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MATRICES WITH PRESCRIBED ROWS AND INVARIANT FACTORS(*)

By

Ion Zaballa(**)

1.- Introduction

Let $\mathbb{F}$ be an arbitrary field and $\mathbb{F}^{m\times n}$ the vector space over $\mathbb{F}$ of m by n matrices with elements in $\mathbb{F}$. In this paper we settle the following problem:

"*Let* $G \in \mathbb{F}^{n\times n}$ *be a matrix partitioned as follows:*

$$G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A \in \mathbb{F}^{p\times p}$, $D \in \mathbb{F}^{q\times q}$ *and* $n=p+q$. *Under what conditions do there exist matrices* C *and* D *such that* G *has prescribed invariant polynomials?*"

If instead of the blocks A and B we prescribe only the block A, the solution is known: E. Marques de Sã [13] and R.C. Thompson [14] gave the solution in terms of the interlacing inequalities which involve the invariant polynomials of G and A. However, these inequalities are not sufficient to sove the present problem. We will have to consider the Hardy, Littlewood and Polya majorization inequality which is denoted by the symbol $\prec$ and is defined as in [9,p.5 and 9].

Moreover, the above problem is closely connected with the Control Theory of Linear Systems. If $A \in \mathbb{F}^{p\times p}$ and $B \in \mathbb{F}^{p\times q}$, we define the controllability indices of the pair (A,B) just like in [1,p.275] and [12, Def. 1 and Remark 2.1].

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(**) Author's address: Departamento de Matemáticas. Escuela Universitaria de Magisterio. 01006.Vitoria-Gasteiz.Alava.

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Finally, if α is a polynomial with coefficients in $\mathbb{F}$ we denote its degree by $d(\alpha)$, and the symbol :> will means "divides".

2.- The Main Results

In these condition we can give the solution of the problem in the following way:

Theorem.- *Let* $A \in \mathbb{F}^{pxp}$, $B \in \mathbb{F}^{pxq}$, $G \in \mathbb{F}^{nxn}$ *and* $n=p+q$. *Let* $\alpha_1 :> \ldots :> \alpha_p$ *and* $\gamma_1 :> \ldots :> \gamma_n$ *be the invariant factors of* $[\lambda I_p - A, -B]$ *and* $\lambda I_n - G$, *respectively. Let* $k_1 \geqslant \ldots \geqslant k_q \geqslant 0$ *be the controllability indices of the pair* (A,B). *Then there exist matrices* $C \in \mathbb{F}^{qxp}$ *and* $D \in \mathbb{F}^{qxq}$ *such that* G *is similar to*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

if and only if the following relations hold:

$$\gamma_i :> \alpha_i :> \gamma_{i+q} \qquad i=1,\ldots,p.$$

$$(k_1+1,\ldots,k_q+1) \prec (d(\sigma_q),\ldots,d(\sigma_1))$$

where $\sigma_j = \frac{\beta^j}{\beta^{j-1}}$, $j=1,\ldots,q$, $\beta^j = \beta_1^j \ldots \beta_{p+j}^j$ *and* $\beta_i^j = \text{l.c.m.}(\alpha_{i-j}, \gamma_i)$, $i=1, \ldots, p+j$, $j=0,1,\ldots,q$.

This Theorem has some important consequences, we emphasize the following:

Corollary.- *With the same notations as in the Theorem, let* (A,B) *be a completely controllable pair. Then there exist matrices* $C \in \mathbb{F}^{qxp}$ *and* $D \in \mathbb{F}^{qxq}$ *such that* G *is similar to*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

if and only if the following relations hold:

$$\gamma_i = 1 \quad i=1,\ldots,p$$

$$(k_1+1,\ldots,k_q+1) \prec (d(\gamma_{p+q}),\ldots,d(\gamma_{p+1}))$$

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THIS PAPER IS GOING TO APPEAR I LINEAR ALGEBRA AND ITS APPLICATIONS.