frac{\partial}{\partial q^\sigma_{(k)}}\,, \tag{2.1}$$

is the so-called *semispray* on $J^k\pi$ (here is the alternative name for $\Gamma^{(k+1)}$ from). In other words, the horizontal form $h_{\Gamma^{(k+1)}}$ defines the one-dimensional π_k-horizontal *semispray distribution* $H_{\Gamma^{(k+1)}}$ on $J^k\pi$ realizing the direct sum decomposition

$$TJ^k\pi = H_{\Gamma^{(k+1)}} \oplus V_{\pi_k}J^k\pi\ .$$

The $(k+1)$-th order equation represented by a $(k+1)$-connection $\Gamma^{(k+1)}$ on

π is the $((k+1)m+1)$-dimensional submanifold $\Gamma^{(k+1)}(J^k\pi) \subset J^{k+1}\pi$ defining a (generally nonlinear) system of ordinary differential equations (O.D.E.) *in normal form*, i.e. explicitly solved with respect to the highest derivatives:

$$q^\sigma_{(k+1)} = \Gamma^\sigma_{(k+1)}(t, q^\sigma, \ldots, q^\sigma_{(k)}) \ . \tag{2.2}$$

A section $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ is called an *integral section* (*path*) of $\Gamma^{(k+1)}$ if it is the solution of $\mathcal{E}^{\Gamma^{(k+1)}} = \Gamma^{(k+1)}(J^k\pi)$, i.e. if $j^{k+1}\gamma = \Gamma^{(k+1)} \circ j^k\gamma$. The point is that (2.2) can be represented by the *Pfaffian system*

$$dq^\sigma_{(i)} = q^\sigma_{(i+1)}\, dt, \qquad dq^\sigma_{(k)} = \Gamma^\sigma_{(k+1)}\, dt, \tag{2.3}$$

for $i = 0, \ldots, k-1$, expressing the fact that $H_{\Gamma^{(k+1)}}$ is a subdistribution of the canonical *Cartan distribution* $C_{\pi_{k,k-1}}$ on $J^k\pi$ (contact forms). In accordance with (2.3), $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^k\gamma(U)$ is an integral manifold of $H_{\Gamma^{(k+1)}}$.

3. Prolongations and Fields of Paths

For a given $(k+1)$-connection on π, the compatibility of the corresponding equations with the underlying structures allows us to present a suitable description of the prolongation of the equation $\mathcal{E}^{\Gamma^{(k+1)}} \subset J^{k+1}\pi$. Notice that the case of $k = 0$ for a (first-order) connection $\Gamma\colon Y \to J^1\pi$ on π is included, as well.

Definition 3.1. Let $k \geq 0$ and $\Gamma^{(k+1)}\colon J^k\pi \to J^{k+1}\pi$ be a $(k+1)$-connection on π. The *r-th prolongation* of the equation $\mathcal{E}^{\Gamma^{(k+1)}}$ is defined to be the submanifold $\mathcal{E}^{\Gamma^{(k+1)}(r)} = \operatorname{Im}\Gamma^{(k+1)(r)} \subset J^{k+r+1}\pi$, where $\Gamma^{(k+1)(r)}$ is the last term of the sequence of sections

$$\left(\Gamma^{(k+1)(0)}, \Gamma^{(k+1)(1)}, \ldots, \Gamma^{(k+1)(r)}\right)$$

recurrently defined for each $\ell = 1, \ldots, r$ by

$$\Gamma^{(k+1)(\ell)} = J^1(\Gamma^{(k+1)(\ell-1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}\colon J^k\pi \to J^{k+\ell+1}\pi \tag{3.1}$$

with $\Gamma^{(k+1)(0)} := \Gamma^{(k+1)}$.

Note first that $\Gamma^{(k+1)(2)}$ defined by (3.1) can be rewritten to (ι's are omitted for the brevity sake)

$$\begin{aligned}\Gamma^{(k+1)(2)} &= J^1(J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \Gamma^{(k+1)}, \mathrm{id}_X) \circ \Gamma^{(k+1)} \\ &= J^1(J^1(\Gamma^{(k+1)}, \mathrm{id}_X), \mathrm{id}_X) \circ J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \Gamma^{(k+1)} \\ &= J^2(\Gamma^{(k+1)}, \mathrm{id}_X) \circ J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \Gamma^{(k+1)} \ ,\end{aligned}$$

where we have used the properties of prolongations and the fact that

$$\Gamma^{(k+1)(1)} = J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \Gamma^{(k+1)} \in J^{k+2}\pi \ .$$

