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Algebras of Real Analytic Functions; Homomorphisms and Bounding Sets

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In this paper we are interested in subsets of a real Banach space on which different classes of functions are bounded. If $A(E)$ be an algebra on a Banach space E, a subset B of E is said to be A*–bounding* if $\sup_{x\in B}|f(x)|<\infty$ for all $f\in A(E)$. In the literature, bounding sets have been extensively studied with respect to the classes of continuous and holomorphic functions, see [9, 7]. In [1] it is shown that if a subset B of a real Banach space E satisfies that each C^{∞}–function on E is bounded on B, then B is relatively compact. We shall be especially interested here in algebras $A(E)$ between $\mathcal{P}(E)$, the continuous polynomials on E, and $\mathcal{A}(E)$, the real analytic functions on E. The close interplay between the homomorphisms on $A(E)$ and the A–bounding sets will play an important role in this article.

In what follows $A(E)$ will be an algebra of continuous real functions on a real Banach space E that contains the dual E'. Futhermore, we require the algebra $A(E)$ to have the additional property that given two Banach spaces E and F and a continuous linear map $T: E\longrightarrow F$, then $f\circ T\in A(E)$ whenever $f\in A(F)$.

Let $\operatorname{Hom} A(E)$ denote the set of all homomorphisms on $A(E)$. We say that $A(E)$ is *single–set evaluating* if for each $\phi\in\operatorname{Hom} A(E)$ and every $f\in A(E)$ we have $\phi(f)\in f(E)$. Obviously, if $A(E)$ is single–set evaluating, then for each $\phi\in\operatorname{Hom} A(E)$ and finite set $\{f_1,\dots,f_n\}$ in $A(E)$, there is a point $a\in E$ such that $\phi(f_i)=f_i(a)$ for all $i=1,\dots,n$. It is also clear that every inverse–closed algebra $A(E)$, i.e., $1/f\in A(E)$ whenever $f\in A(E)$ and $f(x)\neq 0$ for all $x\in E$, is single–set evaluating. An algebra $A(E)$ is *sequentially evaluating* if for each

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$\phi \in \operatorname{Hom} A(E)$ and each sequence (f_n) in $A(E)$ there is a point $a \in E$ with $\phi(f_n) = f_n(a)$ for all n.

By $\mathcal{P}_f(E)$ we mean the algebra of all continuous polynomials of finite type on E; that is, the algebra generated by E'. Let $\mathcal{AE}(E)$ denote the set of all functions $f: E \longrightarrow \mathbb{R}$ such that there exists a sequence $(p_n) \in \mathcal{P}(^nE)$ with $f(x) = \Sigma_{n \in \mathbb{N}} p_n(x)$ for all $x \in E$. Then $\mathcal{AE}(E)$ is an algebra with $\mathcal{P}(E) \subset \mathcal{AE}(E) \subset \mathcal{A}(E)$, where the last inclusion follows from [2].

By $\mathcal{R}A(E)$ we denote the smallest inverse–closed algebra which contains $A(E)$. Hence every element in $\mathcal{R}A(E)$ is of the form f/g, where $f, g \in A(E)$ and $g(x) \neq 0$ for all $x \in E$. For the case of $A(E) = \mathcal{P}(E)$ we shall denote $\mathcal{RP}(E) = \mathcal{R}(E)$, the algebra of rational functions on E. It is not difficult to check that for each Banach space E the algebra $\mathcal{RAE}(E)$ is a proper subalgebra of $\mathcal{A}(E)$.

1. Bounding Sets

Since we always require the algebra $A(E)$ to contain the dual E' and $A(E)$ is contained in $C(E)$, every relatively compact set is A–bounding and every A–bounding set is bounded. Let $E_{A(E)}$ be the set E endowed with the weakest topology making all $f \in A(E)$ continuous. One of the motivations for studying A–bounding sets relies on the fact that if the A–bounding sets in E are relatively compact, then E and $E_{A(E)}$ have the same convergent sequences; that is, $x_n \longrightarrow x$ in E if and only if $f(x_n) \longrightarrow f(x)$ for all $f \in A(E)$.

