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PROBLEMS ON DISTORTION UNDER CONFORMAL MAPPING

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Recently a number of outstanding new theorems concerning conformal mappings have been proved. The need to understand these results and how they connect with some other known results have uncovered certain interesting questions and conjectures, and a' the same time, have thrown new ligth on older questions. The aim of this paper is to describe these results and questions though focusing mainly on the problems which await solution.

In what follows Ω will denote a simply conected domain in the plane and f will denote a conformal mapping from Ω onto the unit disk Δ. For conformal mappings from Δ onto Ω we reserve the letter Ψ. We shall also assume for convenience that Ω is a Jordan domain, although almost every result (or question) which we are about to describe holds (or makes sense) for general simply connected domain.

How does f distort the subsets of the boundary of Ω ?. Or in other terms if we know something about the size of a subset E of $\partial\Omega$, what can we say about the size of f(E)?. When $\partial\Omega$ is a rectifiable Jordan curve then, as a consequence of the classical Riesz' theorem, [8], we know that arc length on $\partial\Omega$ and the measure $\omega(E)$=length(f(E))/2π are absolutely continuous with respect to each other. The measure $\omega(E)$ is the so-called harmonic measure. Therefore, if the length of E is zero then the length of f(E) is also zero. In general, that is, when $\partial\Omega$ is not rectifiable, this does not happen, and, actually, it can occur that a set E of zero length is mapped by f onto a set of full linear measure in $\partial\Delta$. The paper [23] contains examples of this and some other possible occurences.

A classical estimate of harmonic measure which is due to Beurling and Nevanlinna, see [25], [24], claims that if E is a closed subset of $\partial\Omega$ then

$$(1) \qquad \omega(E) \leq C_{\Omega} \sqrt{\text{diam } E} \ ,$$

where diam E denotes the diameter of E, this inequality is sharp, that is, the exponent $\frac{1}{2}$ is best posible.

If A is a subset of the plane and $\alpha\in(0,2]$ we define its α-dimensional content of A, denoted by $M_{\alpha}(A)$, as the infimum of the sums $\sum r_j^{\alpha}$ obtained from all coverings of A by disks of radii r_j. The dimension of a set A is then defined as the infimum of α for which $M_{\alpha}(A)=0$. Then a segment has dimension 1, the middle-third Cantor set has dimension $\frac{\log 2}{\log 3}$, and the snowflake curve has dimension $\frac{\log 4}{\log 3}$. See [3], [18], [19]. We shall denote the dimension of a set A by D(A).

The inequality (1) shows then that if $E\subset\partial\Omega$ has $D(E)< \frac{1}{2}$ then $\omega(E)=0$. Although, (1) is a sharp inequality this consequence es not so. Carleson in [4] showed that one just need to assume that $D(E)<\beta$, where β is some absolute constant, $\beta> 1/2$, to deduce that $\omega(E)=0$. And, finally, Makarov in a brilliant paper [20] has shown that if $D(E)< 1$ then $\omega(E)=0$. Or, in other terms, if $M_{\alpha}(E)=0$, for some $\alpha< 1$, then $\omega(E)=0$. As we have remarked above this does not hold for $\alpha=1$.

This result is very closely related to a question that was first considered by Hayman and Gehring. The size distortion that f produces about a point $z\in\Omega$ is determined by $|f'(z)|$. Thus the means of $|f'|$ do measure the global distortion under f. Let $J_p=\iint_{\Omega}|f'(z)|^p dxdy$, $z=x+iy$. Obviously, $J_2=\pi$. It is not difficult to show that $J_p<\infty$, if $\frac{4}{3} < p\leq 2$; and $\frac{4}{3}$ is sharp. Also, the elementary distortion theorem of univalent functions [8] do show that $J_p<\infty$, if $2\leq p< 3$. Using the machinery that Carleson develop in [4], Brennan, in [2], was able to show that $J_p<\infty$, if $p < 3+\tau$, where τ is some positive absolute constant. For a simpler proof of Brennan's result see [27]. For the inverse of the Koebe function one has $J_4=\infty$. Brennan conjectured that in this respect the Koebe function is extremal. Thus

♀1. Is it true that $J_p < \infty$, if $2 \leq p < 4$?

It could even be true that f', although not in L^4, is of weak-type L^4. This possibility could be restated as follows

♀2. Does it hold that

$$\text{area}(f(A)) \leq C_f \ \text{area}(A)^{1/2} \ ,$$

for every closed subset A of Ω ?

