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The Construction of Cell-to-Cell Mappings from Frobenius-Perron Operators†

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From ergodic theory it is well-known how to associate a Markov operator with a dynamical system. This operator, which represents the probabilistic properties of the deterministic system, is called Frobenius-Perron operator. In the case of a discrete time dynamical system this concept is applied to construct cell-to-cell mappings which have been used in the numerical analysis of dynamical systems.

1. Introduction

In recent years computer experiments in studying nonlinear dynamics have become more and more prominent. Since computers produce round-off errors, it is natural to study approximations of a given dynamical systems. This is done by employing the statistical point of view provided by the ergodic theory of deterministic systems, cf. [5]. With each dynamical system a Frobenius-Perron operator, which reflects the probabilistic properties of the deterministic system, is associated. This operator in turn induces a Markov process: Instead of studying individual trajectories of a given deterministic system, one examines the evolution of probability distributions on the phase space of the dynamical system.

Partitioning the phase space of the dynamical system into a finite number of so-called cells, leads to an approximation of the corresponding Frobenius-Perron operator. This approximation is a stochastic matrix whose entries are the transition probabilities from one cell to another. The proof of convergence of such approximations to the original Frobenius-Perron operator is outlined when refining the partitions.

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The stochastic matrices thus obtained may be interpreted as cell-to-cell mappings, whereas the cells of the corresponding partitions may be thought of as round-off errors of a computer. Cell-to-cell mappings have been used in the numerical analysis of dynamical systems, cf. [3, 6].

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2. Dynamical Systems and Frobenius-Perron Operators

Let X be a compact subset of $\mathbb{R}^d$ which for simplicity is assumed to be rectangular, i.e.

$$x_{min}^{(i)} \leq x^{(i)} \leq x_{max}^{(i)} \text{ for } x = (x^{(1)}, \dots, x^{(d)}) \in X, \quad i = 1, \dots, d.$$

Let $\mathcal{B}$ denote the Borel σ-algebra of X generated by the open subsets of X and $\text{Prob}(X)$ the set of all probability measures on X, i.e. the set of all measures with $\nu(X) = 1$. In particular, let $\mu \in \text{Prob}(X)$ denote the usual Lebesgue measure normalized to 1. Consider a discrete time dynamical system induced by the continuous transformation $S : X \longrightarrow X$. Picking an initial point $x_0 \in X$, the successive states of the system defined by S at times $1, 2, \dots$ are given by the trajectory

$$x_0, S(x_0), S^2(x_0) = S(S(x_0)), \dots .$$

Instead of looking at single trajectories, it turns out to be useful to look at a distribution of all possible trajectories. This can be done by associating an operator $\mathcal{P}_S : \text{Prob}(X) \longrightarrow \text{Prob}(X)$ defined by

$$\mathcal{P}_S \nu(B) = \nu(S^{-1}(B)), \quad B \in \mathcal{B} \tag{2.1}$$

with the transformation S. $\mathcal{P}_S$ is called **Frobenius-Perron operator on measures** corresponding to S, cf. [5].

It can be shown that $\mathcal{P}_S$ is an example of a Markov operator. The iterates of $\mathcal{P}_S$ describe the evolution of probability distributions on X. In particular, iterates of $\mathcal{P}_S$ can reproduce a trajectory of the transformation S in the following way. Let $x_0 \in X$ be fixed and consider the Dirac measure δ_{x_0} supported on the single point x_0. Since $S^{-1}(B) = \{x \in X \mid S(x) \in B\}$, then

$$\mathcal{P}_S \delta_{x_0}(B) = \delta_{x_0}(S^{-1}(B)) = \delta_{S(x_0)}(B) \quad \text{for } B \in \mathcal{B} \tag{2.2}$$

and by induction $\mathcal{P}_S^k \delta_{x_0} = \delta_{S^k(x_0)}$. In order to obtain single trajectories, it is thus sufficient to start from a Dirac measure. If in addition

