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On the Quasinormability of $\mathscr{H}_b(U)$

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Dedicated to the memory of Leopoldo Nachbin, 1921 – 1993.

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INTRODUCTION

Let E be a complex locally convex space, U a balanced open subset of E and let us denote, as usual, by $\mathscr{H}_b(U)$ the space of all holomorphic functions on U which are bounded on the U–bounded subsets of U (a bounded subset B of U is U–bounded if there is an open neighborhood V of 0 in E such that $B+V\subset U$). We always consider the space $\mathscr{H}_b(U)$ endowed with its natural topology of uniform convergence on U–bounded sets.

The quasinormability of the space $\mathscr{H}_b(U)$, when U is a balanced open subset of a Banach space E, has been studied in [1] and [10]. In the recent papers [4], [7] [8] and [15], a good deal of information and new results about the quasinormability of $\mathscr{H}_b(U)$ in the general setting of complex locally convex spaces (including, Fréchet, (LF) and (LB) spaces) are given. Here we give a result which includes all the known results about the quasinormability of $\mathscr{H}_b(U)$ in the Fréchet spaces setting and which gives some new examples of Fréchet spaces for which $\mathscr{H}_b(U)$ is quasinormable.

THE RESULTS

We start by recalling that a locally convex space E is said to be *quasinormable* if for every open neighborhood U of 0 in E there is another open neighborhood V of 0 in E such that for every $\lambda>0$ there is a bounded subset B in E such that $V\subset B+\lambda U$. This definition was given in Grothendieck's thesis and it is known that every Banach space and every Fréchet–Schwartz space are

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quasinormable, but there are Fréchet spaces which are not quasinormable. In particular the Fréchet–Montel spaces which are not Schwartz spaces are not quasinormable [9].

For $n \in \mathbb{N}$, $\mathcal{P}(^nE)$ will denote the space of n–homogeneous continuous polynomials on E and τ_b (respectively τ_ω) the strong topology (respectively the Nachbin ported topology) on $\mathcal{P}(^nE)$. τ_b is defined by the family of seminorms

$$P \in \mathcal{P}(^nE) \longrightarrow |P|_B = \sup\{|P(z)| : z \in B\}$$

when B ranges over the family of all bounded subsets B in E. τ_ω is defined by

$$(\mathcal{P}(^nE), \tau_\omega) = \operatorname{ind} \mathcal{P}(^nE_\alpha)$$

where α ranges over the family of all continuous seminorms on E and, for a continuous seminorm α on E, $\mathcal{P}(^nE_\alpha)$ denotes the Banach space of all n–homogeneous continuous polynomials on the normed space E_α associated with α, endowed with the norm of the supreme on the unit ball of E_α.

PROPOSITION [8]. *For a Fréchet space E and a balanced open subset U of E, $\mathcal{H}_b(U)$ is quasinormable if and only if $(\mathcal{P}(^nE), \tau_b)$ is quasinormable for every $n \in \mathbb{N}$.*

All the results by Dineen in the above mentioned papers [7], [8], concerning the quasinormability of $\mathcal{H}_b(U)$ for Fréchet spaces, were obtained using the above proposition and showing that for all the spaces he considers the strong topology τ_b on $\mathcal{P}(^nE)$ agrees with the τ_ω topology. As $(\mathcal{P}(^nE), \tau_\omega)$ is an inductive limit of Banach spaces it is a quasinormable space ([9]). Hence $\mathcal{H}_b(U)$ is quasinormable for all those spaces.

