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NOTE ON NONTRIVIAL SOLUTIONS TO NONLINEAR VOLTERRA EQUATIONS

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We consider the equation (see |3|):

(1) $$u(x)=\int_0^x k(x-s)g(u(s))ds \qquad (x\geqslant 0)$$

where

(k) $k:[0,+\infty)\longrightarrow[0,+\infty)$ is such that $\int_0^x k(s)ds>0$ for $x>0$

and

(g) g is a continuous nondecreasing function such that $g(0)=0$, $g(u)>0$ for $u>0$ and $g^{(u)}/u\longrightarrow+\infty$ as $n\longrightarrow 0+$.

Let us note, that $u=0$ is the trivial solution to (1). But we consider continuous solutions u such that $u(x)>0$ for $x>0$. These are so-called nontrivial solutions. We ask about the uniqueness and the possibility of the extension to the maximal interval of nontrivial solutions.

If we assume additionaly

(g_1) $g(u)/u$ is strictly decreasing and $g(u)/u\longrightarrow 0$ as $n\rightarrow+\infty$

Then we can show two following theorems (|1|, |2|).

<u>Theorem 1</u>. Assume (k), (g) and (g_1) are statified. If equation (1) has a nontrivial solution on $[0,x_0]$, $(x_0>0)$, then it is unique and nondecreasing on $[0,x_0]$.

<u>Theorem 2</u>. Assume (k), (g) and (g_1) are satisfied. If equation (1) has a nontrivial solution on $[0,x_0]$, $(x_0\ 0)$, then it can be extended to the unique nontrivial solution on $[0,+\infty)$.

But we can prove Theorem 1 under weaker assumptions:

<u>Theorem 1'</u>. Assume (k) and (g) are satisfied. If equation (1) has a nontrivial solution on $[0,x_0]$, then it is unique on $[0,x_0]$. Moreover it is nondecreasing.

We can show this theorem in two steps:

Lemma 1. If equation (1) has a nontrivial solution u on $[0,x_0]$, $(x_0>0)$, then there exists $\delta_0>0$ ($\delta_0\leq x_0$) such that u is unique on $[0,\delta_0]$. Moreover u is nondecreasing on $[0,\delta_0]$.

The proof of this lemma can be found in |2|.

Lemma 2. Let u_i (i=1,2) be two nontrivial solutions to (1) on $[0,x_0]$, $(x_0,0)$ such that $u_1(x)=u_2(x)$ on $[0,\delta_0]$, ($\delta_0\in(0,x_0)$). Then $u_1=u_2$ on $[0,x_0]$.

Proof of Lemma 2 (for the comparison see |1|).

Let

$$M=\max\left\{\max_{s\in[\delta,x_0]}(g(u_i(s))/u_i(s)):i=1,2\right\}$$

and

$$m=\min\left\{\min_{s\in[\delta,x_0]}(g(u_i(s))/u_i(s)):i=1,2\right\}$$

We get

$$m\,u_i(x)\leq g(u_i(x))\leq M\,u_i(x)$$

for $x\in[\delta,x_0]$ and i=1,2. Hence

$$|g(u_1(x))-g(u_2(x))|\leq(M-m)|u_1(x)-u_2(x)|$$

for $x\in[\delta,x_0]$. The result follows by standard methods.

Corollary 1. The nontrivial solution u is nondecreasing on $[0,x_0]$.

Let $[0,\alpha)$, ($\alpha>0$) or $\alpha=+\infty$) be the maximal interval of existence of the nontrivial solution u. By Theorem 1' it is easy to see that u is the unique nontrivial solution to (1) on $[0,\alpha)$. Moreover it is nondecreasing.

Corollary 2. If $[0,\alpha)$ is the maximal interval of the existence of the nontrivial solution u then either $\alpha<\infty$ and $\lim_{x\to\alpha-}u(x)=+\infty$ or $\alpha=+\infty$.

Now we present an example. We consider equation (1) with k(x)=2 and

$$g(\cdot)=\begin{cases}u^{1/2} & \text{for } u\in[0,1]\\ u^2 & \text{for } u>1\end{cases}$$

We can easy compute that

$$u(x)=\begin{cases} x^2 & \text{for } x\in[0,1] \\ 1/(3-2x) & \text{for } x>1 \end{cases}$$

is the unique nontrivial solution to (1) with the maximal interval equal to $[0,3/2)$.

We can formulate:

Theorem 2'. Assume (k) and (g) are satisfied. If equation (1) has a nontrivial solution on $[0,x_0]$ and for $x_1>x_0$ is

$$K(x_1)\leq \sup_{u\in[0,\infty)} \{(u/g(u))\},\ (K(x)=\int_0^x k(s)ds)$$

then the solution u can be extended to $[0,x_1]$.

Corollary 2. If $\lim_{x\to+\infty} K(x)\leq \sup_{u\in[0,\infty)} (u/g(u))$ then every nontrivial solution to (1) (if there exists) can be extended to $[0,+\infty)$.

For similar proofs look |2|.

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