$$|S(y) - S(z)| \leq C\ |y - z| \quad \text{for } y, z \in X$$

and $C < 1$, then the classical contraction principle shows that all trajectories $(S^k(x_0))$ converge to the unique fixed point $x_\star = S(x_\star)$. It can be proven that in this case for arbitrary $\nu \in \mathrm{Prob}(X)$, the sequence $(\mathcal{P}_S^k \nu)$ converges to $\delta_{x_\star}$ with respect to the *weak* topology*[1], where $\delta_{x_\star}$ is the unique so-called stationary distribution, i.e. $\delta_{x_\star} = \mathcal{P}_S \delta_{x_\star}$, cf. [5] for details. Notice that this observation may easily be generalized to period-k cycles. Clearly, $\{x_0, \dots, x_{k-1}\}$ is a period-k cycle of S if and only if $\frac{1}{k} \sum_{i=0}^{k-1} \delta_{x_i}$ is a stationary distribution of $\mathcal{P}_S$ with $\mathcal{P}_S \delta_{x_i} = \delta_{x_{i+1}}$, $i = 0, \dots, k-1$.

If S is a nonsingular transformation, which means that the inverse image $S^{-1}(B)$ of any set $B \in \mathcal{B}$ of Lebesgue measure zero has Lebesgue measure zero, then $\mathcal{P}_S$ can also transform densities. Recall that a function $f : X \longrightarrow \mathbb{R}$ is integrable in the sense of Lebesgue, if $\int_X f(x)\ \mu(dx) < \infty$ and denote the set of all such functions by $L^1 = L^1(\mu)$. This space endowed with the norm

$$\|f\|_{L^1} := \int_X |f(x)|\ \mu(dx) < \infty$$

is a Banach space. Let ν be an absolutely continuous measure w.r.t. the Lebesgue measure μ, that is $\nu(B) = \int_B f(x)\ \mu(dx)$, $B \in \mathcal{B}$ for some $f \in L^1$, $f \geq 0$, then

$$\nu(S^{-1}(B)) = \int_{S^{-1}(B)} f(x)\ \mu(dx).$$

Since S is nonsingular, by the theorem of Radon-Nikodym, there exists a unique $P_S f \in L^1$ such that the right hand side of (2.1) may be rewritten as

$$\nu(S^{-1}(B)) = \int_B P_S f(x)\ \mu(dx) \text{ for } B \in \mathcal{B}. \tag{2.3}$$

For arbitrary $f \in L^1$, we define $P_S f := P_S(f^+) - P_S(f^-)$, where

$$f = f^+ - f^- \text{ with } f^+ = \sup(f, 0), \quad f^- = \sup(-f, 0).$$

It is easy to check that P_S is linear. Thus, if the transformation $S : X \longrightarrow X$ is nonsingular, then there exists a unique linear operator $P_S : L^1 \longrightarrow L^1$, given by

$$\int_B P_S f(x)\ \mu(dx) = \int_{S^{-1}(B)} f(x)\ \mu(dx) \text{ for } B \in \mathcal{B}.$$

[1] Some authors refer to this topology as the weak topology.

P_S is called **Frobenius-Perron operator** corresponding to S. It is a matter of routine to check that P_S satisfies the properties of a Markov operator, that is

$$\begin{aligned} &(i) \qquad P_S f \geq 0 \text{ and } \|P_S f\|_{L^1} = \|f\|_{L^1} \text{ for } f \in L^1,\ f \geq 0; \\ &(ii) \qquad \|P_S f\|_{L^1} \leq \|f\|_{L^1} \text{ for } f \in L^1. \end{aligned} \tag{2.4}$$

Thus, P_S is a particular example of a Markov operator. Analogous properties for $\mathcal{P}_S$ hold true, cf. [5]. Note that if $S^n := S\circ \overset{n}{\dots} \circ S$ and P_S is the Frobenius-Perron operator associated with S, then $P_{S^n} = P_S^n$ is the Frobenius-Perron operator corresponding to S^n. It follows from (2.4,i) that P_S may be restricted to an operator on densities $P_S : \mathcal{D} \longrightarrow \mathcal{D}$, where

$$\mathcal{D} := \{f \in L^1 \mid f \geq 0 \text{ and } \|f\|_{L^1} = 1\}$$

is the space of all densities. In the case where S is nonsingular, P_S may be used to characterize irregular behavior of S such as ergodicity, mixing and exactness, see [5].