Long ago, Pommerenke and Schiffer, [26],[28], showed that if E is closed subset of $\partial\Omega$

$$(2) \qquad \text{cap}_0(f(E)) \leq C_f . \text{cap}_0(E)^{1/2} \ .$$

Hereafter, cap_0 means logarithmic-capacity. In general, if $0 \leq \alpha < 2$, the α-capacity of a closed subset A of $\mathbb{C}$ is defined as follows. The α-energy of A is

$$V_\alpha(A) = \inf\left\{ \iint_{A \times A} \frac{1}{|x-y|^\alpha} \, d\mu(x) d\mu(y) \ ; \ \mu \in \mathcal{P}(A) \right. \ ,$$

where $\mathcal{P}(A)$ is the cone of probabilities supported by A. then

$$\text{cap}_\alpha(A) = V_\alpha(A)^{-1/2}$$

when $\alpha=0$ the Kernel $\frac{1}{|z|^\alpha}$ is replaced by $\log \frac{1}{|z|}$ in the definition of V_0 and then

$$\text{cap}_0(A) = e^{-V_0(A)}$$

See [3].

The inequality (2) is true also for subsets E of Ω an this, in particular, shows that the answer to ♀2 is yes when A is a disk. For the relation between conformal mapping and logarithmic capacity the reader may consult [26, chapter 11].

Recently D. Hamilton has shown than a version of (2) holds for α-capacities. Namely, if A is a closed subset of Ω , then

(3) $$\mathrm{cap}_{\alpha}(f(A)) \leq C_f \, \mathrm{cap}_{\alpha/2}(A)^{1/2} \quad , \quad 0 \leq \alpha < 2 \; .$$

See [16].

Notice that the constant in (3) does not depend on α . If (3) were to hold with the $\frac{\alpha}{2}$ -capacity of the right hand side replaced by α-capacity then the answer to Q2 would be yes.

We have remarked above that Q1 and Q2 are related to the results of Makarov. Actually, one can show, using the argument of the proof of Makarov's theorem (Makarov's own proof) and one idea from an unpublished portion of a paper of Pommerenke, that the following extension of the Beurling-Nevanlinna estimate (1) holds

$$\text{if} \quad f' \in L^{\beta}(\Omega, dxdy) \quad \underline{\text{then for}} \quad E \subset \partial\Omega$$

(4) $$\omega(E) \leq C_f(q).M_q(E)^{1/2-q}$$

$$\underline{\text{for any}} \quad q < \frac{\beta-2}{\beta-1}$$

See [11].

If Brennan's conjecture holds then (4) holds for any $q < \frac{2}{3}$. (4) does not hold for $q > 2/3$ because otherwise, since $M_q(E) \leq (\mathrm{diam}\, E)^q$, we could replace the exponent in (1) by q/2-q and this number exceeds 1/2 and thus it would be impossible.

The inequality (4) is a quantitative (sharp) version of the absolute continuity of ω with respect to M_q. If the answer to the following question is yes then we would have a nice sharp extension of the Beurling-Nevanlinna estimate.

Q3: <u>Does it hold that, for</u> $E \subset \partial\Omega$,

$$\omega(E) \leq C_f [M_{2/3}(E)^{3/2}]^{1/2}$$

Consider now the inverse Ψ of the conformal mapping f. The question Q1 can be restated in terms of Ψ and then we are asking if

(6) $$\iint_{\Delta} \frac{1}{|\Psi'(z)|^p}\, dxdy < \infty ,$$

for every $p < 2$.

To simplify the writing we introduce the following notation. If g is a measurable function in Δ , and $\alpha \in \mathbb{R}$,

$$I_\alpha(g,r)=\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^\alpha d\theta .$$

By the elementary distortion theorems we know that $|\Psi'(z)| \geq c(1-|z|)$. Thus a brutal estimate gives

$$I_1\left(\frac{1}{|\Psi'|},r\right)=0\left(\frac{1}{1-r}\right)$$

and so one is lead to expect a smaller order of growth. That this is the case is the content of Brennan's theorem; that is,

(7) $$I_1\left(\frac{1}{|\Psi'|},r\right)=0\left(\frac{1}{1-r}\right)^\beta$$

for some universal constant $\beta<1$.

The mean growth of $|\Psi'|$ is one more way of measuring the distortion under Ψ and f.