Using the concept of interchangeable double limit property, we obtain:

THEOREM 1. *Let E be a Banach space. Then every $\mathcal{R}$–bounding subset of E is relatively $\sigma(E, E')$–compact.*

COROLLARY 2. *Let E be a Banach space. If E has the Dunford–Pettis property, then the $\mathcal{R}$–bounding and the relatively $\sigma(E, E')$–compact subsets of E coincide.*

Examples of Banach spaces with the Dunford–Pettis property are $C(K)$ for any compact K and $L^1(\mu)$. This class is also closed under formation of preduals, if they exist, and hence the important Banach spaces $c_0(\Gamma)$, $\ell_1(\Gamma)$ and $\ell_\infty(\Gamma)$ all have this property.

Since the Schur property implies the Dunford–Pettis property, it follows from Corollary 2 that for a Banach space E with the Schur–property, every

$\mathcal{R}$–bounding set is relatively compact in E. In our next theorem we show that this is also true if E is a super–reflexive Banach space.

By [8], $E = \mathrm{Hom}\,\mathcal{R}(E)$ for E separable. Therefore every $\mathcal{R}$–bounding set in E is relatively compact in $E_{\mathcal{P}(E)}$. Since $E_{\mathcal{P}(E)}$ is angelic, each $\mathcal{R}$–bounding set in E is relatively sequentially compact in $E_{\mathcal{P}(E)}$. If we, in addition, assume that E is a Λ–space [4] (i.e., the weak–polynomial convergence for sequences implies the norm convergence, since a sequence (x_n) in E is weak–polynomial convergent to x if and only if $p(x_n - x) \longrightarrow 0$ for every $p \in \mathcal{P}(E)$), then we may state

PROPOSITION 3. *In separable Λ–spaces the $\mathcal{R}$–bounding sets are relatively compact.*

Recall from [4] that the separable space ℓ_p is a Λ–space for $1 \leqslant p < \infty$.

In [5] a Banach space E is defined to be in the class $\mathcal{W}_p$ $(1 < p < \infty)$ when for each bounded sequence (x_n) in E there exist a $x \in E$ and a subsequence (x_{nk}) such that $\Sigma_{k=1}^{\infty} |l(x_{nk} - x)|^p < \infty$ for all $l \in E'$. Using a convenient characterization of super–reflexivity, Castillo–Sánchez proved in [5] that every super–reflexive Banach space is in the class $\mathcal{W}_p$ for some p $(1 < p < \infty)$. Recall that a Banach space is super–reflexive if and only if its dual is super–reflexive. The spaces $L^p(\mu)$ are super–reflexive for $1 < p < \infty$ and any measure μ.

THEOREM 4. *Assume that E' is in the class $\mathcal{W}_p$ for some p $(1<p<\infty)$ (e.g. that E is super–reflexive). Then every $\mathcal{R}$–bounding set is relatively compact in E.*

A subset $B \subset E$ is said to be *limited*, if each $\sigma(E',E)$–null sequence (l_n) converges to zero uniformly on B. This concept translates to the language of bounding sets. Indeed, if $W^*(E) = \{\Sigma_{k=1}^{\infty}(l_n)^n : (l_n) \longrightarrow 0 \text{ in } \sigma(E',E)\}$, then obviously $B \subset E$ is limited if and only if B is W^*–bounding. Furthermore, since $W^*(E) \subset \mathcal{AE}(E) \subset \mathcal{A}(E)$ we have the relation

$$\mathcal{A}\text{–bounding} \implies \mathcal{AE}\text{–bounding} \implies \text{limited}.$$

The limited and the relatively compact sets in E coincide when E is isomorphic to a subspace of $C(K)$, where K is a compact, sequentially compact Hausdorff space [6]. All WCG spaces have this property as well as every weak Asplund space. Hence, for large classes of Banach spaces E every limited set is $\mathcal{R}$–bounding. Note also that Bourgain and Diestel [3] proved that in Banach spaces with no copy of ℓ_1 limited sets are relatively weakly compact.

In [12] Schlumprecht has constructed a complex Banach space E which contains a subset that is limited in E but not bounding with respect to holomorphic functions on E. Using this example we obtain a Banach space E for which

$$\text{limited} \not\Rightarrow \mathcal{AE}\text{-bounding}.$$

The original Tsirelson space T' and c_0 provide examples of $\mathcal{R}$–bounding subsets that are not limited. Hence in general

$$\mathcal{R}\text{-bounding} \not\Rightarrow \text{limited}.$$

On the other hand, by Phillips' lemma, $B_{c_0} \subset \ell_\infty$ is a limited set that is not $\mathcal{R}$–bounding. Thus also

$$\text{limited} \not\Rightarrow \mathcal{R}\text{-bounding}.$$

By means of the next theorem, when investigating the bounding sets in Banach spaces, one can restrict to one of the simplest sets in ℓ_∞.