3. Simple Cell-to-Cell Mappings

This section deals with the construction of simple cell-to-cell mappings from Frobenius-Perron operators on measures. Let $\{B_{Ni}\}$ be a partition of the phase space X into N pairwise disjoint hypercubes $B_{Ni} \in \mathcal{B}$, $i = 1, \dots, N$, where $X = \bigcup_{i=1}^N B_{Ni}$. In the sequel these hypercubes will be called *cells*. Let $z_{Ni} \in B_{Ni}$ denote the middle point[2] of the cell B_{Ni}, $i = 1, \dots, N$. Then the Dirac measures $\delta_{z_{N1}}, \dots, \delta_{z_{NN}}$ span a measure space

$$\text{Prob}_N(X) := \left\{ \sum_{i=1}^N \alpha_i \delta_{z_{Ni}} \;\middle|\; \sum_{i=1}^N \alpha_i = 1 \text{ and } \alpha_i \geq 0,\ i = 1, \dots, N \right\}$$

which is called the space of discrete probability measures associated with the partition $\{B_{Ni}\}$. Each probability measure ν on X can now be approximated by a discrete probability measure by means of the mapping

$$\Pi_N : \begin{cases} \text{Prob}(X) & \longrightarrow \quad \text{Prob}_N(X) \\ \quad \nu & \longmapsto \quad \nu_N := \sum_{i=1}^N \nu(B_{Ni}) \delta_{z_{Ni}} \end{cases} \tag{3.1}$$

Observe that by construction $\nu_N(B_{Ni}) = \nu(B_{Ni})$ for $i = 1, \dots, N$ and in particular $\nu_N(X) = \nu(X) = 1$. Therefore, Π_N is a projection, $\Pi_N \circ \Pi_N = \Pi_N$.

[2]Each other point inside of B_{Ni} could equally well be chosen.

Let $\imath_N : \mathrm{Prob}_N(X) \hookrightarrow \mathrm{Prob}(X)$ denote the natural inclusion and consider the linear operator $\mathcal{P}_N$ defined by

$$\mathcal{P}_N : \mathrm{Prob}_N(X) \longrightarrow \mathrm{Prob}_N(X), \quad \mathcal{P}_N := \Pi_N \circ \mathcal{P}_S \circ \imath_N .$$

Using the basis $\delta_{z_{N1}}, \dots, \delta_{z_{NN}}$ of $\mathrm{Prob}_N(X)$, a coordinate representation of $\mathcal{P}_N$ can be constructed as follows. Since $\{B_{Ni}\}$ is a partition of the phase space X, for each point $x \in X$ there exists a unique i_x for which $x \in B_{Ni_x}$. The image of any Dirac measure δ_x, $x \in X$ under the map (3.1) then calculates as

$$\Pi_N \delta_x = \delta_{z_{Ni_x}}, \text{ where } x \in B_{Ni_x}.$$

Since $\mathcal{P}_S \delta_x = \delta_{S(x)}$, the coordinate representation of $\mathcal{P}_N$ is thus given by

$$\mathcal{P}_N \delta_{z_{Nj}} = \sum_{i=1}^{N} p^i_{Nj} \delta_{z_{Ni}}, \text{ where } p^i_{Nj} = \begin{cases} 1 & \text{if } S(z_{Nj}) \in B_{Ni} \\ 0 & \text{if } S(z_{Nj}) \notin B_{Ni} \end{cases}$$

Observe that for each j_0 there is exactly one i_0 such that $S(z_{Nj_0}) \in B_{Ni_0}$. This implies that each column of the matrix (p^i_{Nj}) has exactly one entry equals to 1 whereas all other entries are 0. Thus, (p^i_{Nj}) is the transpose of a stochastic matrix. Since the transition probabilities p^i_{Nj} do not explicitly depend on time, $\mathcal{P}_N$ corresponds to a finite Markov chain, cf. [4]. In other words, $\mathcal{P}_N$ is nothing else but a simple cell-to-cell mapping.

