NONLINEAR VOLTERRA EQUATIONS
AND PHYSICAL APPLICATIONS

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1. INTRODUCTION.

Let us consider two examples leading to nonlinear Volterra integral equations of the same form:

Example 1.1. **Subsolutions of a nonlinear diffusion problem**

Let the plane xOy be the impermeable base. The bottom of a cylindrical reservoir is on this plane. We assume that the z-axis is the axis of the reservoir. The radius of the reservoir is equal to one. Let h(r,t) denote the height of the saturated region at distance r from the axis of the reservoir and at the time t>0. The process of the infiltration from the reservoir into the surrounding soil is described by the radial Boussinesq equation (see [8],[15])

$$(1.1) \qquad D_t h = r^{-1} D_r(r D_r(h^2)) \quad \text{for} \quad r>1 \text{ and } t>0$$

with conditions

$$(1.2) \qquad h(r,0) = 0 \quad \text{for} \quad r>1$$

$$(1.3) \qquad h(1,t) = 1 \quad \text{for} \quad t>0.$$

We can consider a more general equation than (1.1), namely

$$(1.4) \qquad D_t h = Lh$$

where

$$(1.5) \qquad Lh = r^{-1} D_r(r D_r(h^\alpha)) \quad (\alpha>1)$$

It may be shown that the equation (1.5) with conditions (1.2) and (1.3) has a unique so-called weak solution (for details see [16],[17] and [18]).

It is important for applications that the weak solution is classical at these points (r,t) for which $h(r,t)>0$. Moreover it is shown that supp $h(.,t)=\langle 1,r_0(t)\rangle$ for $t>0$, where $r_0(t)$ is a continuous nondecreasing function. With regards to applications it is interesting to give some information, even approximate, about the shape and range of the support of the function h. To get this information we can construct an auxiliary function approximating the exact solution h from below. This is a so-called subsolution (see [16],[17]). We can construct a subsolution $\underline{h}$ having the form (see [41]):

(1.6)
$$\underline{h}(r,t) = \begin{cases} F(t)f(r[A(t)]^{-1/2}) , & r\leq[A(t)]^{1/2} \\ 0 , & r>[A(t)]^{1/2} \end{cases}$$

The above function will be a subsolution if f is a sufficiently smooth function satisfying the problem:

(1.7)
$$s^{-1}(s(f^\alpha)')' = -\frac{1}{2}sf' \quad \text{for} \quad s\in(0,1]$$

with

(1.8)
$$f(1)=0, \quad \lim_{s\to1^-}[f^\alpha(s)]'=0$$

and the function A satisfies the differential equation

(1.9)
$$A(t)f\left(\frac{1}{[A(t)]^{1/2}}\right)=1 , \quad A(0)=1.$$

To solve (1.7)-(1.8) we can use the substitution

(1.10)
$$f(s) = v(-\log s)$$

Using (1.10) we reduce (1.7) and (1.8) to

(1.11)
$$(v^\alpha)'' = \frac{1}{2}e^{-2x}v' \quad \text{for} \quad x\geq0$$

with conditions

(1.12)
$$v(0)=0 \quad \text{and} \quad \lim_{x\to0^+}[v(x^\alpha)]'=0$$

Integrating twice (1.11) and using (1.12) we get

(1.13)
$$[v(x)]^\alpha = \int_0^x e^{-2s}[\frac{1}{2}+(x-s)]v(s)ds \quad (x\in[0,\infty]).$$

If we substitute

(1.14)
$$v(x) \longrightarrow e^{-2x/(\alpha-1)} v(x)$$

into (1.13), we get

(1.15)
$$[v(x)]^{\alpha} = \int_0^x k(x-s)v(s)ds$$

where

(1.16)
$$k(x) \doteq [\tfrac{1}{2}+x]e^{2\alpha x/(\alpha-1)}.$$

Let us note that the trivial solution u=0 satisfies equation (1.15). But, with respect to a physical meaning of the problem, we are looking for continuous solutions v of (1.15) such that v(x)>0 for x>0. We can substitute

(1.17)
$$v = u^{1/\alpha}$$

into (1.15) to obtain

(1.18)
$$u(x) = \int_0^x k(x-s)[u(s)]^{1/\alpha} ds .$$

This is the nonlinear Volterra equation of convolution type. Sometimes, for mathematical considerations, the form (1.15) is more convenient. An equation similar to (1.15) was thoroughly studied in [15] and [41]. Having some information about g we can give estimates of A. Subsolutions having the form (1.16) are interesting with respect to numerical aspects.

Example 1.2. Propagation of shock-waves in gas filled tubes

We consider as in [23] shock waves in gas filled tubes. We are looking for the axial components of the particle velocity behind the shock wave. We choose the coordinate system such that the x-axis is directed along the axis of the tube. Moreover the shock-wave front passes through the origin of the x-axis at the time t=0. Let $c_s \geq c_0$ (c_0 = sound speed) denote the speed of the shock-wave front. Let $t_s(x)$ denote the time at which the shock-wave front passed the tube cross-section referred to by x. We denote by v(x,t) the axial component of the particle velocity behind the wave front at the point x and the time t. This function must satisfy the following equation (see [23]):

(1.19)
$$D_t v + c_s D_x v = -(B_1 v + (c_0-c_s))D_x v + \tfrac{1}{2}B_2 \int_0^{t-t_s(x)} D_t v(x,t-s)k(s)ds$$

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where B_1, $B_2 > 0$ are physical parameters and the kernel k describes a dependence between axial and radial components of the particle velocity. At applications concerning shock-waves the kernel has usually the form $k(x) = x^{\gamma-1}$, ($\gamma > 0$). Moreover the function v must satisfy the condition

(1.20)
$$v(x, t_s(x)) = (c_s - c_0)/B_1 .$$

It describes the discontinuity of the axial velocity of the wavefront.