THEOREM 5. *Let $A(E)$ be an algebra on a Banach space E that contains the algebra $\mathcal{R}(E)$. Then each A–bounding set is relatively compact in E if there exists some function in $A(\ell_\infty)$ that is unbounded on the set of unit vectors in ℓ_∞.*

Theorem 5 is directly applicable to the algebra $C^\infty(E)$ of all C^∞–functions on E in the usual Fréchet sense. Indeed, the function $f: \ell_\infty \longrightarrow \mathbb{R}$ which assigns an arbitrary point $x = (x_1, x_2, \dots) \in \ell_\infty$ the value

$$f(x) = \eta(x_1) + 2\mu(x_1)\eta(x_2) + 3\mu(x_1)\mu(x_2)\eta(x_3) + \dots$$
$$\dots + k\mu(x_1)\mu(x_2)\dots\mu(x_{k-1})\eta(x_k) + \dots,$$

where η and μ are non–negative C^∞–functions on $\mathbb{R}$ such that

$$\eta(t) = \begin{cases} 1, & \text{for } t = 1 \\ 0, & \text{for } t \leqslant 3/4 \end{cases} \quad \text{and} \quad \mu(t) = \begin{cases} 1, & \text{for } t = 0 \\ 0, & \text{for } t \geqslant 1/4, \end{cases}$$

is locally finite and thus an element in $C^\infty(\ell_\infty)$. By construction, $f(e_n) = n$ for all n. Therefore we have

COROLLARY 6. *In every Banach space the C^∞–bounding sets are relatively compact.*

With use of complexification and the deep results for holomorphically bounding sets in the complex Banach space ℓ_∞ due to Dineen and Josefson [7], we study the bounding sets for the algebras $\mathcal{AE}(\ell_\infty)$ and $\mathcal{RAE}(\ell_\infty)$.

PROPOSITION 7. *Every function* $f = \Sigma_{n=0}^{\infty} p_n \in \mathcal{AE}(\ell_\infty)$ *converges uniformly on each bounded subset of* c_0. *Especially, every bounded set in* c_0 *is an* $\mathcal{AE}$*–bounding subset of* ℓ_∞.

By means of the Phillips' lemma, there is, as stated before, a limited set in ℓ_∞ that is not $\mathcal{R}$–bounding. With Proposition 7 in hand, this result is sharpened to the subclass of $\mathcal{AE}$–bounding sets; that is,

$$\mathcal{AE}\text{–bounding} \not\Rightarrow \mathcal{R}\text{–bounding}.$$

On the othed hand we have

COROLLARY 8. *Every weakly compact set in* c_0 *is* $\mathcal{RAE}$*–bounding in* ℓ_∞, *in particular, the unit vectors* $\{e_n : n \in \mathbb{N}\}$ *form an* $\mathcal{RAE}$*–bounding set in* ℓ_∞.

Remark. The set B_{c_0} cannot be $\mathcal{RAE}$–bounding in ℓ_∞, since it isn't even $\mathcal{R}$–bounding.

2. EVALUATING PROPERTIES OF HOMOMORPHISMS

The principal interest in this section is to investigate the evaluating properties of the homomorphisms, such as the single–set and sequentially evaluating properties as well as their complete reduction to point evaluations, defined on the algebras $\mathcal{P}_f(E)$, $\mathcal{RP}_f(E)$, $\mathcal{P}(E)$, $\mathcal{R}(E)$, $\mathcal{AE}(E)$, $\mathcal{RAE}(E)$ and $\mathcal{A}(E)$ for various classes of Banach spaces E.

Although the algebra $\mathcal{AE}(\mathbb{R})$ is not inverse–closed, it is single–set evaluating. In fact, every homomorphism on $\mathcal{AE}(\mathbb{R})$ is even a point evaluation. Indeed, let $\phi \in \operatorname{Hom}\mathcal{AE}(\mathbb{R})$ and take $f \in \mathcal{AE}(\mathbb{R})$. For the function $p \in \mathcal{AE}(\mathbb{R})$, where $p(x) = x$ for all $x \mathbb{R}$, set $\alpha = \phi(p)$. Expand f in a Taylor series at α. Then $f(x) = f(\alpha) + (x - \alpha)g(x)$, where $g \in \mathcal{AE}(\mathbb{R})$. Hence $\phi(f) = f(\alpha) + 0 \cdot \phi(g) = f(\alpha)$.