Now it is clear that the target space of $\Gamma^{(k+1)(2)}$ must be $J^{k+3}\pi$, and repeating the procedure one gets another sequence defining $\Gamma^{(k+1)(r)}$:

$$J^k\pi \xrightarrow{\Gamma^{(k+1)}} J^{k+1}\pi \xrightarrow{J^1(\Gamma^{(k+1)},\mathrm{id}_X)} J^{k+2}\pi \to \cdots J^{k+r}\pi \xrightarrow{J^r(\Gamma^{(k+1)},\mathrm{id}_X)} J^{k+r+1}\pi.$$

Evidently, *the equation* $\mathcal{E}^{\Gamma^{(k+1)(r)}}$ *consists of* $(k+r+1)$*-jets of integral sections of* $\Gamma^{(k+1)}$; in fact

$$\begin{aligned}(3.2)\quad j^{k+r+1}\gamma &= j^r(j^{k+1}\gamma) = j^r(\Gamma^{(k+1)} \circ j^k\gamma) = J^r(\Gamma^{(k+1)}, \mathrm{id}_x) \circ j^{k+r}\gamma \\ &= J^r(\Gamma^{(k+1)}, \mathrm{id}_X) \circ J^{r-1}(\Gamma^{(k+1)}, \mathrm{id}_X) \circ j^{k+r-1} = \cdots = \Gamma^{(k+1)(r)} \circ j^k\gamma \ .\end{aligned}$$

It carries all the information on the initial equation together with 'higher-order consequences' of it. In fact, $\Gamma^{(k+1)(r)}$ represents the family of O.D.E. obtained by differentiating the original equations $0, 1, \ldots, r$ times with respect to the independent variable, which in coordinates means the system

$$(3.3)\quad q^\sigma_{(k+1)} = \Gamma^\sigma_{(k+1)},\ q^\sigma_{(k+2)} = D(\Gamma^\sigma_{(k+1)}), \ldots, q^\sigma_{(k+r+1)} = D^{(r)}(\Gamma^\sigma_{(k+1)}) \ ,$$

where $D^{(r)} = D^{k+r,k+r-1} \ldots D^{k+1,k}$ $(= \frac{d^r}{dt^r})$.

DEFINITION 3.2. Let $k \geq 0, r \geq 1$. A $(k+1)$-connection $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ will be called a *field of paths* of a $(k+r+1)$-connection $\Gamma^{(k+r+1)}\colon J^{k+r}\pi \to J^{k+r+1}\pi$ if on V holds

$$(3.4)\qquad \Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} = \Gamma^{(k+1)(r)} \ .$$

Evidently, if γ is an integral section of $\Gamma^{(k+1)}$, then by (3.2)

$$\Gamma^{(k+r+1)} \circ j^{k+r}\gamma = \Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} \circ j^k\gamma = \Gamma^{(k+1)(r)} \circ j^k\gamma = j^{k+r+1}\gamma \ ,$$

which means that if γ is an integral section of a field of paths $\Gamma^{(k+1)}$ of $\Gamma^{(k+r+1)}$, then it is the integral section (a path) of $\Gamma^{(k+r+1)}$. In other words, $H_{\Gamma^{(k+1)}}$ defines a foliation of V such that each leaf of this foliation is an integral section of $\Gamma^{(k+r+1)}$. It can be shown that (3.4) is equivalent to

$$(3.5)\qquad D_{\Gamma^{(k+r+1)}} \circ \Gamma^{(k+1)(r-1)} = T\Gamma^{(k+1)(r-1)} \circ D_{\Gamma^{(k+1)}}$$

and the local expression of (3.4), (3.5) reads

$$\Gamma^{\sigma}_{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} = D^{(r)}(\Gamma^{\sigma}_{(k+1)}) \circ \Gamma^{(k+1)(r-1)} .$$

Globally speaking, each field of paths represents a (local) order-reduction of the given equation. In this respect, the problem of finding the integral sections of a given integrable higher-order connection can be transferred to the problem of looking for and then solving of its fields of paths.

4. The Method of Fields of Paths

The description of the method goes within the framework of the 2-fibered manifold

$$J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X ,$$

where $r \geq 1$.