We are looking for so-called asymptotic solutions of the problem (1.19)-(1.20) having the form:

(1.21)
$$v(x,t) = v(t - t_s(x)) .$$

Moreover we suppose that c_s is constant. In this case we get:

(1.22)
$$t_s(x) = x/c_s .$$

Hence our asymptotic solutions will have the form:

(1.23)
$$v(x,t) = v(t - x/c_s) .$$

In this case the problem (1.19)-(1.20) will reduce to the following one:

(1.24)
$$\frac{B_1}{c_s}\left([v(x) - (c_s - c_0)/B_1]^2\right)' = B_2 \int_0^x v'(x-s)k(s)ds$$

where $v = v(x)$ is the unknown function such that:

(1.25)
$$v(0) = (c_s - c_0)/B_1 .$$

Substituting

(1.26)
$$v \longrightarrow \frac{B_2 c_s}{B_1}[v - (c_s - c_0)/B_1]$$

into (1.24) and (1.25) we get

(1.27)
$$\left([v(x)]^2\right)' = \int_0^x v'(x-s)k(s)ds$$

and

(1.28)
$$v(0) = 0 .$$

Integrating (1.27) and using (1.28) we get

(1.29)
$$[v(x)]^2 = \int_0^x k(x-s)v(s)ds$$

or, after the substitution

(1.30)
$$v = u^{1/2} ,$$

we obtain

(1.31) $$u(x) = \int_0^x k(x-s)[u(s)]^{1/2} ds \quad x \in [0, \infty] \ .$$

Nonnegative nontrivial solutions of (1.29) were considered in [23] and [45].

In both examples we have reduced the considered problems to nonlinear Volterra equations having the form (1.18). Let us note that the trivial solution satisfies this equation. But from a physical point of view only continuous solutions u of (1.18) or (1.31) such that $u(x) > 0$ for $x > 0$ are interesting. We shall denote the class of functions satisfying the above conditions by M. The aim of this work is to give a survey of the methods and results concerning the solvability of the equation (1.18) in the class M.

Moreover we shall study more general equations that (1.18), namely

(1.32) $$u(x) = \int_0^x k(x-s)g(u(s))ds$$

where $g:\overline{R}_+ \longrightarrow \overline{R}_+$, $(\overline{R}_+ = [0,\infty))$, is a nondecreasing concave function such that $g(0)=0$. We shall assume without loss of generality that the equation (1.18) or (1.32) is considered on [0,1]. Moreover we assume:

(k)   k is a nonnegative measurable function on [0,1] such that

$$\int_0^x k(s)ds > 0 \quad \text{for} \quad x \in (0,1] \ .$$

If it is necessary we shall make additional assumptions concerning k.

2. WEIGHTED METRICS METHOD.

In this part we shall study methods which are very useful with respect to the problems of Examples 1.1 and 1.2. Moreover these methods may be applied for finding approximate solutions to the considered nonlinear Volterra equations. At first, we shall study the equation (1.18), but is is convenient to consider it under the form (1.15).

Now we present the following theorem:

Theorem 2.1. Let **k** satisfy assumptions (k) and

(2.1) $$k(x) \geq C\, x^{\gamma-1} \quad \text{for} \quad x \in [0,1],$$

C >0 a constant. If v∈M is a solution of (1.15) then

(2.2) $$\left[cB\left(\gamma,\frac{\gamma-1+\alpha}{\alpha-1}\right)\right]^{\frac{1}{\alpha-1}} x^{\gamma/(\alpha-1)} \leq v(x) \leq \left[\int_0^x k(s)ds\right]^{\frac{1}{\alpha-1}}$$

for x∈[0,1], where B denotes the beta function.

To obtain the right-hand side of (2.2) we can use methods from [35] and [36]. For special cases the left-hand side of (2.2) was proved in [3], [33], [34] and [35]. But to get the above formula we can apply methods from [43]. Let 0<a<b. We denote

$$m_0 = \min_{x \in [a,b]} v(x) \quad \text{and} \quad \beta_0 = 0 .$$

We can show

$$v(x) \geq m_n (x-a)^{\beta_n} \quad \text{for } x \in [a,b] \quad (n=0,1,2,\dots)$$

where

$$\beta_{n+1} = \alpha^{-1}[\beta_n + \gamma]$$

$$m_{n+1} = [m_n B(\gamma,\beta_n+1)]^{1/\alpha} .$$

It may be shown that

$$\lim_{n\to\infty} \beta_n = \gamma/(\alpha-1)$$

and

$$\lim_{n\to\infty} m_n = \left[B\left(\gamma,\frac{\gamma-1+\alpha}{\alpha-1}\right)\right]^{1/(\alpha-1)}$$

We get

(2.3) $$v(x) \geq \left[B\left(\gamma,\frac{\gamma-1+\alpha}{\alpha-1}\right)\right]^{1/(\alpha-1)} (x-\alpha)^{1/(\alpha-1)} \quad \text{for } x \in [a,b].$$

Since (2.3) is true for any a and b we infer that the left-hand side of (2.2) is true.

Now we shall consider three cases in which weighted metrics method can be applied.

§ 2.1. Now we assume

$$k \in C^n[0,1], \quad k(0) = \ldots = k^{n-1}(0) = 0$$

(k_1) and

$$k^n(x) \geq k^n(0) \text{ on } [0,1].$$

In this case the following inequality is satisfied:

(2.4) $$k^n(0)x^n \leq k(x) \leq k_0 x^n$$

where

$$k_0 = \max_{x \in [0,1]} k^n(x)$$

On the base of (2.2) we get

(2.5) $$\left[k^n(0) B\left(n+1, \frac{n+\alpha}{\alpha-1}\right) \right]^{1/(\alpha-1)} x^{(n+1)/(\alpha-1)} \leq v(x) \leq k_0 x^{(n+1)/(\alpha-1)}$$

if $v \in M$ is a solution of (J 18). We denote by Ω the set of functions from M satisfying the inequality (2.5). Let S denote the following operator

(2.6) $$S(f)(x) = \left[\int_0^x k(x-s)f(s)ds \right]^{1/\alpha}$$

for $f \in M$. We can show as in [3], [34] and [35] that S transforms Ω into Ω. We can introduce in Ω the following distance (see [3],[33],[34] and [35]):

(2.7) $$\rho(v_1, v_2) = \sup_{0 < x \leq 1} \left(|v_1(x) - v_2(x)| \, x^{-(n+1)/\alpha-1} e^{-\beta x} \right)$$

where

(2.8) $$\beta = \frac{1}{k^n(0)} \sup_{0 \leq x \leq 1} \frac{\bar{k}^n(x) - k^n(0)}{x} .$$

In formula (2.8) we put $\bar{k}^{(n)}(x) = \sup_{s \in [0,x]} k^{(n)}(s)$ and $x_0 \in [0,1]$ such that

(2.9) $$\bar{k}^{(n)}(x_0) < \bar{k}^{(n)}(0)\alpha.$$

Let us note that the metrics introduced by Bielecki (see [9]) are similar in some sense to (2.7).