In the next example we consider the algebra $\mathcal{AE}_b(E)$ of all functions $f = \Sigma_{n \in \mathbb{N}} p_n\ \mathcal{AE}(E)$ such that $\|p_n\|^{1/n} \longrightarrow 0$ as $n \longrightarrow \infty$. By taking into account the Josefson–Nissenzweig theorem [6], we observe that $\mathcal{AE}(E) = \mathcal{AE}_b(E)$ precisely when E is finite dimensional. The complex analog of $\mathcal{AE}_b(E)$ is the algebra of all holomorphic functions of bounded type.

EXAMPLE. Let E be a reflexive Banach space such that $\mathcal{P}_f(^nE)$ is dense in $\mathcal{P}(^nE)$ with respect to the norm topology for every n. Then every homomorphism on $\mathcal{AE}_b(E)$ is a point evaluation, and $\mathcal{AE}_b(E)$ endowed with the topology generated by the norms $\{\|\cdot\|_k : k \in \mathbb{N}\}$ where $\|f\|_k = \Sigma_{n=0}^{\infty} \|p_n\| k^n$, is a Fréchet algebra.

The assumptions on E in the example above are satisfied e.g. when E is the original Tsirelson space T' or a finite dimensional space.

In the next results some properties of sequential evaluation of homomorphisms are obtained.

PROPOSITION 9. *Let $\phi \in \operatorname{Hom} \mathcal{R}(E)$ and let (p_n) be a sequence of polynomials on a Banach space E such that their degree is uniformly bounded. Then there is a point $a \in E$ such that $\phi(p_n) = p_n(a)$ for all n.*

THEOREM 10. *Let $A(E)$ be a single-set evaluating algebra containing $\mathcal{AE}_b(E)$. Let $\phi \in \operatorname{Hom} A(E)$, let $f \in A(E)$ and let (f_n) be a sequence in $\mathcal{AE}(E)$. Then there is a point $a \in E$ such that $\phi(f) = f(a)$, $\phi(f_n) = f_n(a)$ for all n. In particular, the restriction $\phi_{|\mathcal{AE}(E)}$ is sequentially $\mathcal{AE}_b(E)$-continuous.*

COROLLARY 11. *For every Banach space E the inverse-closed algebra $\mathcal{RAE}(E)$ is sequentially evaluating.*

Next we give conditions under which each homomorphism on $\mathcal{RAE}(E)$ is a point evaluation.

PROPOSITION 12. *Let (E_n) be a sequence of Banach spaces such that each E_n admits a continuous linear injection into $\ell_q(\Gamma)$ for some $q < \infty$ and some set Γ of non-measurable cardinality. If E equals $(\oplus_n E_n)_{\ell_p}$ or $(\oplus_n E_n)_{c_0}$, then $E = \operatorname{Hom} \mathcal{RAE}(E)$.*

Examples of Banach spaces admitting a continuous linear injection into some $\ell_q(\Gamma)$ are all super-reflexive spaces [10] as well as those with weak*-separable duals.

THEOREM 13. *Let E be a weakly Lindelöf Banach space not containing a copy of ℓ_1 with the Dunford-Pettis property. Then*

$$E = \operatorname{Hom} \mathcal{R}(E) = \operatorname{Hom} \mathcal{RAE}(E) .$$

Examples of spaces satisfying the assumptions in Theorem 13 are $c_0(\Gamma)$ and $C(K)$ for a scattered Corson compact K [11].

In contrast, we now give some negative results.

PROPOSITION 14. *The algebra $\mathcal{AE}(\ell_\infty)$ is not single-set evaluating.*

If E is finite dimensional, then $\mathcal{P}_f(E) = \mathcal{P}(E)$ by which $\mathcal{P}(E)$ is sequentially evaluating. Since E' contains a point separating sequence, every homomorphism on $\mathcal{P}(E)$ is a point evaluation. If E is infinite dimensional the situation is entirely different.

PROPOSITION 15. *Let E be an infinite dimensional Banach space. Then $\mathcal{P}(E)$ is not a single–set evaluating algebra.*

PROPOSITION 16. *Let E be a non–reflexive Banach space. Then neither $\mathcal{P}_f(E)$ nor the inverse–closed algebra $\mathcal{RP}_f(E)$ are sequentially evaluating.*

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