The map $\Bbbk\colon J^1\pi_k \times_{J^k\pi} J^1\pi_{k+r,k} \to J^1\pi_{k+r}$, defined for $\psi \in \mathcal{S}_{\mathrm{loc}}(\pi_k)$ and $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$, $\operatorname{Im}\psi \subset \operatorname{Dom}\varphi$, by $\Bbbk(j^1_x\psi, j^1_{\psi(x)}\varphi) = j^1_x(\varphi\circ\psi)$, locally does not effect the coordinates $t, q^\sigma, \dots, q^\sigma_{(k+r)}, q^\sigma_{;}, \dots, q^\sigma_{(k);}$, and its action on the rest of the coordinates reads

$$q^\sigma_{(k+1);} = z^\sigma_{(k+1)} + \sum_{i=0}^{k} z^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i);}, \dots, q^\sigma_{(k+r);} = z^\sigma_{(k+r)} + \sum_{i=0}^{k} z^{\sigma(i)}_{(k+r)\lambda} q^\lambda_{(i);}$$

with z's being the induced coordinates on $J^1\pi_{k+r,k}$. Clearly, there is a natural candidate for a morphism between $\pi_{k+r,k}$ and $(\pi_k)_{1,0}$ over $J^k\pi$; namely, denote by

$$\Phi_0 = \iota_{1,k} \circ \pi_{k+r,k+1}\colon J^{k+r}\pi \to J^1\pi_k$$

the composition, whose coordinate expression is

$$q^\sigma_{;} = q^\sigma_{(1)}, \dots, q^\sigma_{(k);} = q^\sigma_{(k+1)} . \tag{4.1}$$

Then the affine morphism $\Bbbk_{\Phi_0}\colon J^1\pi_{k+r,k} \to J^1\pi_{k+r}$, defined by the composition

$$J^1\pi_{k+r,k} \xrightarrow{(\pi_{k+r,k})_{1,0}\times \mathrm{id}} J^{k+r}\pi \times J^1\pi_{k+r,k} \xrightarrow{\Phi_0\times\mathrm{id}} J^1\pi_k \times J^1\pi_{k+r,k} \xrightarrow{\Bbbk} J^1\pi_{k+r}$$

determines the affine subbundle $A_{\pi_{k+r,k}}$ consisting of the points $z \in J^1\pi_{k+r}$ satisfying

$$\iota_{1,k} \circ \pi_{k+r,k+1} \circ (\pi_{k+r})_{1,0}(z) = J^1(\pi_{k+r,k}, \mathrm{id}_X)(z) . \tag{4.2}$$

Following the terminology of the well-known situation of $\dim X = n$ and $r = 1$, such elements will be called $\pi_{k+r,k}$-*semiholonomic jets*; the local expression of (4.2) is just (4.1). In fact,

$$\text{(4.3)} \qquad \mathbb{k}_{\Phi_0}\colon \left\{ \begin{array}{ll} q^\sigma_{;} = q^\sigma_{(1)} & q^\sigma_{(k+1);} = z^\sigma_{(k+1)} + \sum_{i=0}^{k} z^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)} \\ \vdots & \vdots \\ q^\sigma_{(k);} = q^\sigma_{(k+1)} & q^\sigma_{(k+r);} = z^\sigma_{(k+r)} + \sum_{i=0}^{k} z^{\sigma(i)}_{(k+r)\lambda} q^\lambda_{(i+1)} \ , \end{array} \right.$$

hence there is a canonical inclusion $J^{k+r+1}\pi \subset A_{\pi_{k+r,k}}$, which corresponds to the associated vector bundle

$$\overline{A}_{\pi_{k+r,k}} = V_{\pi_{k+r,k}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X) \subset V_{\pi_{k+r}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X) \ ;$$

in particular, $A_{\pi_{k+1,k}} \equiv J^{k+2}\pi$ (due to $\dim X = 1$).

Notice first some properties of the sections of $\pi_{k+r,k}$, called *jet fields*; again, we work with global sections for the simplicity only, the same applies (under appropriate restrictions) for the local ones.

A section $\gamma \in \mathcal{S}_U(\pi)$ will be called an *integral section* (or a *path*) of a jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ if it is the solution of the equation $\mathcal{E}^\varphi = \varphi(J^k\pi) \subset J^{k+r}\pi$, i.e. if $\varphi \circ j^k\gamma = j^{k+r}\gamma$ on U. In this respect, φ will be called *integrable* if there is an integral section of φ through each point of Y. In coordinates, the equations of φ are

$$\text{(4.4)} \qquad q^\sigma_{(k+1)} = \varphi^\sigma_{(k+1)}, \dots, q^\sigma_{(k+r)} = \varphi^\sigma_{(k+r)}$$

with the *components* of φ being functions on $J^k\pi$.

For an arbitrary jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$, there is a distinguished associated projection; namely, by

$$\text{(4.5)} \qquad \Gamma^{(k+1)}_\varphi = \pi_{k+r,k+1} \circ \varphi$$

we get a $(k+1)$-connection $\Gamma^{(k+1)}_\varphi$ on π. In coordinates, $\Gamma^\sigma_{(k+1)} = \varphi^\sigma_{(k+1)}$. The following assertion is evident from (4.4) and (3.3).