The set Ω with the metric ρ is a complete metric space. Having introduced β by the formula (2.8) we can show the following lemma:

Lemma 2.1. Let k satisfy assumptions (k_1) and β be given by (2.8). Then for $x \in [0,1]$

(2.10)
$$k(x)e^{-\beta x} \le \overline{k}^{(n)}(x_1)x^n.$$

The proof of this lemma is based on the inequality $k(x) \le \overline{k}^{(n)}(x_0)x^n$.

Next we apply a method similar to that one from [33] and obtain (2.10) (see also [3]).

After some not so difficult computations, using (2.5) and Lemma 2.1. we get (see [3]):

Lemma 2.2. For v_1, $v_2 \in \Omega$

(2.11)
$$\rho(S(v_1),S(v_2)) \le \frac{\overline{k}^{(n)}(x_0)}{k^{(n)}(0)\alpha} \rho(v_1,v_2).$$

Since $\dfrac{\overline{k}^{(n)}(x_0)}{k^{(n)}(0)\alpha} < 1$, by the Banach fixed point theorem, we obtain (see [3], [33], [34] and [35]):

Theorem 2.2. Let k satisfy assumptions (k_1). Then equation (1.18) has a unique solution v belonging to M. This solution may be found using the method of the successive approximations in Ω with respect to ρ.

§ 2.2. Secondly we consider (see [3])

(k_2)
$$k(x) = cx^{\gamma-1} + l(x)$$

where $c>0$, $\gamma>0$, and $l(x)$ is a continuous nonnegative function such that $x^{1-\gamma}l(x)$ goes to 0 as x goes to 0^+.

Using (2.2), in this case we get the following a priori estimate for a solution v of (1.18) belonging to M:

(2.12)
$$\left[cB(\gamma,\frac{\gamma-1+\alpha}{\alpha-1})\right]^{1/(\alpha-1)} x^{\gamma/(\alpha-1)} \le v(x) \le$$

$$\le \left[(c+c_0)B(\gamma,\frac{\gamma-1+\alpha}{\alpha-1})\right]^{1/(\alpha-1)} x^{\gamma/(\alpha-1)}$$

where $c_0 = \max\limits_{x \in [0,1]} l(x)$.

Let Ω_1 denote the set of functions of M satisfying the above inequality. The operator S defined by (2.6) transforms Ω_1 into Ω_1. Now we formulate the following analytical lemma ([3]):

Lemma 2.3. Let k satisfy assumptions (k_1). For any $\varepsilon > 0$ a number $x_1 \in (0,1]$ exists such that

(1.13)
$$k(x)e^{-\beta_1 x} \leq (c+\varepsilon)x^{\gamma-1} \quad (x \in [0,1]),$$

where

(2.14)
$$\beta_1 = \frac{1}{x_1}\log\left[\frac{2}{\varepsilon}x_1^{-\gamma+1}\max_{x \in [x_1,1]}l(x)\right]$$

We define the following metric

$$\rho_1(v_1,v_2) = \sup_{0 \leq x \leq 1}|v_1(x)-v_2(x)|x^{-\gamma/(\alpha-1)}e^{-\beta_1 x}$$

for $v_1, v_2 \in \Omega_1$. The space Ω_1 with the metric ρ_1 is a complete metric space.

Applying Lemma 2.3. we can show ([3])

Lemma 2.4. For $v_1, v_2 \in \Omega_1$, $\rho_1(s(v_1),s(v_2)) \leq \frac{c+1}{\alpha c}\rho_1(v_1,v_2)$.

We can put $\varepsilon < c(\alpha-1)$. Using the Banach fixed point theorem we get ([3]):

Theorem 2.3. Let the kernel k satisfy assumptions (k_2). Then equation (1.18) has a unique solution v belonging to M. This solution may be found as the limit of successive approximations in Ω_1 with respect to ρ_1.

For $\gamma \in (0,1)$ we can find similar results in [22]. A survey concerning equation (1.18) will be given in [2]. Some results about the smoothness of solutions of (1.18) are presented in [21].

§ 2.3. One can ask if the above methods may be used for more general nonlinear Volterra equations. The example below will show that it is possible for some cases. We consider the equation

(2.17)
$$w(v(x)) = \int_0^x k(x-s)v(s)ds \ , \quad x \in [0,1]$$

where

$w: \overline{R}_+ \longrightarrow \overline{R}_+$ is a strictly increasing convex function,

(W) $w \in C^1(\overline{R}_+) \cap C^2(\overline{R}_+)$, $w(0)=w'(0)=0$

and $\int_0^1 w'(s)\overline{s}^1 \, ds < \infty$.

(In the case of equation (1.32) this last integral condition means
$\int_0^1 \dfrac{du}{g(u)} < \infty$.)

There exist constants c_1, c_2, $0 < c_1 \le c_2 < c_1+1$ such that

(W_+) $\quad \dfrac{w_1(x)}{w'(x)} \le c_2$ and $\left[\dfrac{w_1(x)}{w'(x)} x\right]' \ge c_1$ for $x>0$,

where $w_1(x) = \int_0^x w'(s)s^{-1}ds < \infty$.

For $w(x)=x^\alpha$ $(\alpha>1)$ the assumption (W_+) is satisfied with $c_1=c_2=1/(\alpha_1-1)$.