4. Approximation of Markov Operators on Densities

Since the construction of generalized cell-to-cell mappings is somewhat more involved, we will consider approximations of arbitrary Markov operators on densities first. Let $L^\infty = L^\infty(\mu)$ denote the Banach space of all functions $g : X \longrightarrow \mathbb{R}$ which are essentially bounded, i.e. $|g(x)| \leq c$ for μ-almost all $x \in X$. Since the measure μ is finite, $L^\infty \subset L^1$. If $f \in L^1$ and $g \in L^\infty$, then the product fg is integrable and the Lebesgue integral defines the continuous bilinear pairing

$$L^1 \times L^\infty \longrightarrow \mathbb{R}, \quad \langle f, g \rangle = \int_X f(x) g(x) \; \mu(dx). \tag{4.1}$$

Consider a partition $\{B_{Ni}\}$ of the phase space X into N hypercubes (cells) B_{Ni}, $i = 1, \dots, N$ as before. Since the B_{Ni} are measurable, the characteristic functions $\chi_{B_{Ni}}$ and the normalized characteristic functions $\bar{\chi}_{B_{Ni}} := \frac{1}{\mu(B_{Ni})} \chi_{B_{Ni}}$ are functions in both L^∞ and L^1. In particular, the normalized characteristic functions span the N-dimensional linear subspace $\mathcal{E}_N = [\bar{\chi}_{B_{N1}}, \dots, \bar{\chi}_{B_{NN}}]$ of L^1. Since the B_{Ni} are disjoint, the pairing (4.1) yields

$$\langle \bar{\chi}_{B_{Ni}}, \chi_{B_{Nj}} \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \tag{4.2}$$

stating that $\chi_{B_{N1}}, \dots, \chi_{B_{NN}} \in L^\infty$ is the dual basis of $\bar{\chi}_{B_{N1}}, \dots, \bar{\chi}_{B_{NN}} \in L^1$. It is seen from (4.2) that the mapping

$$\pi_N : L^1 \longrightarrow \mathcal{E}_N, \quad \pi_N f := \sum_{i=1}^{N} \langle f, \chi_{B_{Ni}} \rangle \, \bar{\chi}_{B_{Ni}} \tag{4.3}$$

is a projection, $\pi_N \circ \pi_N = \pi_N$. Moreover, it is shown in [7] that the projection (4.3) is continuous and $\|\pi_N\|_{L^1} = 1$. Note that $\langle f, \chi_{B_{Ni}} \rangle = \int_{B_{Ni}} f(x) \, \mu(dx)$ is the mean value of the function f over the set B_{Ni}. In particular, for arbitrary $f \in L^1$, $f \geq 0$, one has

$$\|\pi_N f\|_{L^1} = \sum_{i=1}^{N} \langle f, \chi_{B_{Ni}} \rangle \cdot \int_X \bar{\chi}_{B_{Ni}}(x) \, \mu(dx) = \sum_{i=1}^{N} \langle f, \chi_{B_{Ni}} \rangle \cdot 1 = \|f\|_{L^1}. \tag{4.4}$$

Now let $\imath_N : \mathcal{E}_N \hookrightarrow L^1$ denote the inclusion map. Consider the linear operator P_N defined by

$$P_N : \mathcal{E}_N \longrightarrow \mathcal{E}_N, \quad P_N := \pi_N \circ P \circ \imath_N, \tag{4.5}$$

where P is some Markov operator. Since $\mathcal{E}_N$ is finite-dimensional, P_N is a discrete approximation of P associated with the partition $\{B_{Ni}\}$. Setting $f_N = \pi_N f$ for $f \in L^1$, it follows from (4.4) and property (2.4,i) of Markov operators that

$$P_N f_N \geq 0 \text{ and } \|P_N f_N\|_{L^1} = \|P f_N\|_{L^1} = \|f_N\|_{L^1} \quad \forall f_N \in \mathcal{E}_N, \; f_N \geq 0. \tag{4.6}$$

Property (2.4,ii) of Markov operators then states that

$$\|P_N f_N\|_{L^1} \leq \|f_N\|_{L^1} \quad \forall f_N \in \mathcal{E}_N.$$

Therefore, $P_N : \mathcal{E}_N \longrightarrow \mathcal{E}_N$ is again a Markov operator. Using the basis of $\mathcal{E}_N$, P_N has the coordinate representation

$$P_N \bar{\chi}_{B_{Nj}} = \sum_{i=1}^{N} p_{Nj}^{i} \, \bar{\chi}_{B_{Ni}}, \quad j = 1, \dots, N.$$

This implies that (4.6) is equivalent to

$$p^i_{Nj} \geq 0 \text{ and } \sum_{i=1}^{N} p^i_{Nj} = 1. \tag{4.7}$$

Thus, (p^i_{Nj}) is a stochastic matrix[3] with transition probabilities p^i_{Nj}. The second equality in (4.7) is known as the *Markov property* of a stochastic matrix which states that the system will never "die". Since the transition probabilities p^i_{Nj} do not explicitly depend on time, here again P_N corresponds to a finite Markov chain, cf. [4].