Proposition 4.1. *A jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is integrable if, and only if, it is the prolongation of its projection $\Gamma^{(k+1)}_\varphi$, i.e. $\varphi = \Gamma^{(k+1)(r-1)}_\varphi$.*

As for higher-order connections, there is a one-dimensional π_{k+r-1}-horizontal distribution H_φ on $J^{k+r-1}\pi$ naturally associated with φ. In fact, $H_\varphi = \operatorname{span}\{D_\varphi\}$, where the generator D_φ is defined by

$$D_\varphi = D^{k+r,k+r-1} \circ \varphi \circ \pi_{k+r-1,k} \ ,$$

i.e. locally

$$D_\varphi = \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} q^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} + \sum_{i=k}^{k+r-1} \varphi^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} \ . \tag{4.6}$$

As to be expected, it can be proved by direct calculations in coordinates that *a section $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of φ if, and only if, $j^{k+r-1}\gamma(U)$ is an integral manifold of H_φ.*

Remark 4.1. It should be stressed that for $r \geq 2$, the integrability of H_φ is *not* equivalent with the integrability of φ in the above presented sense. Nevertheless, the integral section of φ could be defined to be $\psi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r-1})$ such that $\psi(U)$ is an integral manifold of H_φ. Of course, now the equations must be considered on $J^1\pi_{k+r-1}$.

Let Σ be a connection on π_{k+r}. Then it can be called $\pi_{k+r,k}$-*semiholonomic*, if

$$\Sigma \colon J^{k+r}\pi \to A_{\pi_{k+r,k}} \ ,$$

which by (4.1) means just

$$\Sigma^\sigma_{;} = q^\sigma_{(1)}, \ldots, \Sigma^\sigma_{(k);} = q^\sigma_{(k+1)} \ . \tag{4.7}$$

Then the corresponding π_{k+r}-horizontal distribution is spanned by

$$D_\Sigma = \frac{\partial}{\partial t} + \sum_{i=0}^{k} q^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} + \sum_{i=k+1}^{k+r} \Sigma^\sigma_{(i);} \frac{\partial}{\partial q^\sigma_{(i)}} \ , \tag{4.8}$$

which can be then called the $\pi_{k+r,k}$-*semispray* on $J^{k+r}\pi$. In this respect, if $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is a jet field, then H_φ is nothing but a (special type of) $\pi_{k+r-1,k-1}$-semispray distribution on $J^{k+r-1}\pi$, i.e. φ can be identified with a (special type of) $\pi_{k+r-1,k-1}$-semiholonomic connection on π_{k+r-1}, which completes the ideas of Remark 4.1.

Our particular concern is with connections on $\pi_{k+r,k}$. Let $\Xi \colon J^{k+r}\pi \to J^1\pi_{k+r,k}$ be such a connection. The corresponding local equations are

$$z^\sigma_{(i)} = \Xi^\sigma_{(i)}, z^\sigma_{(i)\lambda} = \Xi^\sigma_{(i)\lambda}, \ldots, z^{\sigma(k)}_{(i)\lambda} = \Xi^{\sigma(k)}_{(i)\lambda} \tag{4.9}$$

for $i = k+1, \dots, k+r$, with Ξ's from $\mathcal{F}(J^{k+r}\pi)$. The horizontal form

$$h_\Xi = D_\Xi \otimes dt + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} \otimes dq^\lambda_{(i)} : J^{k+r}\pi \to TJ^{k+r}\pi \otimes \pi^*_{k+r,k}(T^*J^k\pi)$$

is defined by

$$\begin{aligned}
D_{\Xi 0} &= \frac{\partial}{\partial t} + \sum_{i=k+1}^{k+r} \Xi^\sigma_{(i)} \frac{\partial}{\partial q^\sigma_{(i)}} \\
D_{\Xi\lambda} &= \frac{\partial}{\partial q^\lambda} + \sum_{i=k+1}^{k+r} \Xi^\sigma_{(i)\lambda} \frac{\partial}{\partial q^\sigma_{(i)}} \\
&\ \vdots \\
D^{(k)}_{\Xi\lambda} &= \frac{\partial}{\partial q^\lambda_{(k)}} + \sum_{i=k+1}^{k+r} \Xi^{\sigma(k)}_{(i)\lambda} \frac{\partial}{\partial q^\sigma_{(i)}} \,.
\end{aligned} \tag{4.10}$$

The $\pi_{k+r,k}$-horizontal distribution H_Ξ spanned by (4.10) generates the decomposition

$$TJ^{k+r}\pi = H_\Xi \oplus V_{\pi_{k+r,k}} J^{k+r}\pi \,.$$

The point is that *the integral sections* (if any) *of a connection* Ξ *on* $\pi_{k+r,k}$ *are (local) jet fields* from $\mathcal{S}(\pi_{k+r,k})$ satisfying $j^1\varphi = \Xi \circ \varphi$, the local expression of which can be easily derived by (4.9).