Let $w(x)=x^\alpha+x^{\alpha 1}$, where $2 \le \alpha-1 \le \alpha_1 < \alpha$. After simple calculations we see that (W_+) is fulfilled for $c_1=1/(\alpha-1)$ and $c_2=1/(\alpha_1-1)$.

These examples show that the equation (2.17) is a generalization of (1.18).

Moreover we assume that

(k_3) $\qquad k \in C^1[0,1]$, $k(x) \ge k(0)$.

We can show the following lemma:

Lemma 2.5. Let w satisfy (W) and (W_+), and k satisfy (k_3). If $v \in M$ is a solution of (2.17) then

(2.18) $\qquad \underline{v}(x) \le v(x) \le \overline{v}(x)$ for $x \in [0,1]$

where

(2.19) $\qquad \underline{v}(x) = w_1^{-1}(k(0)x)$

and

(2.20) $\qquad \overline{v}(x) = w_1^{-1}\left(\int_0^x k(s)ds\right)$

(w_1^{-1} denotes the inverse function of w_1).

Let Ω_2 denote the set of functions of M satisfying the inequality (2.18). Let S_1 be the operator

(2.21) $$S_1(f)(x) = w^{-1}(\int_0^1 k(x-s)f(s)ds)$$

for $f \in M$.

It is easy to prove that $S_1 : \Omega_2 \longrightarrow \Omega_2$. Using simple calculations we can show the following lemma:

Lemma 2.6. Let (W), (W$_+$) and (k$_3$) satisfied. For $x \in [0,1]$

(2.22) $$\int_0^x [\bar{v}(s)-\underline{v}(s)]ds \leq \frac{1}{c_1+1} x[\bar{v}(x)-\underline{v}(x)] .$$

Let $x_2 \in [0,1]$ be such that

(2.23) $$\bar{k}(x_2) \leq \frac{1}{c_1+1} k(0) \quad (\bar{k}(x)= \max_{s \in [a,x]} k(s))$$

and let β_2 be defined by

(2.24) $$\beta_2 = \frac{1}{k(0)} \sup_{s \in [x_2,1]} \frac{k(s)-k(0)}{s}$$

We can prove an inequality similar to (2.10), namely

(2.25) $$k(x)e^{-\beta_2 x} \leq \bar{k}(x_2)$$

for $x \in [0,1]$.

Now we define the metric

(2.26) $$\rho_2(v_1,v_2)= \sup_{x \in (0,1]} |v_1(x)-v_2(x)| \, |\bar{v}(x)-\underline{v}(x)|^{-1} e^{-\beta_2 x}$$

for $v_1,v_2 \in \Omega_2$. The set Ω_2 is a complete metric space with ρ_2.

We obtain:

Lemma 2.7. For $v_1,v_2 \in \Omega_2$

(2.27) $$\rho_2(S_1(v_1),S_1(v_2)) \leq \frac{\bar{k}(x_2)c_2}{k(0)c+1} \rho_2(v_1,v_2).$$

Using (2.23), Lemma 2.7. and Banach fixed point theorem we get a theorem similar to Theorem 2.2.:

Theorem 2.4. Let w satisfy (W), (W$_+$) and k satisfy (k$_3$). Then equation (2.17) has a unique solution v belonging to M. This solution may be found as a limit of successive approximations in Ω_2 with respect to ρ_2.

Remark 2.1. The assumption (W_+) is only needed to prove that S_1 is a contraction. Assuming only (W) we can prove the existence of a solution belonging to M.

We have a lot of open questions concerning weighted metrics method. For example we may ask about a generalization of the above metrics. Here we have shown how to construct these metrics only at a few particular cases.

At the end of this part we must underline that at some physical problems we get nonlinear integral Volterra equations of nonconvolution type, namely

$$(2.28) \qquad [v(x)]^\alpha = \int_0^x k(x,s)v(s)ds \qquad (\alpha>1)$$

(see [15], [37]). Using similar metrics we can show the existence and uniqueness of nonnegative nontrivial solutions. Applying some properties of those metrics it is easy to find an approximate solution close to the exact one. Moreover the error may be estimated. For example, the methods used in [15] and [37] can be applied to the problems concerning nonlinear diffusion treated at works [24], [25] and [29]. Let us note that for the equation (2.28) another techniques may be applied (see [7], [39]).

Moreover, using weighted metrics method, discrete nonlinear equations similar to (1.18) may be studied (see [4], [5]).

With respect to numerical applications, the methods presented above are convenient.

3. PROJECTIVE METRIC METHODS.

The method of Hilbert projective metrics may be applied to different mathematical problems (see [9], [12], [28], [30], [31], [32], [44] and [46]). Here we shall present this method only with respect to applications to nonlinear integral equations having the form

$$(3.1) \qquad u(x) = \int_0^x k(x-s)[u(s)]^p ds \qquad p\in(0,1)$$

(for details see [11] and [13]).

We can show the following theorem (for a comparison see [11]):

Theorem 3.1. Assume

(i)   The kernel k satisfies (k)

(ii)   There exist a function w∈M and constants $0 < m_1 \leq m_2 < \infty$ such that

(3.2) $$m_1 w(x) \leq \int_0^x k(x-s)[w(s)]^P ds \leq m_2 w(x) \quad x \in [0,1] .$$

Then equation (3.1) has a unique continuous solution u with the property

(3.3) $$\bar{m}_1 w(x) \leq u(x) \leq \bar{m}_2 w(x) \quad x \in [0,1],$$

were $0 < \bar{m}_1 \leq \bar{m}_2 < \infty$.

Remark 3.1. This theorem does not give any answer about if the solution u is unique in M. In the next part we shall show that this solution is unique.

Remark 3.2. Theorem 3.1 may be proved for any interval $[0,\alpha]$, $\alpha > 0$.

Remark 3.3. The above theorem can be extended very easily to a nonconvolution kernel ([11]).

