EXTRACTA MATHEMATICAE 1, 3-5 (1986)

APPROXIMATION OF SYMMETRIC CONVEX BODIES

Ricardo Faro Rivas

Dpto. de Matemáticas. Fac. de Ciencias
Univ. de Extremadura. 06071 Badajoz. SPAIN

Pra Alicia, a miña filla

1. In R^2 we consider a norm $\|\ \|$ with unit ball B and unit sphere S. A homogeneous polynomial in R^2 of degree 2n is a function P: $R^2 \longrightarrow R$, that is the restriction to the diagonal of a 2n-linear function. Let P be a homogeneous polynomial, we call the associated polynomial body and polynomial sphere of P to the sets $B_P = \{x \in R^2 : P(x) \leq 1\}$, $S_P = \{x \in R^2 : P(x) = 1\}$.

2. In |2| we introduced three approximation criteria: area, width and radius and two types of approximation: exterior approximation (i.e. when we only consider polynomials P such that $B \subset B_P$) and interior approximation ($B_P \subset B$). And we prove the theorems about existence of best approximation. We call $\mathcal{A}_a^e(B)$, $\mathcal{A}_w^e(B)$, $\mathcal{A}_r^e(B)$ -for the exterior approximation- and $\mathcal{A}_a^i(B)$, $\mathcal{A}_w^i(B)$ and $\mathcal{A}_r^i(B)$ - for the interior approximation- the sets of the best approximation with the three criteria. In |3| we only consider the width and radius cases and we proved that $\mathcal{A}_w^e(B)$, $\mathcal{A}_r^e(B)$, $\mathcal{A}_w^i(B)$ and $\mathcal{A}_r^i(B)$ are singletons; that $S_P \cap S$ has at least 2n+2 points when B_P is in one of this four sets -**contact theorems**-; that S_P moves away from S in the maximal form in n+1 directions at least -**draw-back theorems**-; and that the four best approximation operators are continuous.

3. Now we consider the area criterion in the <u>exterior approximation</u>. We have that the unicity of best approximation is a simple consequence of the inequality (with the equality iff P=Q)

$$\langle 1 \rangle \qquad m[B_{tP+(1-t)Q}] \leq tm[B_P] + (1-t)m[B_Q]$$

where m is the Lebesgue measure.

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4. We have using <1> the contact theorem for the area case, that is a generalization of the ellipses of Loewner result (see Day |1|).

THEOREM (1). Let $B_P \in \mathcal{A}_a^e(B)$, then $S_P \cap S$ has at least 2n+2 points.

We prove this by assuming that $S_P \cap S$ has less than 2n+2 points and constructing another polynomial body external to B that encloses an area strictly smaller than that of B_P.

And we have too that

THEOREM (2). Let $B_P \in \mathcal{A}_a^e(B)$, and $B'=[S_P \cap S]$ (the convex enveloppe of $S_P \cap S$). Then we have that $B_P \in \mathcal{A}_a^e(B')$.

COROLLARY (3). Let E be the ellipse of minimal area circumscribed to B. Then if $E \cap S = \{x,y,-x,-y\}$, with $x \neq y$, we have that x and y are Birkhoff orthogonal.

5. We have too the continuity of the best approximation operator. If we call $\mathcal{C}_s$ to the family of all the compact, symmetric convex bodies with non-empty interior we have the

THEOREM (4). Let $B,B_1,B_2,\ldots \in \mathcal{C}_s$, and $B_{P_n} \in \mathcal{A}_a^e(B_n)$. If we have that $B_n \longrightarrow B$ (with the Hausdorff metric), and there exists t>0 such that $B_{P_n} \subset tB$ for all n, then we have that $P_n \longrightarrow P$ with $B_P \in \mathcal{A}_a^e(B)$.

For this result we use the continuity of $P \longrightarrow m[B_P]$ in the definite positive homogeneous polynomials, the convergence dominated theorem and the unicity of best approximation.

6. We have in the <u>interior approximation</u> -with the area criterion- that for the contact theorem we needed the inequality opposite to <1>, but this is not possible and we solve introducing some functions, calculating its derivatives and applying the convergence dominated theorem. This theorem is like (1).

7. For the continuity of the best approximation operator we obtain

THEOREM (5). Let $B,B_1,B_2,\ldots \in \mathcal{C}_s$, and $B_{P_n} \in \mathcal{A}_a^i(B_n)$. Now if we have that $B_n \longrightarrow B$ (in the Hausdorff metric), then there exists a limit P of P_n such that $B_P \in \mathcal{A}_a^i(B)$.

8. In the unicity problem we have that for n=1 the result is known

-due to **Loewner** (see Day |1|)- and very simple, but for $n \geqslant 2$ do not know the answer, and it seems not to be simple. But we can give a restricted answer following **Gruber and Kenderov** |4| in his polygonal approximation. We have that if we call

$$\mathcal{C}_{kn} = \{ C \in \mathcal{C}_s : \exists\, B_P, B_Q \in \mathcal{A}^i_a(C), \text{ with } m[B_P \Delta B_Q] \geqslant 1/k \}$$

(where P and Q are of degree 2n), then in $\mathcal{C}_s - \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{C}_{kn}$ we have the unicity of best approximation and

THEOREM (6). The set $\bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \mathcal{C}_{kn}$ is of first category **(with the topo-** logy of the Hausdorff metric, that is equivalent to the topology of the difference symmetric metric).

9. Finally we have a result dual to (2) which we call the caged amoeba theorem

THEOREM (7). Let $B_P \in \mathcal{A}^i_a(B)$ **(with** P **of** degree **2n and** $\mathcal{C}_{kn} = \phi$ **for all** $k \in \mathbb{N}$, **for example** n=1) **and** let $B_t \in \mathcal{C}_s$ **the convex that we construct from** B **with the tangents at the points of** $S_P \cap S$. **Then we have that** $B_P \in \mathcal{A}^i_a(B_t)$.

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