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Compact Polynomials Between Banach Spaces

RAQUEL GONZALO [1] AND JESÚS ANGEL JARAMILLO [1]

Dpto. de Análisis Matemático, Univ. Complutense, 28040 *Madrid, Spain*

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The classical Pitt theorem [25] asserts that every bounded linear operator from ℓ_p into ℓ_q is compact whenever $q<p$. This result was extended by Pelczynski [23] who showed in particular that every N–homogeneous polynomial from ℓ_p into ℓ_q is compact if $Nq<p$. Our aim in this note is giving conditions on Banach spaces X and Y in order to obtain that every polynomial of a given degree N from X into Y is compact.

We recall that, if X and Y are (real or complex) Banach spaces, an N–homogeneous polynomial $P:X \longrightarrow Y$ is a map of the form $P(x)=T(x,\ldots,x)$, where $T:X\times\cdots\times X \longrightarrow Y$ is a continuous linear map. We shall say that P is compact if P maps the unit ball of X into a relatively compact set of Y, and P is weakly sequentially continuous if P maps weakly convergent sequences in X into norm convergent sequences in Y. These classes of polynomials have been extensively studied, both from the point of view of Banach space theory and also in infinite–dimensional holomorphy, especially in connection with compact holomorphic mappings (cf. [1,2,3,5,12,15,23,24,26] and references therein). It follows from [5] that if X does not contain a copy of ℓ_1 then every weakly sequentially continuous N–homogeneous polynomial $P:X \longrightarrow Y$ is compact, and also that if X contains a copy of ℓ_1 and Y is infinite dimensional then for each $N \geqslant 2$ there exists a non–compact N–homogeneous polynomial $P:X \longrightarrow Y$. Therefore we shall be mainly concerned with weak sequential continuity of polynomials.

We will show that polynomials preserve weak summability of sequences, and we shall deduce that every N–homogeneous polynomial $P:X \longrightarrow Y$ is weakly sequentially continuous if $N\cdot u(Y) < l(X)$, where $l(X)$ and $u(Y)$ are indexes defined in relation with certain properties of weak summability (the existence of

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upper and lower estimates) of sequences in X and Y.

1. Lower and Upper Estimates

Let X be a Banach space over $\mathbb{K}$ (where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) and let $1\leqslant p,q\leqslant\infty$. We shall say that a sequence $\{x_n\}$ in X has an upper p-estimate (respectively, a lower q-estimate) if there exists a constant $C>0$ such that for every $n\in\mathbb{N}$ and every $a_1,\dots,a_n\in\mathbb{K}$,

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant C\Big[\sum_{i=1}^{n}|a_i|^p\Big]^{1/p}$$

$$\Big[\text{respectively, } \Big\|\sum_{i=1}^{n} a_i x_i\Big\| \geqslant C\Big[\sum_{i=1}^{n}|a_i|^q\Big]^{1/q}\Big],$$

where the right-hand side means $\sup_i|a_i|$ if $p=\infty$ (or $q=\infty$). In particular, a normalized basic sequence $\{x_n\}$ has an upper ∞-estimate if, and only if, it is equivalent to the unit vector basis of c_0.

A sequence $\{x_n\}$ in X is said to be weakly p-summable if for every $x^*\in X^*$ we have that $\{x^*(x_n)\}\in\ell_p$ (or $\{x^*(x_n)\}\in c_0$ when $p=\infty$). It is worth noting that a sequence has an upper p-estimate if, and only if, it is weakly p^*-summable, where p^* is such that $\frac{1}{p}+\frac{1}{p^*}=1$ (see for instance [7]).

REMARK 1.1. Following Pelczynski [23], a sequence $\{x_n\}$ in X is said to be $\tau_{1/p}$-null if there exists a constant $C>0$ such that for every $n\in\mathbb{N}$, every $\zeta_1,\dots,\zeta_n\in\mathbb{K}$ with $|\zeta_j|=1$ and every $k_1<\dots<k_n$, we have

$$\Big\|\sum_{i=1}^{n}\zeta_i x_{k_i}\Big\|\leqslant C\,n^{1/p}.$$

It is easy to see that the $\tau_{1/p}$-null sequences coincide with the hereditarily p-Banach-Saks sequences in the sense of [8]; that is, sequences $\{x_n\}$ for which there exists a constant $C>0$ such that for every $n\in\mathbb{N}$ and every $k_1<\dots<k_n$, we have

$$\Big\|\sum_{i=1}^{n} x_{k_i}\Big\|\leqslant C\,n^{1/p}.$$

It is shown in [8] that if a sequence is hereditarily p-Banach-Saks then it has an upper r-estimate for every $r<p$.