How to prove this theorem ?. Let K_w denote the following cone

(3.4) $$K_w = \left\{ u \in C[0,1] : \inf_{x \in (0,1]} u(x)/w(x) \geq 0 \right\} .$$

It may be shown that $u \in \overset{\circ}{K}_w$ (the interior of K_w) if and only if there are positive constants $\overset{\circ}{m}_1$, $\overset{\circ}{m}_2$ such that

$$\overset{\circ}{m}_1 w(x) \leq u(x) \leq \overset{\circ}{m}_2 w(x) \quad x \in [0,1]$$

Let T be the following operator:

(3.5) $$T(f)(x) = \int_0^x k(x-s)[f(s)]^P ds$$

for $f \in K_w$. We obtain $T: K_w \longrightarrow K_w$.

We want to define the Hilbert projective metric. For $u_1, u_2 \in \overset{\circ}{K}_w$ we put

$$M(u_1 | u_2) = \sup_{x \in (0,1]} u_1(x)/u_2(x)$$

and

$$m(u | u) = \inf_{x \in (0,1]} u_1(x)/u_2(x).$$

The Hilbert projective metric is defined in $\overset{\circ}{K}_w$ by

(3.6) $$d_w(u_1, u_2) = \log(M(u_1 | u_2)/m(u_1 | u_2)).$$

We may prove:

Lemma 3.1. For $u_1, u_2 \in \overset{\circ}{K}_W$

(3.7) $$d_W(T(u_1), T(u_2)) \leq p\,d(u_1, u_2).$$

Let $E_W = \left\{ u \in \overset{\circ}{K}_W : \|u\| = 1 \right\}$, where $\|u\| = \sup\limits_{x \in [0,1]} |u(x)|$.

The set E_W is a complete metric space with d_W.

Let

$$T_0(u) = T(u) / \|T(u)\|_W$$

where

$$\|T(u)\|_W = \sup\limits_{x \in [0,1]} \frac{T(u)(x)}{w(x)}.$$

We can show

Lemma 3.2. (i) $T_0 : E_W \longrightarrow E_W$

(ii) For $u_1, u_2 \in E_W$, $d_W(T_0(u_1), T_0(u_2)) \leq p\,d_W(u_1, u_2)$.

Using the above lemma and Banach fixed point theorem we can finish the proof of theorem 3.1.

Now we shall apply theorem 3.1 to three examples:

Example 3.1. We assume that the kernel k is a measurable function such that

$$c_1 x^{\gamma-1} \leq k(x) \leq c_2 x^{\gamma-1} \quad \text{a.e. on } [0,1]$$

where $c_1, c_2 > 0$ and $\gamma > 0$. In this case we can put $w(x) = x^{\gamma/(1-p)}$. For such w we get

$$c_1 B\left(\gamma, \frac{\gamma p}{1-p} + 1\right) w(x) \leq T(w)(x) \leq c_2 B\left(\gamma, \frac{\gamma p}{1-p} + 1\right) w(x).$$

By Theorem 3.1 the equation (3.1) has a unique continuous solution u with the property

$$\overline{m}_1 x^{\gamma/(1-p)} \leq u(x) \leq \overline{m}_2^{\gamma/(1-p)}, \quad x \in [0,1],$$

where $0 < \overline{m}_1 \leq \overline{m}_2 \leq \infty$.

Example 3.2. Let $k:[0,1] \longrightarrow [0,1]$ be defined by the formula:

$$k(x) = \begin{cases} 1 & \text{if } x \in [2^{-(2n+1)}, 2^{-2n}] \quad (n=0,1,2,\ldots) \\ 0 & \text{elsewhere} \end{cases}$$

We try to put $w(x) = x^{1/(1-p)}$. For $x \in (2^{-2(n+1)}, 2^{-2n}]$, $n=0,1,2\ldots$ we get

$$\int_0^x k(x-s)s^{p/(1-p)}ds \geq \int_0^{1/4^n 6} s^{p/(1-p)}ds = (1-p)[1/4^n 6]^{1/(1-p)} .$$

On the base of the above inequality we can write

(3.8) $$T(w)(x) \geq (1-p)6^{-1/(1-p)}w(x).$$

Moreover we have

(3.9) $$T(w)(x) \leq \int_0^x s^{p/(1-p)}ds = (1-p)w(x).$$

By (3.8), (3.9) and Theorem 3.1. we infer that equation (3.1) has a unique continuous solution u satisfying the inequality

$$(1-p)6^{-1/(1-p)}x^{1/(1-p)} \leq u(x) \leq (1-p)x^{1/(1-p)}.$$

Example 3.3. Now we consider equation (3.1) with

(3.10) $$k(x) = \exp(-1/x^\alpha) \quad \alpha > 0.$$

In 1987 P.J. Bushell applying the Laplace asymptotic formula showed that

(3.11) $$T(w)(x) \cong w(x) \quad \text{as} \quad x \longrightarrow 0^+$$

where

(3.12) $$w(x) = cx^{(1+\alpha/2)/(1-p)}e^{-\nu/px^\alpha}$$

$$\nu = (n/(1-\mu))^{\alpha+1}, \quad \mu = p^{1/(\alpha+1)}$$

and

$$c = \mu^{p(1+\alpha/2)/(1-p)^2}\left[2\pi p/\left\{\alpha(\alpha+1)\nu\left((1-\mu)^{-2}+\mu^{-2}\right)\right\}\right]^{1/2(1-p)}$$

For $\varepsilon > 0$ given $\alpha > 0$ exists such that

$$(1-\varepsilon)w(x) \leq T(w)(x) \leq (1+\varepsilon)w(x)$$

for $x \in [0,a]$. By theorem 3.1. and remark 3.2. we infer that there exists a continuous solution u of (3.1) in [0,a] such that

$$\overline{m}_1 w(x) \leq u(x) \leq \overline{m}_2 w(x), \qquad x \in [0,a]$$

with positive constants m_1, m_2.

As we have seen above, Hilbert projective metric method works without a priori estimates. But to show the existence of a nontrivial solution we must find a nontrivial function w such that (3.2) is satisfied. And this is the most complicated part of the method. At the end of this part we must underline that the projective metric method may be used with some nonlinear equations which are nonhomogeneous (see [43]).

4.MONOTONE OPERATOR METHODS.

