EXTRACTA MATHEMATICAE 3, n.2, 64-66 (1988)

NONSELFADJOINT BOUNDARY VALUE PROBLEMS AT RESONANCE. NONLINEARITIES WHICH MAY GROW LINEARLY

M. Arias

Departamento de Matemática Aplicada. Universidad de Granada
18071. Granada. SPAIN.

1980 Mathematics Subject Classification, 35G30.

Let Ω be a finite complete measure space and $H=L^2(\Omega)$ be the space of Lebesgue measurable square integrable functions on Ω. We consider the existence of solutions of the equation

(1) $$Lu - \lambda u = g(.,u) + f,$$

where $L: \operatorname{dom} L \subset H \longrightarrow H$ is a Fredholm linear operator with index zero and compact generalized right inverse, K, λ is an eigenvalue of L, $f \in H$ and $g: \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear function satisfying Caratheodory conditions which grows at most linearly.

Recently, Iannacci and Nkashama, [3], have obtained an existence Theorem for (1) when L is selfadjoint, $\lambda=\lambda_n$ is the n-th eigenvalue of L and the nonlinear function g lies between two consecutives eigenvalues and may be unbounded and reach the eigenvalue $\lambda_{n+1}-\lambda_n$ on a subset of positive measure.

When L is nonselfadjoint, there exist few results about (1). Ahmad, [1](see, also [2]) studies (1) when L is an uniformly elliptic linear operator of second order with Dirichlet conditions on the boundary of a bounded open set Ω of $\mathbb{R}^n$, $\lambda=\lambda_1$ is the principal eigenvalue and $g(.,u)\equiv g(u)$ is locally Lipchitzian on $\mathbb{R}$. He shows that there exists a constant $d_o>\lambda_1$ depending only on L ($d_o=\lambda_2$ if L is selfadjoint) such that if $\limsup_{|u|\to\infty} (g(u)/u)<d_o-\lambda_1$, (and f satisfies a Landesman Lazer type condition), (1) is solvable.

We present some abstract results for (1) valid for L nonselfadjoint which generalize in certain cases those of [1], [3].

Let L be satisfying the previous conditions. It is not restrictive to assume $\lambda=0 \in \sigma(L)$(= eigenvalues of L) and consider the equation

(2) $$Lu = g(.,u) + f, \qquad u \in \operatorname{dom} L.$$

Suppose moreover that

(C1) For any $\phi \in \ker L$, $\phi \neq 0$, there exists $\Psi \in \ker L^* - \{0\}$ with $\phi(x)\Psi(x) \geq 0$ a.e. Ω and $|N(\phi)-N(\Psi)|=0$,

where L^* is the adjoint operator of L, $N(f)=\{x\in\Omega \mid f(x)=0\}$ and $|.|$ denotes the measure of Ω.

THEOREM.- *Let* $d_0 = \sup\{d \geq 0 /$ *if* $\mu\in H$, $0 \leq \mu \leq d$ *a.e.* Ω *and* ϕ *is solution of* $Lu=\mu u$, *then* $\phi \in \ker L\}$ *(one can prove* $d_0>0$*). Suppose*

(H1) There exist $\gamma, k, \alpha, \beta \in H$, $0 \leq \gamma \leq d < d_0$ *a.e.* Ω *such that* $|g(x,u)| \leq \gamma(x)|u| + k(x)$, $u\in\mathbb{R}$, $g(x,u) \geq \alpha(x)$, $u \geq 0$, $g(x,u) \leq \beta(x)$, $u \leq 0$, *a.e.* $x\in\Omega$.

(H2) For any $\Psi \in \ker L^* - \{0\}$, $\int_\Omega f\,\Psi > \int_\Omega \bar{g}(-\infty)\Psi^- - \int_\Omega \underline{g}(+\infty)\Psi^+$ *where* $\bar{g}(-\infty)(x) = \limsup_{u\to-\infty} g(x,u)$, $\underline{g}(+\infty)(x) = \liminf_{u\to+\infty} g(x,u)$, $\Psi^+(x) = \max(\Psi(x),0)$, $\Psi^-(x) = \max(-\Psi(x),0)$.

Then, equation (2) has at least one solution.

The proof of this Theorem is based in topological methods and it is essential to obtain apriori bounds. These apriori bounds are obtained by using a contradiction argument and comparing with a linear problem.

The following Proposition sets up the relation between the previous Theorem and the results in [1], [3].

PROPOSITION.- *Suppose, also*

(C2) $0 < \lambda_1 = \inf\{Re\lambda > 0 / \lambda\in\sigma(L)\} \in \sigma(L)$ *and* $\|(L - \frac{\lambda_1}{2}I)^{-1}\|_{H\to H} = \frac{2}{\lambda_1}$.

Then, $d_0 = \lambda_1$ *and we can replace the condition* $0 \leq \gamma \leq d < d_0$ *in (H1) by* $0 \leq \gamma < \lambda_1$, *a.e.* Ω.

REMARK.- If L is selfadjoint, conditions (C1) and (C2) are trivially satisfied (when L has positive eigenvalues) and (H1) allows to the nonlinearity g to lie between two consecutives eigenvalues of L.

EXAMPLES.-

Let Ω be a bounded open set of $\mathbb{R}^n$, $T>0$, $H=L^2((\mathbb{R}/T\mathbb{Z})\times\Omega)$ and λ_n the n-th eigenvalue of Dirichlet problem for the Laplacian operator on Ω (we denote by A). Define

L: dom L $\subset$ H $\longrightarrow$ H by

1.- dom L = { $u \in H^{1,2}((\mathbb{R}/T\mathbb{Z}) \times \Omega)$ / $u=0$ in $\partial\Omega$ in the sense of traces},

$Lu := u_t - \Delta u - \lambda_n u.$

2.- dom L = { $u \in H^{2,2}((\mathbb{R}/T\mathbb{Z}) \times \Omega)$/ $u=0$ in $\partial\Omega$ in the sense of traces},

$Lu := u_{tt} - \Delta u + \eta u_t - \lambda_n u, \qquad \eta \neq 0.$

In both examples, if f and g verify (H1) and (H2) with $0 \leq \gamma < \lambda_{n+1} - \lambda_n$, a.e. $(\mathbb{R}/T\mathbb{Z}) \times \Omega$, $Lu = g(t,x,u)+f$ has solution.

These results for the heat and telegraph equation, respectively, cannot be obtained from the results in [3] since L is not selfadjoint.

REFERENCES

[1] AHMAD, S., Nonlinear Analysis, V. 10,N.2,(1986),147-156.

[2] CESARI, L., PUCCI, P., Nonlinear Analysis, V.9, N.11, (1985), 1227-1241.

[3] IANNACCI, R., NKASHAMA, M.N., Nonlinear Analysis, V.11, n.4, (1987), 445-473.