Recall that a sequence $\{x_n\}$ in a Banach space is called seminormalized if there exist constants $k,K>0$ so that $k\leqslant\|x_n\|\leqslant K$ for each n. According to [21], a Banach space X is said to have property S_p (for some $1\leqslant p\leqslant\infty$) if every weakly null seminormalized basic sequence in X has a subsequence with an upper p–estimate. In a similar way, we shall say that X has property T_q (for some $1<q\leqslant\infty$) if every weakly null seminormalized basic sequence in X has a subsequence with a lower q–estimate. It is plain that a Schur space has properties S_p and T_q for every $1\leqslant p\leqslant\infty$ and every $1<q\leqslant\infty$. So we shall say that a Banach space has property T_1 if, and only if, it is a Schur space. On the other hand, every Banach space has properties S_1 and T_∞. We also note that properties S_p and T_q are inherited by closed subspaces.

The lower and upper indexes of a Banach space X are now defined as:

$$l(X)=\sup\{p\geqslant 1 : X \text{ has property } S_p\}\in[1,\infty]$$

$$u(X)=\inf\{q\geqslant 1 : X \text{ has property } T_q\}\in[1,\infty].$$

Our next result will show some duality between properties S_p and T_q. In [20] the result is obtained for $p=\infty$.

PROPOSITION 1.2. *Let X be a Banach space without any copy of ℓ_1 and let $1\leqslant p\leqslant\infty$. If X has property S_p then X^* has property T_p^*, where $\frac{1}{p}+\frac{1}{p^*}=1$.*

The converse of Proposition 1.2 is not true. See [20] for the case $p=\infty$. In the case that $1<p<2$, a subspace X of $L_p[0,1]$ is constructed in [18] such that every weakly–null seminormalized basic sequence in X has a subsequence that is equivalent to the unit vector basis of ℓ_p, but the equivalence constant cannot be chosen uniformly for all sequences in question. Therefore X has properties S_p and T_p but it follows from [21] that X^* has not property S_{p^*}.

EXAMPLES AND REMARKS 1.3. 1) If X is not a Schur space, then $l(X)\leqslant u(X)$.

2) For $1<p<\infty$, ℓ_p has properties S_p and T_q, and $l(\ell_p)=u(\ell_p)=p$. More generally, if M is an Orlicz function satisfying the Δ_2–condition at 0 [22], the Orlicz sequence space ℓ_M satisfies that $l(\ell_M)=\alpha_M$, $u(\ell_M)=\beta_M$ (see [19]), where α_M and β_M are the Boyd indexes of M.

3) For $1<p<\infty$, $L_p[0,1]$ has properties $S_{\min(2,p)}$ and $T_{\max(2,p)}$, and $l(L_p[0,1])=\min(2,p)$, $u(L_p[0,1])=\max(2,p)$. On the other hand, $l(L_1[0,1])=1$ and $u(L_1[0,1])=2$.

4) The James space J and the dual J^* have properties S_2 and T_2.

5) It follows from the results of [17] and [8] that if X is superreflexive then $1<l(X)\leqslant u(X)<\infty$. On the other hand, if $1<p<\infty$, the space $X=(\oplus_k \ell_1^k)_{\ell_p}$ satisfies $l(X)=u(X)=p$, although X is not superreflexive.

6) Property S_∞ is equivalent to the hereditary Dunford Pettis property [7]. For the original Tsirelson space T^* we have $l(T^*)=u(T^*)=\infty$ [10], although T^* has not property S_∞. For the dual space T, $l(T)=u(T)=1$.

7) If $1\leqslant p<\infty$, and X has property S_∞ then so does $\ell_p(X)$ [9].

8) If X has property S_∞ then so does $c_0(X)$ [7,9].

Next we will relate properties S_p and T_q with type and cotype. First note that, in the case of a Banach space with unconditional basis, type p implies property S_p ([11],[21]) and cotype q implies property T_q. Using the theory of spreading models and some ideas along the lines of the results of Farmer and Johnson in [13], we obtain the following:

THEOREM 1.4. *Let $\{x_n\}$ be a weakly null seminormalized basic sequence in a Banach space X and let $1<p<\infty$.*

1) *If $\{x_n\}$ admits a spreading model whose fundamental sequence has an upper $p-$estimate, then there exists a subsequence of $\{x_n\}$ with an upper $r-$estimate for every $r<p$.*

2) *If $\{x_n\}$ admits a spreading model whose fundamental sequence has a lower $p-$estimate, then there exists a subsequence of $\{x_n\}$ with a lower $r-$estimate for every $r>p$.*

And, as a consequence, we also have:

