EXTRACTA MATHEMATICAE 2, n.3, 108-110 (1987)

SMOOTH TORAL ACTIONS ON PRINCIPAL BUNDLES AND CHARACTERISTIC CLASSES

NIEVES ALAMO AND FRANCISCO GOMEZ

Departamento de Algebra, Geometría y Topología. Facultad de Ciencias. Universidad de Málaga. Campus Teatinos. Apartado 59. Málaga 29080.

1980 AMS Subject Classification: 57R20

This is an extract of a preprint which will be published elsewhere.

The purpose of our work is to find explicit formulae for the computation of some characteristic classes of smooth principal bundles $\mathcal{P}: P \to B$, in terms of local invariants at a "singular subset" A_G of B, associated to a smooth action of a compact Lie group G on $\mathcal{P}$. This singular subset, A_G, is defined as the set of points x in B whose isotropy subgroups G_x have dimension at least one.

The starting point for our research is the following result:

Let $\alpha \in H^{2p}(B;k)$ *be a characteristic class of* $\mathcal{P}$ *with* $2p > n-r$ ($n = \dim B$, $r = \dim G$), *where* H^* *denotes singular cohomology and* k *is any field of characteristic zero. There exists then* $\beta \in H^{2p}(B, B\text{-}A_G; k)$ *such that* $j^*(\beta) = \alpha$, *where* j^* *is the homomorphism induced in cohomology by the inclusion* $j: B \to (B, B\text{-}A_G)$. *In particular, if the action of G on B is almost free,* α *must be zero.* (See [6] and [9] for the particular case of vector bundles)

If B is compact and oriented, we can see more explicitly the dependence of the characteristic classes on the singular set A_G, because in this case we have a commutative diagram

$$\begin{array}{ccc} H^{2p}(B, B - A_G) & \xrightarrow{j^*} & H^{2p}(B) \\ \gamma \uparrow \cong & & \cong \uparrow \text{Poincaré duality} \\ H_{n-2p}(A_G) & \longrightarrow & H_{n-2p}(B) \end{array}$$

(see lemma 14, section 10, chapter 6 of [10], for the definition of γ), and then, if $2p > n-r$, the Poincaré dual of α can be represented by a cycle z of A_G. The general problem is to find an explicit formula giving such a z. This kind of residue formula should involve only the restriction of $\mathcal{P}$ to A_G, the action of G on this restriction, the embedding of A_G in B ("normal bundle" of A_G in B), and the action of G in this "normal bundle".

We shall restrict ourselves to study the case of G being a torus. This restriction includes the equivalent formulation, for compact manifolds, of infinitesimal isometries.We further assume that the action of G on B has finite orbit type (i.e. the action has only a finite number of isotropy subgroups).This is the case, for instance, when B is compact (see [8]).

We have: $A_G = \bigcup_{F \in \mathcal{F}} F$, where $\mathcal{F}$ is the family of connected components of the fixed point sets under the action of all subtori H of G, with $\dim H \geq 1$, appearing as 1-component of isotropy subgroups under the action of G on B.

We need to assume a hypothesis concerning the "genericity" of the action:

Definition 1. The action of G on B is called *generic* if for each connected component of A_G, there exist r subtori of G of dimension one, $S_1,\dots,S_r$, such that they generate G, i.e. $S_1 \dots S_r = G$, and any subtorus of dimension one appearing as 1-component of isotropy subgroup on that connected component, is one of the S_i (cf. 2.10 pag 42 of [1]).

In particular, the genericity assumption implies that, each element of $\mathcal{F}$ which is fixed by a subtorus of dimension s, is contained in exactly s elements of $\mathcal{F}$ which are fixed only by subtori of dimension 1.

To construct characteristic classes, we use the Chern-Weil homomorphism of $\mathcal{P}$,

$$w_{\mathcal{P}} : \mathrm{Sym}\,(\underline{K})_I \to H^*_{dR}(B)$$

(see for instance [7]), where $\mathrm{Sym}\,(\underline{K})_I$ is the graded algebra of multilinear symmetric functions in the Lie algebra $\underline{K}$ of the structure group K of $\mathcal{P}$, invariant under the adjoint representation of K in its Lie algebra $\underline{K}$.