5. Generalized Cell-to-Cell mappings

We are ready to construct generalized cell-to-cell mappings from Frobenius-Perron operators on densities. Let $P = P_S$ be the Frobenius-Perron operator associated with the nonsingular transformation $S : X \longrightarrow X$. The transition probabilities p^i_{Nj} of the corresponding approximation P_N can now be calculated in terms of measures of the sets $B_{Ni} \in \mathcal{B}_N$ and their inverse images under S

$$S^{-1}(B_{Ni}) = \{x \in X \mid S(x) \in B_{Ni}\}.$$

PROPOSITION 5.1. *The matrix representation (p^i_{Nj}) of the discrete Frobenius-Perron operator P_N associated with the nonsingular transformation $S : X \longrightarrow X$ and the partition $\{B_{Ni}\}$ is given by*

$$p^i_{Nj} = \frac{\mu(S^{-1}(B_{Ni}) \cap B_{Nj})}{\mu(B_{Nj})}, \quad i, j = 1, \dots, N.$$

The proof of Proposition 5.1 can be found in [7]. The transition probabilities provided by Proposition 5.1 allow the following interpretation. Given a system governed by the nonsingular transformation S,

$$S^{-1}(B_{Ni}) \cap B_{Nj} = \{x \in X \mid S(x) \in B_{Ni} \text{ and } x \in B_{Nj}\}$$

is the set of all states $x \in X$ which pass from cell B_{Nj} to cell B_{Ni} within one period. Therefore,

$$p^i_{Nj} = \frac{\mu(S^{-1}(B_{Ni}) \cap B_{Nj})}{\mu(B_{Nj})}$$

may be interpreted as the conditional probability that the system is in cell B_{Ni}, knowing that the system is in cell B_{Nj} one period before. Here, the

[3] Actually, (p^i_{Nj}) is the transpose of a stochastic matrix.

Markov property of (p_{Nj}^i) means that no trajectory will leave the phase space X.

If $S : X \longrightarrow X$ is measure preserving, $\mu(B) = \mu(S^{-1}(B))$ for all $B \in \mathcal{B}$, and invertible, then $\mu(B) = \mu(S(B))$ for all $\in \mathcal{B}$. In this case the transition probabilities p_{Nj}^i of P_N satisfy

$$p_{Nj}^i = \frac{\mu(S^{-1}(B_{Ni}) \cap B_{Nj})}{\mu(B_{Nj})} = \frac{\mu(B_{Ni} \cap S(B_{Nj}))}{\mu(S(B_{Nj}))}, \quad i,j = 1,\ldots,N. \tag{5.1}$$

The transition probabilities (5.1) have been used in the works of the authors Hsu and Kreuzer, see [3, 6] and references therein.

The transition probabilities provided by Proposition 5.1 will, in general, be difficult to calculate, since one has to find the inverse image of each set of a given partition under the transformation S. From the point of view of numerical analysis, it is therefore necessary to have good approximations for these probabilities. The basic idea of the method introduced here is due to Kreuzer [6]. We will outline next that this method provides good approximations of the transition probabilities given in Proposition 5.1.

Consider a partition $\{B_{Ni}\}$ of the phase space X into N pairwise disjoint sets B_{Ni}. Let B_{Mj}, $j = 1,\ldots,M$, where $M \geq N$, be a refinement of this partition such that for each $i = 1,\ldots,N$,

$$B_{Ni} = \bigcup_{k=1}^{M_i} B_{Mi_k}, \quad B_{Mi_k} \cap B_{Mi_l} = \emptyset \text{ whenever } i_k \neq i_l \tag{5.2}$$

and $\sum_{i=1}^N M_i = M$. Choose $x_{Mj} \in B_{Mj}$, $j = 1,\ldots,M$ and define a measure μ_M on X by

$$\mu_M(B) := \sum_{j=1}^M \mu(B_{Mj})\delta_{x_{Mj}}(B), \quad B \in \mathcal{B}.$$