Here we will use the above proposition to obtain a result which gives more examples of Fréchet spaces for which $\mathcal{H}_b(U)$ is quasinormable. With this aim we recall that a Fréchet space E has the $(\mathrm{BB})_{n,s}$–property ($n \in \mathbb{N}$, $n \geqslant 2$) if every bounded subset B of the completion of the n–fold symmetric projective tensor, $\hat{\otimes}_{s,n,\pi} E$, of E, there is a bounded subset C in E such that B is contained in the closed convex hull of the tensor product $\otimes_{s,n} C$. This property is a generalization given by Dineen of the well–known (BB)–property introduced by Taskinen [18] in connection with the "Problème des topologies" of Grothendieck. A Fréchet space has the $(\mathrm{BB})_\infty$–property if it has the $(\mathrm{BB})_{n,s}$–property for every $n \in \mathbb{N}$, $n \geqslant 2$ ([7]).

THEOREM. *For all Fréchet spaces E with the* $(BB)_\infty$*–property* $\mathcal{H}_b(U)$ *is quasinormable for every balanced open subset U of E.*

Proof. Let E be a Fréchet space with the $(BB)_\infty$–property. The natural algebraic isomorphism between $\mathcal{P}(^nE)$ and the topological dual of the n–fold symmetric projective tensor product $\hat{\otimes}_{s,n,\pi}E$ ([17]) is a topological isomorphism for the corresponding strong topologies if (and only if) E has the $(BB)_{n,s}$–property [7]. Hence for spaces with the $(BB)_{n,s}$–property $(\mathcal{P}(^nE),\tau_b)$ is topologically isomorphic to the strong dual of the Fréchet space $\hat{\otimes}_{s,n,\pi}E$ and hence a (DF)–space. As (DF)–spaces are quasinormable ([11]), we have that $(\mathcal{P}(^nE),\tau_b)$ is quasinormable for every $n \in \mathbb{N}$ and hence, by the above Proposition, $\mathcal{H}_b(U)$ is quasinormable for every balanced open subset U of E.

Remark. All known examples of Fréchet spaces for which $\mathcal{H}_b(U)$ is quasinormable are spaces with a T–Schauder decomposition and another condition. Every space with T–Schauder decomposition has the $(BB)_\infty$–property ([6]), hence our Theorem includes them.

EXAMPLES. Let us consider a Köthe sequence space $\lambda_p(A)$, $1 \leqslant p < \infty$, or $p = 0$. These spaces have the $(BB)_\infty$–property ([6]) and hence $\mathcal{H}_b(U)$ is quasinormable for every balanced open subset U of such spaces.

We remark that if we consider Köthe sequence spaces $E = \lambda_p(A)$, $1 \leqslant p < \infty$, which are not distinguished ([12]) then the strong dual $E_\beta' = (\mathcal{P}(^1E),\tau_b)$ of E is not barrelled and then τ_b is different from τ_ω on $\mathcal{P}(^1E)$ (τ_ω is the barrelled topology associated with τ_b on $\mathcal{P}(^nE)$ for E metrizable [5, p. 132]). Hence we have examples of Fréchet spaces E such that $\mathcal{H}_b(U)$ is quasinormable not included in the known examples.

Another examples of spaces E such that $\mathcal{H}_b(U)$ is quasinormable (U balanced open subset of E) are the Fréchet–Montel spaces. For them, the strong topology τ_b on $\mathcal{P}(^nE)$ $(n \in \mathbb{N})$ agrees with the compact open topology τ_o, and since $(\mathcal{P}(^nE),\tau_o)$ is a Schwartz space ([8], [13]), it is quasinormable, the result follows again from the above Proposition.

Note also that if we consider the Fréchet–Montel space that Taskinen considers in [18] to solve in the negative the "Problème des Topologies" of Grothendieck, it is a Fréchet–Montel space without the $(BB)_\infty$–property (it does not have the $(BB)_{2,s}$–property) and without the property $\tau_b = \tau_\omega$ on $\mathcal{P}(^nE)$, see [3].

Remark. In [14, Ex. 1.3.3(3)] examples of Fréchet spaces E such that $\mathcal{H}_b(U)$ is not quasinormable for every open subset U of E are given. Another example can be obtained from the strong dual of the DFS space Peris considers in [16].

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