A beautiful argument of Pommerenke [27] based on the Hardy-Stein-Spencer identity shows that for $\alpha \in \mathbb{R}$.

(8) $$I_\alpha(|\Psi'|,r)=0\left(\frac{1}{1-r}\right)^{\pi(\alpha)}$$

where $\pi(\alpha)= - \frac{1}{2} +\alpha+\sqrt{\frac{1}{4}-\alpha+4\alpha^2}$. Interpolating between this result for α small and negative and the trivial estimate $|\Psi'(z)| \geq c(1-|z|)$ gives (7). For $\alpha > 2/5$, the following sharp result is known to hold

(9) $$I_\alpha(|\Psi'|\ ,r)=0\left(\frac{1}{(1-r)^{3\alpha-1}}\right)$$

See [10], [26, chap.5].

It was conjectured for some time that (9) would hold for $\alpha > 1/3$. But Makarov disproved this in [22].

φ4. <u>What are the sharp exponent of growth of the means of</u> $|\varphi'|$?

Let us denote that sharp exponent by $\lambda(\alpha)$, $\alpha \in \mathbb{R}$. Then Brennan's conjecture would follow from $\lambda(-2)=1$.

We should remark that for α small in absolute value the example of Makarov in [22] shows that $\lambda(\alpha) \geq c\alpha^2$, for some constant $c > 0$.

The deepest result delaing with conformal mappings is L. de Branges theorem, [7]: if the Taylor expansion about 0 of φ is $\varphi(z) = \sum_{n=o}^{\infty} a_n z^n$ then

$$|a_n| \leq n|a_1| \tag{9}$$

Here we are not concerned with such sharp estimates but with orders of growth. And so instead of (9) we will be contented with its much simpler predecessor

$$|a_n| \leq k \cdot n \cdot |a_1| \tag{10}$$

where k is an absolute constant.

The first proof of (10) is due to Littlewood: one studies how the 1-mean of $|\varphi'|$ grows and then uses the trivial estimate

$$n|a_n| \leq k \int_{|z|=r_n} |f'(z)||dz| , \tag{11}$$

where k is an absolute constant and $r_n = 1 - \frac{1}{n}$.

The inseparable companion of the class $\mathcal{S}$ is the class Σ, that is, the class of the univalent functions defined in the complement of the unit disk and whose expansion at ∞ starts with the term z.

If $g \in \Sigma$ and $g(z) = z + \sum_{n=o}^{\infty} \frac{b_n}{z^n}$ then the area theorem shows that

$|b_n| \leq \frac{1}{n^{1/2}}$.

The exact order of decay of the coefficients of function in Σ, is still unkonown, and remains as one of the m·in challenges in this area of function theory. Let

$$\sigma_o = \sup \left\{ t : |b_n| = O\left(\frac{1}{n^t}\right) , \quad \text{for each } g \in \Sigma \right\}.$$

Thus $\sigma_o \geq \frac{1}{2}$. It is known that $\sigma_o < 1$. See [26] for the appropiate example. Also $\sigma_o > \frac{1}{2}$, but we shall discuss this latter.

The behaviour of the coefficients of bounded univalent functions is rather similar to that of functions in Σ. We denote by β_o the analogous constant. Then since the area of the image of a bounded univalent function is finite we see that $\beta_o \geq 1/2$. As before $\beta_o < 1$ and $\beta_o > 1/2$.

The argument of Littlewood to show (10) was later improved by Littlewood and Paley to treat odd univalent functions and actually shows that

$$\left.\begin{array}{ll} \text{if} & |\Psi(z)| = O\left(\frac{1}{1-|z|}\right)^{\alpha} \\ \text{then} & \int_{|z|=r} |f'(z)||dz| = O\left(\frac{1}{1-|z|}\right)^{\alpha} \end{array}\right\} \quad LP(\alpha) ,$$

as long as $\alpha > \frac{1}{2}$. See [26, chapter 5].

Recently, A. Baernstein [1] have shown that the result above is true also when $\alpha > \alpha'$, where α' is some absolute constant smaller than 1/2. We define $\alpha_o = \inf\{ \alpha \ni LP(\alpha) \text{ holds}\}$. Thus Baernstein's theorem claims that $\alpha_o < \frac{1}{2}$. It has been pointed out by Baernstein that it is not unlikely that $\beta_o = \sigma_o = 1 - \alpha_o$.