According to Section 3, μ_M defines a discrete probability measure on X. It follows from (5.2) that

$$\mu_M(B_{Ni}) = \sum_{k=1}^{M_i} \mu(B_{Mi_k}) = \mu(B_{Ni}), \quad i = 1,\ldots,N$$

and $\mu_M(X) = \mu(X) = 1$ as before. Using μ_M as an approximation of the original measure μ, gives

$$p_{Nj}^i \approx \frac{\mu_M(S^{-1}(B_{Ni}) \cap B_{Nj})}{\mu_M(B_{Nj})}, \quad i,j = 1,\ldots,N \tag{5.3}$$

for the transition probabilities. In particular, one may choose $M_i = K$ for all i such that $\mu(B_{Ni}) = \frac{1}{N}$, $i = 1, \ldots, N$ and $\mu(B_{Mj}) = \frac{1}{KN}$, $j = 1, \ldots, M$. Then, using (2.2), for each $i, j = 1, \ldots, N$,

$$\begin{aligned}\frac{\mu_M(S^{-1}(B_{Ni}) \cap B_{Nj})}{\mu_M(B_{Nj})} &= \sum_{l,k=1}^{N,K} \tfrac{\mu(B_{Ml_k})}{\mu(B_{Nj})} \delta_{x_{Ml_k}}(S^{-1}(B_{Ni}) \cap B_{Nj}) \\ &= \sum_{k=1}^{K} \tfrac{1}{K} \delta_{x_{Mj_k}}(S^{-1}(B_{Ni})) = \sum_{k=1}^{K} \tfrac{1}{K} \delta_{S(x_{Mj_k})}(B_{Ni}).\end{aligned}$$

Notice that term of the right hand side may be evaluated by a computer. For $K = 1$ one obtains once again a simple cell-to-cell mapping. For $K > 1$ one obtains a general cell-to-cell mapping, where the transition probabilities are the ones first obtain by Kreuzer [6]. In the sense of Proposition 6.1 below, μ_M is a good approximation of μ. Moreover, if S is continuous and nonsingular, it is shown in [7] that the right hand side of (5.3) converges to p^i_{Nj} when M tends to infinity.

6. On the Convergence of the Approximations

In this section we briefly outline the question what happens if the partition of the phase space is refined. This can be achieved by subdividing each hypercube B_{Ni}, $i = 1, \ldots, N$ into smaller hypercubes B_{Mj}, $j = 1, \ldots, M$, where $M \geq N$ and each B_{Ni_0} is the disjoint union of some of the cubes B_{Mj}. Thus the family of hypercubes B_{Mj}, $j = 1, \ldots, M$ constitutes a refinement of the original partition. This process may be repeated for each integer. Each family of hypercubes B_{Ni}, $i = 1, \ldots, N$ generates a σ-subalgebra $\mathcal{B}_N$ of the original Borel σ-algebra such that $\mathcal{B}_N \subset \mathcal{B}_M$ for $N \leq M$. Moreover, the sequence of partitions $\{B_{Ni}\}_{N \in \mathbb{N}}$ converges to points in X in the following sense (cf. [2]): for each $B \in \mathcal{B}$ there is a sequence of sets $B_N \in \mathcal{B}_N$ such that

$$\mu(B_N \triangle B) \to 0 \quad \text{when} \quad N \to \infty, \tag{6.1}$$

where $B_N \triangle B := (B_N \backslash B) \cup (B \backslash B_N)$ denotes the symmetric difference of two sets.

It is now natural to ask whether for a given Frobenius-Perron operator P_S, the sequence of approximations $(P_N)_{N \in \mathbb{N}}$ corresponding to the σ-algebras $(\mathcal{B}_N)_{N \in \mathbb{N}}$ converges to P_S and analogously, whether the sequence $(\mathcal{P}_N)_{N \in \mathbb{N}}$ converges to $\mathcal{P}_S$.

In the first case it is straightforward to show that for each $f \in L^1$ the sequence of μ-integrable functions $(\pi_N f)_{N \in \mathbb{N}}$ is a martingale. In probability theory, $f_N := \pi_N f$ is also referred to as the *conditional expectation* of f. Applying standard results in martingale theory (see e.g. [1]), it is shown in [7] that the sequence $(f_N)_{N\in\mathbb{N}}$ converges to f both μ-almost everywhere and in the mean, that is with respect to the L^1-norm. This observation leads to the following result.