DEFINITION 4.1. A connection Ξ on $\pi_{k+r,k}$ will be called *characterizable*, if the connection $\Bbbk_{\Phi_0} \circ \Xi$ is holonomic. The connection $\Gamma^{(k+r+1)}_\Xi = \Bbbk_{\Phi_0} \circ \Xi$ will be called *characteristic* to Ξ.

By (4.3) and (4.7), $\Bbbk_{\Phi_0} \circ \Xi$ is $\pi_{k+r,k}$-semiholonomic for an arbitrary Ξ and it is holonomic if, and only if,

$$\begin{aligned}
q^\sigma_{(k+2)} &= \Xi^\sigma_{(k+1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)} \\
&\ \vdots \\
q^\sigma_{(k+r)} &= \Xi^\sigma_{(k+r-1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+r-1)\lambda} q^\lambda_{(i+1)} \,,
\end{aligned} \tag{4.11}$$

and the components of the characteristic connection are

$$\Gamma^\sigma_{(k+r+1)} = \Xi^\sigma_{(k+r)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+r)\lambda} q^\lambda_{(i+1)} \,. \tag{4.12}$$

PROPOSITION 4.2. *A $(k+r+1)$-connection $\Gamma^{(k+r+1)}$ on π is the characteristic connection of a connection Ξ on $\pi_{k+r,k}$ if, and only if, the components of $\Gamma^{(k+r+1)}$ and Ξ are related by* (4.12), *which equivalently means $H_{\Gamma^{(k+r+1)}} \subset H_{\Xi}$ or*

$$D_{\Gamma^{(k+r+1)}} = D_{\Xi 0} + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} q^{\lambda}_{(i+1)} \,.$$

Here is the motivation of the above constructions.

PROPOSITION 4.3. *Let Ξ be a characterizable connection on $\pi_{k+r,k}$, and $\Gamma^{(k+r+1)}_{\Xi}$ its characteristic connection. Let $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$ be an integral section of Ξ and $\Gamma^{(k+1)}_{\varphi}$ the $(k+1)$-connection on π, defined by* (4.5). *Then $\Gamma^{(k+1)}_{\varphi}$ is a field of paths of $\Gamma^{(k+r+1)}_{\Xi}$ and*

$$\varphi = \Gamma^{(k+1)(r-1)}_{\varphi} \,. \tag{4.13}$$

Proof. It is easy to see that for an arbitrary jet field φ we have

$$\Bbbk_{\Phi_0} \circ j^1\varphi = \Bbbk_{\iota_{1,k}\circ\Gamma^{(k+1)}_{\varphi}} \circ j^1\varphi =^1 (\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}_{\varphi} \,.$$

Then $J^{k+r+1}\pi \ni \Gamma^{(k+r+1)}_{\Xi} \circ\varphi = \Bbbk_{\Phi_0}\circ\Xi\circ\varphi = \Bbbk_{\Phi_0}\circ j^1\varphi = J^1(\varphi, \mathrm{id}_X)\circ\iota_{1,k}\circ\Gamma^{(k+1)}_{\varphi}$. Since by definition

$$\begin{aligned} &J^1(\pi_{k+r,k+1}, \mathrm{id}_X)(j^{k+r+1}_x\gamma) \\ &\quad = j^1_x(\pi_{k+r,k+1} \circ j^{k+r}\gamma) = j^1_x(j^{k+1}\gamma) = j^{k+2}_x\gamma = \pi_{k+r+1,k+2}(j^{k+r+1}_x\gamma) \,, \end{aligned}$$

we have

$$\begin{aligned} \Gamma^{(k+1)(1)}_{\varphi} &= J^1(\Gamma^{(k+1)}_{\varphi}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}_{\varphi} \\ &= J^1(\pi_{k+r,k+1}, \mathrm{id}_X) \circ J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}_{\varphi} \\ &= J^1(\pi_{k+r,k+1}, \mathrm{id}_X) \circ \Gamma^{(k+r+1)}_{\Xi} \circ \varphi = \pi_{k+r,k+2} \circ \varphi \,. \end{aligned}$$