COROLLARY 1.5. *Let X be a Banach space.*

1) *If X has type $p\in(1,2]$ then X has property S_r for every $r<p$.*

2) *If X has cotype $q\in[2,\infty)$ then X has property T_r for every $r>q$.*

REMARK 1.6. Let $p(X)=\sup\{p:X \text{ has type } p\}$ and $q(X)=\inf\{q:X \text{ has cotype } q\}$. From Corollary 1.5 we have that $p(X)\leqslant\min\{2,l(X)\}$ and $q(X)\geqslant \max\{2,u(X)\}$. Now consider $X=(\oplus_k \ell_4^k)_{\ell_2}$ and $X^*=(\oplus_k \ell_{4/3}^k)_{\ell_2}$. Then $q(X)=4$, $p(X^*)=4/3$, $l(X)=l(X^*)=2$. Therefore in general we have that $\min\{2,l(X)\}$ is different from $p(X)$ and $\max\{2,u(X)\}$ is different from $q(X)$.

2. Polynomials Against Sequences with Upper Estimates

It is shown in [6] (see also [23]) that every N–homogeneous polynomial takes $\tau_{1/p}$–null sequences into $\tau_{N/p}$–null sequences if $N<p$. We will see that an analogous result holds for sequences with an upper p–estimate. The case $p=\infty$ is considered in [14].

THEOREM 2.1. *Let $P:X\longrightarrow Y$ be an N–homogeneous polynomial. Then*

1) *If $N<p<\infty$, P takes sequences with an upper p–estimate in X into sequences with an upper (p/N)–estimate in Y.*

2) *P takes sequences with an upper ∞–estimate in X into sequences with an upper ∞–estimate in Y.*

Theorem 2.1 extends the following result of Aron, Globevnik and Zalduendo:

THEOREM 2.2. ([4],[27]) *Let $X=\ell_p$ $(1<p<\infty)$ or c_0, let $\{e_n\}$ be the usual basis of X, and let P be a scalar valued N–homogeneous polynomial on X.*

1) *If $X=\ell_p$ and $N<p$, then $\{P(e_n)\}\in\ell_{(p/N)^*}$.*

2) *If $X=c_0$, then $\{P(e_n)\}\in\ell_1$.*

REMARKS 2.3. We note that the special case of c_0 and ℓ_p are determining, since in fact Theorem 2.1 can also be derived in a simple way from Theorem 2.2 using the following Lemma.

LEMMA 2.4. *Let $\{y_n\}$ be a sequence in a Banach space Y. Let $N\in\mathbb{N}$ and $N\leqslant p<\infty$. Then the following are equivalent:*

a) *$\{y_n\}$ is weakly $(p/N)^*$–summable.*

b) *There exists a bounded linear operator $T:\ell_{p/N}\longrightarrow Y$ such that $T(e_n)=y_n$, where $\{e_n\}$ is the unit vector basis of $\ell_{p/N}$.*

c) *There exists an N–homogeneous polynomial $P:\ell_p\longrightarrow Y$ such that $P(e_n)=y_n$, where $\{e_n\}$ is the unit vector basis of ℓ_p.*

Now we can easily deduce the following Theorem, which is essentially a reformulation of [23].

THEOREM 2.5. *Let X and Y be Banach spaces.*

1) *If $N\cdot u(Y)<\ell(X)$, then every N–homogeneous polynomial from any subspace of X into Y is weakly sequentially continuous.*

2) *If X has property S_∞ and Y does not contain copy of c_0, then every homogeneous polynomial from any subspace of X into Y is weakly sequentially*

continuous.

Next Corollary deals with reflexivity of the space $\mathcal{P}(^N X, Y)$ of all N−homogeneous polynomials from X into Y. For the original Tsirelson space T^* it was proved in [2] that $\mathcal{P}(^N T^*, \ell_p)$ is reflexive for all N. From Corollary 2.6 we obtain that $\mathcal{P}(^N X, Y)$ is reflexive for all N in the case that X is, for example, a quotient of T^* and Y is superreflexive.

COROLLARY 2.6. *Let X and Y be reflexive spaces, and suppose that every N−homogeneous polynomial from X into Y is weakly sequentially continuous (e.g., if $N \cdot u(Y) < \ell(X)$). Then $\mathcal{P}(^N X, Y)$ is reflexive.*

We do not know wether the condition in Theorem 2.5 (1) is sharp. We have nevertheless a partial answer.

PROPOSITION 2.7. *Let X, Y be Banach spaces, and suppose that X has a weakly null, normalized unconditional basis.*

1) *If $N > u(X)$, then there exists an N−homogeneous polynomial $P: X \longrightarrow Y$ that is not weakly sequentially continuous.*

2) *If Y is not Schur and $N \cdot \ell(Y) > u(X)$, then there exists an N−homogeneous polynomial $P: X \longrightarrow Y$ that is not weakly sequentially continuous.*

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