THEOREM 6.1. *Let $\{B_{Ni}\}_{N\in\mathbb{N}}$ be a sequence of partitions of X which converges to points in the sense of (6.1) as N tends to infinity. Let P be a Markov operator and P_N its discrete approximation corresponding to $\mathcal{B}_N$. Then for each $f \in L^1$,*

$$\|Pf - P_N f_N\|_{L^1} \to 0 \quad \textit{when} \quad N \to \infty.$$

Proof. Let $f \in L^1$ be arbitrary. Using $\|P\|_{L^1} \le 1$ and the definition of P_N, then

$$\begin{aligned} \|Pf - P_N f_N\|_{L^1} &\le \|Pf - Pf_N\|_{L^1} + \|Pf_N - P_N f_N\|_{L^1} \\ &\le \underbrace{\|f - f_N\|_{L^1}}_{\to 0} + \underbrace{\|Pf_N - \pi_N(Pf_N)\|_{L^1}}_{\to 0} \end{aligned}$$

when $N \to \infty$. The convergence of the first term follows from the martingale property just stated. The convergence of the second term can be seen as follows. Let $(h_m)_{m\in\mathbb{N}}$ be a sequence of functions in L^1 which in the L^1-norm converges to $h \in L^1$. By the continuity of π_N (see Eq. (4.3)) and the martingale property,

$$\begin{aligned} \|\pi_N h_m - h_m\|_{L^1} &\le \|\pi_N(h_m - h)\|_{L^1} + \|\pi_N h - h\|_{L^1} + \|h - h_m\|_{L^1} \\ &\le 2\underbrace{\|h - h_m\|_{L^1}}_{\to 0} + \underbrace{\|\pi_N h - h\|_{L^1}}_{\to 0} \end{aligned}$$

when $N \to \infty$ and $m \to \infty$. This proves the convergence in the mean. ∎

In the second case observe that the set of all discrete probability measures is dense in Prob(X) with respect to the weak* topology, cf. [1]. Using the map $\Pi_N : \text{Prob}(X) \longrightarrow \text{Prob}_N(X)$ defined in (3.1), one may state the following.

PROPOSITION 6.1. *Let $\{B_{Ni}\}_{N\in\mathbb{N}}$ be a sequence of partitions of X which converges to points in the sense of (6.1) as N tends to infinity. Then for each measure $\nu \in$ Prob(X),*

$$\Pi_N \nu \longrightarrow \nu \quad \text{weak}^\star \textit{ in Prob}(X) \quad \text{when } N \to \infty.$$

THEOREM 6.2. *Let $\{B_{Ni}\}_{N\in\mathbb{N}}$ be a sequence of partitions of X which converges to points in the sense of (6.1) as N tends to infinity. Setting $\nu_N := \Pi_N \nu$, then for each measure $\nu \in Prob(X)$,*

$$\mathcal{P}_N \nu_N \longrightarrow \mathcal{P}_S \nu \quad \text{weak}^\star \textit{ in } Prob(X) \quad \text{when } N \to \infty.$$

The proofs of Proposition 6.1 and Theorem 6.2 are somewhat more technical and are omitted here (see [7] for details). However, notice that the notion of convergence in Theorem 6.2 is a lot weaker than the one in Theorem 6.1.

7. CONCLUSIONS

Starting with a discrete-time dynamical system, we associated two Frobenius-Perron operators with the deterministic system via ergodic theory. For a given partition of the phase space, cell-to-cell mappings, i.e. Markov chains, were constructed and the convergence of these approximations was outlined when refining the partitions.

The theory of Markov chains may now be used to numerically analyze the dynamical behavior of the system with the help of a computer, cf. Hsu [3] and Kreuzer [6]. However, from the authors point of view the transition probabilities used therein are motivated only intuitively. This paper bridges that gap to some extent.

It is important to realize that simple cell-to-cell mappings may be used to approximate continuous transformations S, whereas by Proposition 5.1, general cell-to-cell mappings are only applicable if in addition the transformation is nonsingular. As the discussion above shows, the transition probabilities of the general cell-to-cell mappings used by Hsu [3] are only applicable if the transformation S is invertible and measure-preserving with respect to the measure μ. These conditions are rather restrictive, since such an invariant measure is not known a priori and, in general, will be difficult to calculate.

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