Q5. $\beta_o = \sigma_o = 1 - \alpha_o$?.

The main references in this connection are chapter 5 of Pommerenke's book [26] and Baernstein's paper [1].

The cornerstone in this area is the following result of Clunie and Pommerenke, [5]: There exist a constant $\lambda<1/2$ such that if Ψ is bounded and univalent

$$(12) \qquad \int_{|z|=r} |\Psi'(z)|\,|dz| = O\left(\frac{1}{1-r}\right)^{\lambda} .$$

More generally, one has, for any Ψ univalent in Δ , that

$$(13) \qquad \int_{|z|=r} \frac{|\Psi'(z)|}{|\Psi(z)|}\,|dz| = O\left(\frac{1}{1-r}\right)^{\lambda}$$

for some $\lambda<1/2$. The smallest λ for which (13) holds shall be denoted by λ_o. It can be shown, see [11], that

$$(14) \qquad \begin{aligned} &\beta_o \geqslant \sigma_o \geqslant 1-\lambda_o \\ &\beta_o \geqslant 1-\sigma_o \geqslant 1-\lambda_o \ . \end{aligned}$$

So we must ask

♀6. Is $\beta_o = 1-\lambda_o$?.

Notice that (14) and (13) give that $\sigma_o > 1/2$.

To get a picture of what f does, one may draw the level curves of $\mathrm{Re}\,\Psi$ and $\mathrm{Im}\,\Psi$, that is, the images under f of $\Omega \cap \{\mathrm{Re}\,w=t\}$ and $\Omega \cap \{\mathrm{Im}\,w=t\}$, $t \in \mathbb{R}$. This drawing is then a 'map' with orthogonal coordinates of the graph of Ψ. In general, the level set $\{z : \mathrm{Re}\,\Psi(z)=t\}$ is not a curve but a countable collection of them. Since the conformal mapping Ψ is in the Hardy class H^p, for any $p<\frac{1}{2}$, ([8]), it is clear that each one of those component curves have definite limits as they approach the boundary of the unit disk. And also, since $\iint_{\Omega} |f'(z)|^2 dx\,dy < \infty$, we have that almost all those level sets have finite length.

It was asked, by Piranian and Weitsman, whether or not all these level sets have finite length. The answer is a resounding yes, and it is the content of the following theorem of

Hayman and Wu [17].

Theorem: There exists a universal constant M_o such that for any straight line L,

$$\text{length}(f(\Omega \cap L)) \leqslant M_o \ .$$

A technically simpler proof has been given by Garnett, Gewing and Jones, [15].

This is a surprising result. One can not stop there, one has to ask which curves L in the plane have the proper: that for any pair (Ω, f)

(11) $$\text{lenght}(f(\Omega \cap L)) \leqslant M(L) \ ,$$

where M(L) is a constant which depends on L but not on Ω or f.

The curves with this property (11) must be what Ahlfors termed regular curves, that is, they must satisfy that

$$\text{length}(L \cap \{z \ : \ |z-a| < r\}) \leqslant C\, r$$

for any $a \in \mathbb{C}$, and $r > 0$, where C is a constant which does not depend or a or r .

Regular curves are also the curves for which the Cauchy integral furnishes a bounded operator from L^2 into L^2, [6]. See also [29].

It has been conjectured, [12], that they are precisely those curves which satisfy (11).

♀7. Do regular curves satisfy (11)?.

In [12] it was shown that regular quasicircles do satisfy (11), and in [14] it is shown that if L is a regular curve then any component of $\Omega \cap L$ is mapped under f into a curve of length at most M(L). See also [13] for more results on the distortion of curves.

The Hayman-Wu theorem claims that $|f'|_{|\Omega \cap \mathbb{R}}$ is in $L^1(dx)$.

But what one actually should expect is that $|f'|_{|\Omega \cap \mathbb{R}}$ is in $L^p(dx)$, for any $p < 2$. Even further it should hold that $|f'|_{|\Omega \cap \mathbb{R}}$ is of weak type L^2. This may be restated into the following question of A. Baernstein.

Q8. Does it hold that

$$\text{length}(f(E)) \leq c(\text{length}(E))^{1/2}$$

for any $E \subset \Omega \cap R$, where c is a constant which does not depend on E?

Hopefully, the problems described above will show that there are still intriguing aspects in the venerable subject of conformal mapping. For more background information on all of this we wish to refer the reader to the lecture notes [11].

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