Now we shall study the equation (1.32). We assume that

(g)
$$\begin{cases} g: \overline{R}_+ \longrightarrow \overline{R}_+ \text{ is a continuous nondecreasing concave function} \\ \text{such that } g(0)=0, \ g(u)/u \text{ is strictly decreasing for } u>0, \\ g(u)/u \longrightarrow \infty \text{ as } u \longrightarrow 0^+ \text{ and } g(u)/u \longrightarrow 0 \text{ as } u \longrightarrow \infty \end{cases}$$

Moreover the kernel k satisfies assumptions (k).

In [13] the following theorem is presented:

Theorem 4.1. Assume that (k) and (g) are satisfied. If there exists a continuous nonnegative nontrivial function q on $[0,x_0]$, $(x_0 \in [0,1])$ such that

(4.1)
$$q(x) \leq \int_0^1 k(x-s)g(q(s))ds \qquad x \in [0,x_0]$$

then equation (1.32) has a unique continuous solution $u \in M$. Moreover u is nondecreasing and

(4.2)
$$u(x) \leq G\left(\int_0^x k(s)ds\right) \qquad x \in [0,1]$$

where G is the inverse function to $u \longrightarrow u/g(u)$.

The theorem is proved in two steps. At first by some monotone methods we can show the existence of the maximal solution. At the proof of the uniqueness a translation invariance property of equation (1.32)

is applied. This part of the proof is a simplified version of ideas used in [36] and [37].

Let us note that the existence of a nontrivial solution follows from classical comparison theorems (see [30], [47]). On the other side the existence of a nontrivial solution satisfying some a priori estimates can be obtained from Krasnoselskii ([26], [27]).

By Theorem 4.1. we can formulate:

Corollary 4.1. If the equation (1.32) has a solution $u \in M$, then u is unique. Moreover it is a nondecreasing function satisfying (4.2).

As an easy consequence of Theorem 4.1. we can present:

Lemma 4.1. Let the functions k_1 (i=1,2) satisfy assumptions (k) and g_1 (i=1,2) satisfy assumptions (g). Moreover $k_1 \leq k_2$ and $g_1 \leq g_2$. If $u_1 \in M$ is the solution of

$$(4.3) \qquad u(x) = \int_0^x k_1(x-s)g_1(u(s))ds$$

then the equation

$$(4.4) \qquad u(x) = \int_0^x k_2(x-s)g_2(u(s))ds$$

has a unique solution $u_2 \in M$ such that $u_1 \leq u_2$.

Moreover , as an extension of Corollary 4.1. we can formulate:

Corollary 4.2. Let the kernel k satisfy assumptions (k) and let the function g satisfy assumptions (g). If equation (1.32) has a nontrivial nonnegative continuous solution on $[0,a]$, $(a \in [0,1])$, then this solution may be extended to a continuous one on $[0,1]$. Moreover this solution is unique and nondecreasing.

Under the assumptions of Corollary 4.2., Theorem 4.1. can be proved for all $x \geq 0$. Using this extended version of theorem 4.1. we get the proof of the corollary.

Remark 4.1. All the results above remain true if we remove the assumption "g concave".

Example 4.1.([13]) Let $\gamma \geq 1$. Assume that assumptions (g) are satisfied. If

(4.5)
$$\int_0^1 \frac{u^{1/\mu}}{[g(u)]^{1/\mu}} \, du \; < \; \infty$$

where $\mu > \gamma$, then the equation

(4.6)
$$u(x) = \int_0^x (x-s)^{\gamma-1} g(u(s)) ds \quad x \in [0,1]$$

has a unique solution $u \in M$.

To show it we introduce the function

$$F(u) = \int_0^u \frac{s^{1/(n-1)}}{[g(s)]^{1/\mu}} \, ds$$

The function

$$u(x) = F^{-1}(c_0 x),$$

where $c_0 = \mu[(\mu-\gamma)/(\mu-1)]^{(\mu-1)/\mu}$, is a solution of the initial value problem

$$u = c_0 u^{1-1/\mu} [g(u)]^{1/\mu}$$
$$u(0) = 0.$$

This solution belongs to M. Moreover

$$\mu u(x) = \left\{ c_0 \int_0^x [g(u(s))]^{1/\mu} ds \right\}^{1/\mu}.$$

Using Hölder's inequality we get

$$\mu u(x) \leq c_0^{1/\mu} \left\{ \int_0^x (x-s)^{\gamma-1} g(u(s)) ds \right\} \left\{ \int_0^x (x-s)^{-(\gamma-1)/(\mu-1)} ds \right\}^{\mu-1}.$$

Hence

(4.7)
$$u(x) \leq \int_0^x (x-s)^{\gamma-1} g(u(s)) ds.$$

By (4.7) and theorem 4.1. we infer that equation (4.6) has a unique solution $u \in M$.

Example 4.2. Once more let us consider the equation

(4.8)
$$u(x) = \int_0^x [u(s)]^p \exp(-1/(x-s)^\alpha) ds \quad \alpha > 0, \; p \in (0,1).$$

In example 3.3. we have asked about the existence of nontrivial solutions for the equation above. By Hilbert projective metric

techniques it was proved that equation (4.8.) has a unique continuous solution u on [0,a], a>0, such that u(x)>0 for x∈(0,a]. By Corollary 4.2. we infer that equation (4.8.) has a unique solution u∈M.

Example 4.3. Let k satisfy assumptions (k) and g satisfy assumptions (g). If moreover $k(x) \geq c_1 \exp(-1/x^{\alpha})$ and $g(u) \geq c_2 u^p$, where c_1, $c_2 > 0$, $\alpha > 0$ and $p \in (0,1)$, then by example 4.2 and lemma 4.1. we infer that equation (1.32) has a unique solution belonging to M.

5. SOME PARTICULAR RESULTS.

One may ask if for every function g satisfying assumptions (g) and for every kernel k satisfying assumptions (k) there exists a solution of (1.32) belonging to M. This is a difficult question and a complete answer is not known. We can present only a particular answer. We shall consider equation (4.6).