Then by (3.1),

$$\Gamma^{(k+1)(2)}_{\varphi} = J^1(\pi_{k+r,k+2}, \mathrm{id}_X) \circ J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}_{\varphi} = \pi_{k+r,k+3} \circ \varphi \,,$$

and the procedure stops with

$$\Gamma^{(k+1)(r-1)}_{\varphi} = J^1(\pi_{k+r,k+r-1}, \mathrm{id}_X) \circ \Gamma^{(k+r+1)}_{\Xi} \circ \varphi = \pi_{k+r+1,k+r} \circ \Gamma^{(k+r+1)}_{\Xi} \circ \varphi = \varphi \,,$$

which proves (4.13). Finally,

$$\Gamma_{\Xi}^{(k+r+1)} \circ \Gamma_{\varphi}^{(k+1)(r-1)} = \Gamma_{\Xi}^{(k+r+1)} \circ \varphi = J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} =$$
$$= J^1(\Gamma_{\varphi}^{(k+1)(r-1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} = \Gamma_{\varphi}^{(k+1)(r)} . \quad \blacksquare$$

In this arrangement, any connection Ξ on $\pi_{k+r,k}$ whose characteristic connection is a given $\Gamma^{(k+r+1)}$ will be called *associated* to it. If, moreover, Ξ is integrable, then it will be called the $\pi_{k+r,k}$*-integral* of $\Gamma^{(k+r+1)}$.

PROPOSITION 4.4. *Let $\Gamma^{(k+r+1)}$ be a $(k+r+1)$-connection on π and $\{a^1, \dots, a^K\}$, where $K = rm$, be a set of independent first integrals of $\Gamma^{(k+r+1)}$, defined on some open $W \subset J^{k+r}\pi$. If the matrix*

$$A = \left(\frac{\partial a^L}{\partial q^{\sigma}_{(\ell)}} \right) , \tag{4.14}$$

where $\ell = k+1, \dots, k+r$, is regular on W, then

$$H_{\Xi} = \mathrm{anih}\{da^1, \dots, da^K\} \tag{4.15}$$

defines an $\pi_{k+r,k}$-integral of $\Gamma^{(k+r+1)}$ on W.

Proof. First it should be stressed that we suppose $W \subset \pi^{-1}_{k+r,0}(V)$, where (V, ψ) is a fibered chart on Y.

By definition, the distribution (4.15) is completely integrable. Let us denote by $(A^{\sigma}_{(\ell)L})$ the inverse matrix to A, where σ and (ℓ) label the rows and L the columns. Then the annihilators of H_{Ξ} are

$$dq^{\sigma}_{(\ell)} + A^{\sigma}_{(\ell)L} \frac{\partial a^L}{\partial t}\, dt + \sum_{i=0}^{k} A^{\sigma}_{(\ell)L} \frac{\partial a^L}{\partial q^{\lambda}_{(i)}}\, dq^{\lambda}_{(i)}$$

and it remains to show that Ξ is characterizable and its characteristic connection is just $\Gamma^{(k+r+1)}$. Since it holds

$$\begin{aligned}
\Xi^{\sigma}_{(\ell)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(\ell)\lambda} q^{\lambda}_{(i+1)} &= -A^{\sigma}_{(\ell)L} \left(\frac{\partial a^L}{\partial t} + \sum_{i=0}^{k} \frac{\partial a^L}{\partial q^{\lambda}_{(i)}} q^{\lambda}_{(i+1)} \right) \\
&= A^{\sigma}_{(\ell)L} \left(\sum_{i=k+1}^{k+r-1} \frac{\partial a^L}{\partial q^{\lambda}_{(i)}} q^{\lambda}_{(i+1)} + \frac{\partial a^L}{\partial q^{\lambda}_{(k+r)}} \Gamma^{\lambda}_{(k+r+1)} \right) \\
&= \begin{cases} q^{\sigma}_{(\ell+1)}, & \text{for } \ell = k+1, \dots, k+r-1 \\ \Gamma^{\sigma}_{(k+r+1)}, & \text{for } \ell = k+r, \end{cases}
\end{aligned}$$

the proof is completed (see (4.11) and (4.12)). $\blacksquare$

5. Regular Variational Equations

Let $E \in \Omega^2(J^s\pi)$, $s \geq 2$, be a $\pi_{s,0}$-horizontal form such that in coordinates

