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CENTRAL CLOSURE OF PRIME NONCOMMUTATIVE JORDAN ALGEBRAS WITH NONZERO SOCLE

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The theory of the extended centroid and central closure of a semiprime nonassociative algebra was developed by Baxter and Martindale in [1]. In this note we present these notions in relation with the structure theorem of prime noncommutative Jordan algebras with nonzero socle (proved by Rodríguez Palacios and the second author in [6] by using other methods).

Let A be a *prime* ($IJ = 0$ implies $I = 0$ or $J = 0$, I and J ideals of A) nonassociative algebra over a field K of characteristic $\neq 2$ and let M(A) be the multiplication algebra of A. A linear mapping $f : U \to A$ where U is a nonzero ideal of A is called *admissible* if f is an homomorphism of M(A)-modules. Consider the set $\mathcal{C} = \{(f,U)\}$ of all admissible mappings. In this set we can define an equivalence relation as follow: $(f,U) \sim (g,V)$ if and only if there exists a nonzero ideal W of A such that $W \subset U \cap V$ and $f_{|W} = g_{|W}$. Write C(A) to denote the set of all equivalence classes [f,U] of admissibles mappings. Baxter and Martindale proved in [1] that C(A) with the operations:

(1) $[f,U] + [g,V] = [f+g, U \cap V]$

(2) $[f,U]\,[g,V] = [fg, g^{-1}(U)]$

is a field called the *extended centroid* of A. This nomenclature makes sense since the mapping $\rho \to [\rho, A]$ is a monomorphism of the centroid Γ of A into C(A). It is shown in [2] that if every nonzero ideal U of A contains an ideal of the form $\rho(A)$, with $0 \neq \rho \in \Gamma$, then C(A) coincides with the field of fractions $F(\Gamma)$ of the centroid.

However, this is not true in general.

Baxter and Martindale also proved that A can be embedded in a C(A)-algebra Q(A), called the *central closure* of A, that is spanned by A. Moreover, Q(A) is prime and *closed*, i.e., the extended centroid of Q(A) coincides with C(A).

Suppose now that A is a noncommutative Jordan algebra (n.c.J.a.) (see [5] for definition and notations). If A is prime and nondegenerate with *socle* S (the socle is the sum of all minimal inner ideals), then S is a simple n.c.J.a. containing minimal inner ideals. By [6, Theorem 4], S is a division algebra, a quadratic algebra over its centre, a (commutative) Jordan algebra or a quasi-associative algebra over its centroid. We have proved in [3]:

LEMMA 1. *Let* A *and* S *be as above. Then*

i) The extended centroid C(A) *coincides with the centroid* Γ *of* S.

ii) The central closure Q(A) *is a prime nondegenerate noncommutative Jordan algebra with the same socle* S.

This lemma together with the structure of the socle given above allow us to prove, via a result of the second author [4], the following

THEOREM 1. *The central closure* Q(A) *of a prime nondegenerate noncommutative Jordan algebra with nonzero socle* S *is one of the following:*

(i) A division algebra.

(ii) A simple quadratic algebra over its centre.

(iii) A (commutative) Jordan algebra.

(iv) A quasi-associative algebra over the centroid Γ *of* S.

Since A is contained in Q(A) and A and Q(A) have the same socle, we can now derive the structure theorem of prime nondegenerate n.c.J.a. A

COROLLARY [6, Theorem 13]. *Let* A *be a prime nondegenerate noncommutative Jordan* K-*algebra with nonzero socle* S. *Then* A *is one of the following:*

(i) A division algebra.

(ii) A simple quadratic algebra over its centre.

(iii) A (commutative) Jordan algebra.

(iv) A K-*subalgebra of a quasi-associative algebra over the centroid* Γ *of* S.

We note that the case (iv) cannot be improved. Indeed, let X be an infinite-dimensional complex vector space and let $\mathcal{F}(X)$ denote the set af all finite rank linear operators on X. Then the real vector space $A = \mathcal{F}(X) + \mathbb{R}1_X$ with the product given by $a._i b = iab + (1-i)ba$ is not quasi-associative, although it is a real subalgebra of the complex quasi-associative algebra $B = \mathcal{F}(X) + \mathbb{C}1_X$ (with the same product).

Finally we proved in [3] that every complex normed prime nondegenerate n.c.J.a. with nonzero socle is closed. Hence it can be derived the structure theorem of these algebras given in [5].

THEOREM 2. *Let* A *be a prime nondegenerate noncommutative Jordan normed complex algebra with nonzero socle. Then* A *is one of the following:*

(i) The complex field

(ii) A simple quadratic normed complex algebra

(iii) A (commutative) Jordan algebra.

(iv) A split quasi-associative algebra $A = B^{(\lambda)}$ *where* B *is a prime associative normed complex algebra with nonzero socle, and* $\lambda \in \mathbb{C}$, $\lambda \neq 1/2$.

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