Remark 5.1. Let us note that for $\gamma \in (0,1)$ equation (4.6) is a nonlinear Abel equation. An interesting survey concerning nonlinear Abel equations can be found at [19].

Under strong assumptions concerning g, equation (4.6) was considered by G. Gripenberg in [20]. There it is shown that equation (4.6) has a nontrivial nonnegative solution if and only if the so-called Osgood-Gripenberg condition is satisfied. But assumptions concerning g may be weakened , and we are able to show the following theorems:

Theorem 5.1. Let $\gamma > 0$. Assume (g) is satisfied. If

$$\int_0^1 \frac{u^{1/(\gamma-1)}}{[g(u)]^{1/\gamma}} \, du < \infty$$

then equation (4.6.) has a unique solution belonging to M.

Theorem 5.2. Let $\gamma \geq 1$. Assume (g) is satisfied. If the equation (4.6) has a solution belonging to M, then (5.1) is satisfied.

We can write the following corollary:

Corollary 5.1. Let $\gamma \geq 1$. Assume (g) is satisfied. Equation (4.6) has a unique solution belonging to M if and only if the condition (5.1) is

satisfied.

The proofs ° of the above theorems are based on interesting functional inequalities. Moreover a comparison theorem is used ([30]).

Remark 5.2. All results above will remain true if we remove the assumption « g concave » .

Let us look at the Osgood-Gripenberg condition (5.1).

Example 5.1. Let $g(u)=u^p$ $(p \in (0,1))$. For such g the Osgood-Gripenberg condition (5.1) is satisfied. This means that for such g the equation (4.6) has a unique solution belonging to M.

Example 5.2. If we put $g(u)=u[\log(\log 1/u)]^\beta$, $\beta > 0$, on $[0,1/2]$ and extend it to a function satisfying (g) then we get:

$$\int_0^{1/2} \frac{u^{-1}}{[\log(\log 1/u)]^{\beta/\alpha}} \, du = +\infty$$

This means that for such function g satisfying (g) equation (4.6) has not nontrivial solutions.

Example 5.3. We put $g(u)=u(\log 1/u)^\beta$, $\beta > 0$, on $[0,1/2]$ and extend it to a function satisfying (g) then we get

$$\int_0^{1/2} \frac{u^{-1}}{[\log 1/u]^{\beta/\alpha}} \, du = \begin{cases} < +\infty & \text{if } \gamma < \beta \\ +\infty & \text{if } \gamma \geq \beta \end{cases}$$

We infer that for $\beta > \gamma$ there exists a nontrivial solution of (4.6), while for $\beta \leq \gamma$ nontrivial solutions do not exist.

The above examples suggest that for a kernel k very smooth near the origin, there may not be nontrivial solutions of the equation (3.1). We suppose that it is possible to find the most general sufficient and necessary conditions for the existence of a nontrivial solution of (1.32).

As the above survey shows, there are a lot of open problems con-cerning the considered equations. Because of the importance for appli-. cations this equation ought to be intensively studied.

Some more general equations than (1.32) are also interesting (see [1], [6]).

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APPENDIX

On an extension of Gripenberg's condition

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1.- In some problems of mathematical physics the nonlinear integral equations of the form

$$(1) \qquad u(x) = \int_0^x k(x-s)\, g(u(s))\, ds \qquad (x \in [0,1])$$

where g is a nondecreasing concave function (g(0)=0) and k is a convex one (k(0)=0), are considered (see |4|, |5|, |7|, |8|). If a physical phenomena is described by the equation (1) then the existence of nontrivial solution of (1) verifies the correctness of the mathematical model. From a mathematical point of view we ask for which k and g exist nontrivial solutions. for the case $k(x) = x^{\alpha - 1}$ ($\alpha > 1$) this problem was considered by G. Gripenberg (|3|). It was shown that equation (1) with this particular kernel has a nontrivial solution if and only if

$$(2) \qquad \int_0^1 \{u/g(u)\}^{1/\alpha} \frac{du}{u} < +\infty$$

But Gripenberg's assumptions do not admit the case $g(u) = u^p$ ($p \in (0,1)$).

In papers |1|, |2| and |6| it is shown that Gripenberg's condition is true for the more general g satisfying the following assumptions:

(g) $g: R_+ \longrightarrow R_+$ ($R_+ = [0, \infty)$) is a continuous, non-decreasing function such that g(0)=0, g(u)/u is strictly decreasing for $u > 0$ and $g(u)/u \longrightarrow \infty$ as $u \longrightarrow 0+$.

At this note we would like to show shortly how to extend the classes of functions g and k for which exist nontrivial solutions of (1).

We shall assume

(k) $k:[0,1] \longrightarrow R_+$ is a convex function such that $k(0)=0$ and $k \in C^1 |0,1|$ $(k(x) \neq 0)$. .

2.- We can show.

Lemma 1: Equation (1) has a nontrivial continuous solution u if and only if a decreasing function $v \in L^1[0,1] \wedge C(0,1]$ exists such that $v(x) > 0$ for $x > 0$ and satisfying the equation

(3) $$v(x) = \left\{ \int_0^x k\left(\int_s^x v(\xi) d\xi \right) g'(s) ds \right\}^{-1}$$

Proof: We know u is an absolutely continuous convex function. We get

(4) $$u'(x) = \int_0^x k(x-s) g'(u(s)) u'(s) ds$$

Since u is strictly increasing we infer that inverse function u^{-1} exists and is differentiable for $x > 0$. We denote by v the derivative of u^{-1}. It is a decreasing function belonging to $L^1[0,1] \cap C(0,1|$ and satisfying (3) with respect to (4).

If a locally integrable function v satisfies (3) then the inverse function to $x \longrightarrow \int_0^x v(s) ds$ must satisfy (1).

Remark 1: If $v \in L^1[0,1] \wedge C(0,1)$ satisfies (3) then

(5) $$\int_0^x K\left(\int_s^x v(\xi) d\xi \right) g'(s) ds = x \quad \text{for} \quad x > 0$$

where

(6) $$K(x) = \int_0^x k(s) ds$$

From here, instead equation (1) we can consider equation (3).