$$E = E_\sigma dq^\sigma \wedge dt, \tag{5.1}$$

$$E_\sigma = A_\sigma + B_{\sigma\nu} q^\nu_{(s)}, \tag{5.2}$$

where $A_\sigma, B_{\sigma\nu} \in \mathcal{F}(J^{s-1}\pi)$. Any such a form E generates naturally the s-th order differential equation

$$\mathcal{E}^E = \{j^s_x\gamma,\ E(j^s_x\gamma) = 0\} \subset J^s\pi \tag{5.3}$$

with the solutions (called the *extremals* of E) being $\gamma \in \mathcal{S}_{\rm loc}(\pi)$ satisfying

$$E \circ j^s\gamma = 0\ . \tag{5.4}$$

The form E and accordingly the equation (5.3) are called *regular* if $\det(B_{\sigma\nu}) \neq 0$, and in such a case, the equation (5.3) can be represented by the s-connection $\Gamma^{(s)}_E$ on π, defined by

$$E \circ \Gamma^{(s)}_E = 0\ , \tag{5.5}$$

called *characteristic* to E. In coordinates, (5.5) means equivalently $E_\sigma \circ \Gamma^{(s)}_E = 0$ or $E_\sigma = B_{\sigma\nu}(q^\nu_{(s)} - \Gamma^\nu_{(s)})$ or $\Gamma^\sigma_{(s)} = -B^{\sigma\nu}A_\nu$, where $(B^{\sigma\nu})$ is the inverse matrix to $(B_{\sigma\nu})$.

A form E is called *variational* if there is an integer r and a *lagrangian* λ on $J^r\pi$, such that E is just the *Euler-Lagrange form* of λ (up to the projection). The point is that (5.2) is just a necessary condition for E to be *locally variational*, which by definition means that there is an open covering $\{W_\alpha\}$ of $J^s\pi$ such that for each α, $E|_{W_\alpha}$ is variational. In this respect, the s-connection $\Gamma^{(s)}$ on π (and accordingly the corresponding horizontal distribution $H_{\Gamma^{(s)}}$ or the equation $\Gamma^{(s)}(J^{s-1}\pi)$) is called *variational* if there is a regular locally variational form E related to $\Gamma^{(s)}$ by (5.5).

In what follows, we suppose $\Gamma^{(s)}_E$ to be the characteristic connection of a given *regular locally variational* form E and $s = 2c$. In such a case, $H_{\Gamma^{(s)}_E}$ is the so-called *Euler-Lagrange distribution* associated to E, which is the characteristic distribution of the distinguished 2-form α_E, called the *Lepagean equivalent* of E; this means

$$H_{\Gamma^{(s)}_E} = \mathrm{anih}\{i_{\zeta^{(s-1)}}\alpha_E;\ \zeta^{(s-1)} \in \mathcal{X}^v_X(J^{s-1}\pi)\}\ .$$

Consequently, there is another important characterization of the solutions of (5.3) (= integral sections of $\Gamma_E^{(s)}$): a section $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ satisfies (5.4) if, and only if,

$$(j^{s-1}\gamma)^* i_{\zeta^{(s-1)}} \alpha_E = 0 \tag{5.6}$$

for an arbitrary $\zeta^{(s-1)} \in \mathcal{X}_X^v(J^{s-1}\pi)$.

A jet field $\varphi \in \mathcal{S}_V(\pi_{s-1,c-1})$ is called a *field of extremals* of E on $V \subset J^{c-1}\pi$, if it is the integral section of α_E, which by definition means

$$\varphi^* \alpha_E = 0 \; ; \tag{5.7}$$

this relation is called the *Hamilton-Jacobi equation*. For an arbitrary field of extremals φ of E, the projection

$$\Gamma_\varphi^{(c)} = \pi_{s-1,c} \circ \varphi$$

is called the *Hamilton-Jacobi field* of E. Clearly, it is a local c-connection on π, and the corresponding π_{c-1}-horizontal distribution $H_{\Gamma_\varphi^{(c)}}$ on $V \subset J^{c-1}\pi$ is called the *Hamilton-Jacobi distribution* of E.