The residue classes.

Let $F \in \mathcal{F}$ be a connected component of the fixed point set by a subtorus of dimension s, and let $F_1,\dots,F_s$ be the elements of $\mathcal{F}$ containing F, and fixed only by subtori of dimension 1, $S_1,\dots,S_s$, respectively.

Choose $h_i \neq 0$ in the Lie algebra $\underline{S}_i$ of S_i , $i = 1,\dots,s$. Give η_{F_i} (normal bundle of F_i in B) the orientation induced by the complex structure associated to h_i, and give $\eta_F = \eta_{F_1|F} \oplus \dots \oplus \eta_{F_s|F}$ the direct sum orientation.(See [6]).

Then, if $\Gamma \in \mathrm{Sym}\,(\underline{K})_I$, and $2m = \mathrm{codim}\, F$, we consider the following "Laurent polynomial" in the indeterminates $X_1,\dots,X_s$:

$$(-2\pi)^m\,(-1)^{s+1}\,\frac{w\,(\mathcal{P}|_F\,,\,\sum_{i=1}^{s} X_i h_i\,,\,\Gamma)}{\prod_{i=1}^{s} w\,(\eta_{F_i}|_F\,,\,X_i h_i\,,\,\mathrm{Pf}_{F_i})}$$

where $w\,(\xi, h, .\,)$ denotes the "generalized" Chern-Weil homomorphism (see [2], or

[6] where the definition differs from the one in this paper by a certain constant factor).

Definition 2. We define the *residue class* $\alpha_\Gamma(F) \in H_{dR}^{2p-2m}(F)$ as the coefficient of the term of degree 0 of the above "Laurent polynomial".

Now, we can state the residue formula as follows:

Theorem. *Let $\mathcal{P}: P \to B$ be a smooth principal bundle with structural group K, and assume that a torus G acts smoothly on $\mathcal{P}$. Suppose that the action of G on B has a finite number of orbit type and it is generic. Then, if* $2p > \dim B - \dim G$,

$$w_{\mathcal{P}}(\Gamma) = \sum_{F \in \mathcal{F}} \int^{-1} \alpha_\Gamma(F) \quad ,$$

for $\Gamma \in \mathrm{Sym}^p(\underline{K})_I$, *and where* $\int^{-1}$ *denotes the inverse of the fiber integral* $\int$ *associate to any tubular neighbourhood U of F in B oriented as above, followed by the canonical homomorphism* $H_{fc}^*(U) \to H^*(B)$.

For the proof, see [2].

Residue formulae for characteristic classes in some particular cases have been found by other authors. See for instance [3], [6],[9] in the real case, [4] in the complex case, and [5] for $\mathbb{Z}/(2)$ coefficients.

REFERENCES

[1] N. ALAMO: Acciones de toros sobre fibrados principales y una fórmula de residuos para algunas clases características. Thesis. University of Málaga, (1987).

[2] N. ALAMO and F. GOMEZ: Smooth toral actions on principal bundles and characteristic classes. Preprint. Forshungsinstitut für Mathematik. Zürich, (1987).

[3] P. BAUM and J. CHEEGER: Infinitesimal isometries and Pontrjagin numbers. *Topology* **8** (1969), 173-193.

[4] R. BOTT: Vector fields and characteristic numbers. *Michigan Math.J.* **14** (2), (1967).

[5] J. DACCACH and A. WASSERMAN: Stiefel-Whitney classes and toral actions. *Topology and its aplications* **21** (1985), 19-26.

[6] F. GOMEZ: A residue formula for characteristic classes.*Topology* **21** (1), (1982), 101-124.

[7] W. GREUB, S. HALPERIN and R. VANSTONE: *Connections, Curvature and Cohomology*, Vol. II. Academic Press (1972).

[8] S. HALPERIN: Real cohomology and smooth transformations groups. Ph.D. Thesis. University of Cornell, (1970).

[9] J.S. PASTERNACK: Folations and compact Lie group actions. *Comm. Math. Helv.* **46** (1971), 467-477.

[10] E. H. SPANIER: *Algebraic Topology*, McGraw-Hill, New York (1966).