3.- At this point we additionaly suppose: (k_1) log $k(x)$ is a concave function.

Remark 2: Under such assumptions the function $k K^{-1}$ is concave.

We can prove:

Theorem 1: If $v \epsilon L^1[0,1] \wedge C(0,1|$ such that $v(x) > 0$ for $x > 0$ is a solution of (3) then $(K^{-1})'(x/g(x))/g(x) \epsilon L^1[0,1]$.

Proof: We can write (3) as

$$v(x) = 1/\int_0^x k \ K^{-1}(K(\int_s^x v(\int)d\int))g'(s)ds$$

snice $k \ K^{-1}$ is concave we get

$$v(x) \geq 1/g(x)k \ K^{-1}(\int_0^x K(\int_s^x v(\int)d\int)g'(s)ds/g(x))$$

By Remark 1 we have

$$v(x) \geq 1/g(x) \ k \ K^{-1}(x/g(x)) = (K^{-1})'(x/g(x))/g(x)$$

The theorem is proved.

Corollary 1: If $v \epsilon L^1[0,1]_\wedge C(0,1|$ is a solution of (3) then

$$(7) \qquad \int_0^t \frac{du}{g(u)k \ K^{-1}(u/g(u))} < +\infty$$

4.- Let

$$(8) \qquad \bar{K}(x) = x \ k(x)$$

Remark 3: The function K is convex.

Theorem 2: If $v \epsilon L^1[0,1] \wedge C(0,1|$ is a solution of (3) then

$$(9) \qquad v(x) \leq 2\bar{K}^{-1}(x/g(x))/x$$

Proof: If $v \epsilon L^1[0,1]_\wedge C(0,1|$ is a solution of (3) then $v(x)$ is decreasing and continuous for $x > 0$. Applying Young inequatily to (3) we obtain

$$v(x) \leq 1/g(x) \ k(\int_0^x \int_s^x v(\int)d\int \ g'(s)ds/g(x))$$

From the above inequality we get

$$v(x) \leqslant 1/g(x) \ k(\int_0^x v(s)g(s)ds/g(x))$$

Since v is decreasing we obtain

$$v(x).k(v(x)\int_0^x g(s)ds/g(x)) \leqslant 1/g(x)$$

Since, by assumptions, $g(s) \geqslant sg(x)/x$ for $s \in [0,x]$ then

$$v(x)k(v(x)x/2) \leqslant 1/g(x)$$

and

$$v(x)x \ k(v(x)x/2)/2 \leqslant x/2g(x)$$

From the last inequality we have

$$v(x)x/2 \ \bar{K}^{-1}(x/2g(x))$$

We infer inequality (9) is true.

Corollary 2: We assume (g), (k) and (k$_1$) are satisfied. If $v \in L^1[0,1] \wedge C(0,1]$ is a solution of (2) then

$$1/g(x)k \ K^{-1}(x/g(x)) \leqslant v(x) \leqslant 2 \ \bar{K}^{-1} \ (x/2g(x))/x$$

Remark 4: Since $\bar{K}^{-1}$ is concave then $\bar{K}^{-1}(x/2g(x))/x \in L^1[0,1]$ if and only if $\bar{K}^{-1}(x/g(x))/x \in L^1[0,1]$.

4.- We can show

Theorem 3: If

(10)
$$\int_0^1 \bar{K}^{-1}(u/g(u)) \ \frac{du}{u} < +\infty$$

then equation (1) has a nontrivial solution.

Proof: Consider the equation

(11)
$$u_\varepsilon(x) = \varepsilon x^2 + \int_0^x k(x-s)g(u_\varepsilon(s))ds \ (x \in [0,1])$$

By |5| this equation has the unique solution u_ε $(\varepsilon \in (0,1))$

such that $u_\varepsilon(x) > 0$ for $x > 0$. This solution is convex. Let u_ε^{-1} denote the inverse function to u_ε. This function is nondecreasing concave and differentiable for $x > 0$. Let v_ε denote $(u_\varepsilon^{-1})'$. The function v_ε satisfies the following equation (see Theorem 1).

$$(12) \quad v_\varepsilon(x) = 1/(2\varepsilon u_\varepsilon^{-1}(x) + \int_0^x k(\int_0^x k(\int_s^x v_\varepsilon(\xi)d\xi)g'(s)ds) \leq$$

$$\leq 1/\int_0^x k(\int_s^x v_\varepsilon(\xi)d\xi)g'(s)ds$$

As in Theorem 2 we can show that

$$v_\varepsilon(x) \leq 2 \; \overline{K}^{-1}(x/2g(x))/x \quad (x \in [0,1])$$

and by Remark 4 we can write

$$(13) \quad u_\varepsilon^{-1}(x) \leq 2 \int_0^x \overline{K}^{-1}(s/2g(s)) \frac{ds}{s} \doteq F^{-1}(x)$$

From (13) we obtain

$$u_\varepsilon(x) \geq F(x) \quad (x \in [0,1])$$

Let $\varepsilon \downarrow 0$. Then $u_\varepsilon(x) \downarrow u(x)$ and u is a solution of (1) (see |6|). since $u(x) \geq F(x)$ we have found a nontrivial solution of (1).

Corollary 3: If

$$(14) \qquad \int_0^1 K^{-1}(u/g(u)) \frac{du}{u} < +\infty$$

Then equation (1) has a nontrivial solution.

Since $K(x) \leq x \; k(x) \doteq \overline{K}(x)$ then by Theorem 3 we infer the corollary is true.

5.- Let us note that in the case $k(x) = x^{\alpha-1} (\alpha > 1)$ conditions (7), (10) or (14) are equivalent to Gripenberg's condition (2). By the condition (10) we can infer that the equation (1) has a nontrivial solution in the case $k(x) = \exp(-1/x^\beta)$ ($\beta \in (0,1)$) and $g(u) = u^p (p \in (0,1))$. But it is known that equation (1) has a nontrivial solution for any $\beta > 0$ (see |2|). It suggest we ought to look for more subtle conditions.

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