If $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ is an integral section of φ, i.e.

$$j^{s-1}\gamma = \varphi \circ j^{c-1}\gamma \; , \tag{5.8}$$

then γ is an integral section of $\Gamma_\varphi^{(c)}$. Conversely, it can be shown that if γ is an integral section of $\Gamma_\varphi^{(c)}$, then for each $\zeta^{(s-1)} \in \mathcal{X}_X^v(J^{s-1}\pi)$ it holds

$$(j^{c-1}\gamma)^* \varphi^* i_{\zeta^{(s-1)}} \alpha_E = (j^{c-1}\gamma)^* i_{\zeta^{(c-1)}} \varphi^* \alpha_E \; , \tag{5.9}$$

where $\zeta^{(c-1)} \in \mathcal{X}_X^v(J^{c-1}\pi)$ is defined by

$$\zeta^{(c-1)}(j_x^{c-1}\gamma) = T\pi_{s-1,c-1}\zeta^{(s-1)}(\varphi(j_x^{c-1}\gamma)) \; . \tag{5.10}$$

Consequently, by (5.9) together with (5.7), $(\varphi \circ j^{c-1}\gamma)^* i_{\zeta^{(s-1)}} \alpha_E = 0$, which means that $\varphi \circ j^{c-1}\gamma = j^{s-1}\overline{\gamma}$, where $\overline{\gamma}$ is an extremal of E. Since

$$j^c\overline{\gamma} = \pi_{s-1,c} \circ j^{s-1}\overline{\gamma} = \pi_{s-1,c} \circ \varphi \circ j^{c-1}\gamma = \Gamma_\varphi^{(c)} \circ j^{c-1}\gamma = j^c\gamma \; ,$$

we get (5.8).

On the whole, $\varphi(V)$ is foliated by the $(s-1)$-jets of the integral sections of $\Gamma_\varphi^{(c)}$, in other words, *a field of extremals φ is just the $(s-c-1)$-th prolongation of the corresponding Hamilton-Jacobi field $\Gamma_\varphi^{(c)}$*, i.e. $\varphi = \Gamma_\varphi^{(c)(s-c-1)}$.

Let $\zeta \in H_{\Gamma_\varphi^{(c)}}$, $\zeta^{(s-1)} \in \mathcal{X}_X^v(J^{s-1}\pi)$ and $z \in V$. Evidently, we can suppose $z = j_x^{c-1}\gamma$ with γ being an integral section of $\Gamma_\varphi^{(c)}$. Then again by (5.9),

$$\begin{aligned} i_{\zeta^{(s-1)}}\alpha_E(\Gamma_\varphi^{(c)(s-c-1)}(z))(T_z\Gamma_\varphi^{(c)(s-c-1)}(\zeta)) &= i_{\zeta^{(s-1)}}\alpha_E(\varphi(z))(T_z\varphi(\zeta)) \\ &= \varphi^* i_{\zeta^{(s-1)}}\alpha_E(z)(\zeta) = \varphi^* i_{\zeta^{(s-1)}}\alpha_E(j_x^{c-1}\gamma)(T_x j^{c-1}\gamma(\xi)) \\ &= (j^{c-1}\gamma)^*\varphi^* i_{\zeta^{(s-1)}}\alpha_E(x)(\xi) = (j^{c-1}\gamma)^* i_{\zeta^{(c-1)}}\varphi^*\alpha_E(x)(\xi) = 0 \end{aligned}$$

with $\xi \in T_xX$ and $\zeta^{(c-1)}$ related to $\zeta^{(s-1)}$ by (5.10). Consequently, $T\varphi(\zeta) \in H_{\Gamma_E^{(s)}}$, which is by (3.5) equivalent to the fact that the *Hamilton-Jacobi field* $\Gamma_\varphi^{(c)}$ *is a field of paths of* $\Gamma_E^{(s)}$, i.e. $\Gamma_E^{(s)} \circ \varphi = \Gamma_\varphi^{(c)(s-c)}$. In fact, the conditions for each Hamilton-Jacobi field are stronger, expressing its relation to E; nevertheless, from our point of view, Hamilton-Jacobi fields represent a (local) reduction of the order of (5.3) from $s = 2c$ to c. On the other hand, the variational character of the equations studied results in the existence of a special kind of (local) integrals of $\Gamma_E^{(s)}$, called the *Jacobi complete integrals*. These are constructed (in sense of Prop. 4.4) through the existence (ensured by the *Jacobi theorem*) of the corresponding system of independent first integrals of $H_{\Gamma_E^{(s)}}$ (*Jacobi charts* of E) – we refer to [8] for more details.

Notice finally that the most classical situation is that of E being the Euler-Lagrange form of a *global regular lagrangian* λ on $J^c\pi$. In such a case, (5.3) or (5.4) are nothing but the *Euler-Lagrange equations* of λ,

$$E_\sigma = \sum_{j=0}^{c}(-1)^j D^{(j)}\left(\frac{\partial L}{\partial q^\sigma_{(j)}}\right) ,$$

and

$$B_{\sigma\nu} = (-1)^c\left(\frac{\partial^2 L}{\partial q^\sigma_{(c)}\partial q^\nu_{(c)}}\right),$$

with $\lambda = L\,dt$.

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