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COMPACTOIDS IN SOME SPACES OF CONTINUOUS FUNCTIONS

J. Martinez-Maurica
Facultad de Ciencias
U. de Cantabria

Samuel Navarro Hernández
Dpto. Matemática. Fac. Ciencia
U. de Santiago de Chile

The aim of this paper is to announce some Ascoli theorems for some subspaces of the space C(X) of all continuous $\mathbb{K}$-valued functions, where X is a Hausdorff zèrodimensional topological space and $\mathbb{K}$ is a complete non-archimedean nontrivially valued field; this is, we will explore the relationships between a certain kind of compact-like sets and equicontinuous sets within C(X).

The first difference with its archimedean analog is the class of compact-like sets we are going to consider. For that it is worth mentioning that (pre)compactness is not very interesting in p-adic analysis: there is no compact convex subset of a locally convex space over $\mathbb{K}$ with more than one point unless $\mathbb{K}$ is locally compact. Altough the field of p-adic numbers $\mathbb{Q}_p$ is locally compact, in many applications it is certainly useful to consider some other valued fields apart from $\mathbb{Q}_p$ (for instance, the non locally compact field $\mathbb{C}_p$ of p-adic complex numbers defined as the completion of the algebraic closure of $\mathbb{Q}_p$).

Quite a number of different variants of precompact sets have been studied in p-adic analysis (see [6]), and it seems for many reasons that the most succesful ones are compactoids defined as follows: a subset A of a locally convex space E is said to be compactoid if for every neighborhood of zero U there exists a finite set $Y\subset E$ such that $A\subset U+c_o(Y)$, where $c_o(Y)$ denotes the absolutely convex hull of Y.

Our first Ascoli theorem is for the space PC(X) of all continuous functions $f\in C(X)$ such that f(X) is a precompact subset of $\mathbb{K}$ endowed with the topology of uniform convergence.

THEOREM 1: A subset $H\subset PC(X)$ is compactoid if and only if the following properties are satisfied:

(a) H(x) is bounded in $\mathbb{K}$ for every $x\in X$.

and (b) For every $\varepsilon>0$, there exists a finite partition $X_1,\ldots,X_n$ of X consisting of clopen sets such that $x,y\in X_i \rightarrow |f(x)-f(y)|\le\varepsilon$ for all $f\in H$ $(i=1,\ldots,n)$.

In this theorem the condition (b) implies equicontinuity of H. Also, if X is compact both properties coincide.

Our theorem 1 is a generalization of a previous one of N. De Grande-De Kimpe [1, theorem 1.8] in which she characterizes compactoids in the space C(X) where X is a compact subset of a nonarchimedean valued field $\mathbb{K}$.

Now assume that X is also locally compact and consider the subspace $C_\infty(X)$ of PC(X) consisting of all continuous functions which vanish at infinity. Compactoids in $C_\infty(X)$ can be characterized in the following way:

THEOREM 2: A subset H of $C_\infty(X)$ is compactoid if and only if

(a) H(x) is bounded in $\mathbb{K}$ for every $x\in X$.

and (b) For every $\varepsilon>0$, there exists a finite number of pairwise disjoint clopen compact sets $P_1,\ldots,P_n$ in X such that $x,y\in P_i \rightarrow |f(x)-f(y)|\le\varepsilon$ for all $f\in H$, $i\in\{1,\ldots,n\}$ and $|f(x)|<\varepsilon$ for every $x\in X-(\bigcup_i P_i)$, $f\in H$.

In order to study compactoids in the space C(X) endowed with the topology of uniform convergence on compact sets we need the following zerodimensional version of the concept of k-space: A zerodimensional space X is called an ultra k-space if $f:X\longrightarrow\{0,1\}$ is continuous when $f|K$ is continuous for each compact $K\subset X$ ($\{0,1\}$ is endowed with the discrete topology).

Every zerodimensional k-space is an ultra k-space. However, there are examples of zerodimensional ultra k-spaces which are not k-spaces as well as zerodimensional spaces which are not ultra k-spaces.

THEOREM 3: Let X be an ultra k-space and let $H\subset C(X)$. Then H is compactoid if and only if H is equicontinuous and

H(x) is bounded in $\mathbb{K}$ for every $x\in X$.

Finally, we are going to consider the case of the strict topology in the space BC(X) of all bounded continuous functions $f:X\longrightarrow\mathbb{K}$. This topology was introduced in the non-archimedean setting by J.B. Prolla [4, chapter 9] in case X is locally compact. For general zerodimensional spaces the strict topology has been studied by A.C.M. Van Rooij [5] and A.K. Katsaras [2].

THEOREM 4: Let X be an ultra k-space. A subset $H\subset BC(X)$ is compactoid for the strict topology if and only if

(a) $\sup\{|f(x)|: f\in H, x\in X\}<\infty$

and (b) H is equicontinuous.

Complete proofs of all these results as well as other related results will appear in [3].

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