EXTRACTA MATHEMATICAE 1, 6-8 (1986)

AN ELEMENTARY PROOF OF THE INVARIANCE AND INVERSION OF CHARACTERISTIC PAIRS

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The purpose of this paper is to give a short and elementary proof of the invariance and the inversion of the characteristic pairs of an irreducible plane algebroid curve, using the concept of saturation given by Campillo in [2]. The inversion and invariance of these pairs is proved by Abhyankar in [1] using direct computations.

Let k be an algebraically closed field, p = charac. k. We will only consider subrings A of $k[[t]]$ containing $k[[x]]$ for some $x \in k[[t]]$ with $0 < \mathrm{ord}_t(x) < \infty$, and such that the quotient field of A is $F = k((t))$. Such an A is a complete local noetherian domain of Krull dimension 1 and its integral closure in F is $\bar{A} = k[[t]]$. Moreover one has $k \subset A$ and k is isomorphic to the residue field of A via the canonical map. If we denote by $v\colon k[[t]] \longrightarrow \mathbb{Z}_+ \cup \{\infty\}$ the order function relative to t, the semigroup of values of A is $S(A) = \{v(z) \mid z \in A,\ z \neq 0\}$ If S is an additive subsemigroup of $\mathbb{Z}_+$ such that $\mathbb{Z}_+ - S$ is a finite set, the monomial ring $A_S = \{\sum_{\gamma \in S} a_\gamma t^\gamma \mid a_\gamma \in k\}$ verifies the above conditions.

Let A (resp. S) as above. The ring A (resp. the semigroup S) is said to be saturated with respect to a non zero element $w \in A$ (resp. $m \in S$) if the following property holds:

(P_w) If $z \in A$, $z_1,\dots,z_r$, $w_1,\dots,w_s \in A - \{0\}$ and $l \in \mathbb{Z}$ are such that zz_i^{-1}, zw_j^{-1}, $(z_1\dots z_r)(w_1\dots w_s)^{-1} w^l \in \bar{A}$ then $z(z_1\dots z_r)(w_1\dots w_s)^{-1} w^l \in A$.

(A_m) If $\gamma,\gamma_1,\dots,\gamma_h \in S$ are such that for $i = 1,\dots,h$ either $\gamma \geq \gamma_i$ or $\gamma_i = m$, then one has $\gamma + e \in S$ where $e = \mathrm{g.c.d.}(\gamma_1,\dots,\gamma_h) \geq 0$).

(*) A.M.S. Subject Clasification (1980) 14-B-05, 14-H-20, 13-J-10.

Proposition.- Let A be a ring as above and let $S = S(A)$. If A is saturated with respect to w with $v(w) = m$ then S is saturated with respect to m. Conversely, if S is saturated with respect to m then A_S is saturated with respect to t^m. Finally, if $t^m \in A$, $m \not\equiv 0 \pmod p$, and A is saturated with respect to t^m then $A = A_S$.

The proof of the two first assertions consists in direct computations from the definitions. For the last one, observe that if A verifies (Pw) for some $w \in A - \{0\}$ and $z \in A$, then the set $A(z) = \{ w \in \bar{A} \mid wz \in A \}$ is a subring of $\bar{A}$ containing A, and so local and complete ($A(z)$ only depends on $\gamma = v(z)$ and its semigroup of values is $S(\gamma) = \{ \gamma' \in \mathbb{Z}_+ \mid \gamma' + \gamma \in S \}$). Now, let $\gamma \in S$ and take $z \in A$ such that $z = t^\gamma$ + higher order terms. One has $z(t^m)^\gamma z^{-m} \in A$, so $(t^\gamma z^{-1})^m \in A(z)$. By Hensel lemma and taking into account that $m \not\equiv 0 \pmod p$ one has $t^\gamma z^{-1} \in A(z)$, so $t^\gamma \in A$. This proves $A_S \subset A$ and the converse is evident.

Consider the case in which $A = k[[x,y]]$ where $n = v(x) \not\equiv 0 \pmod p$ and $m = v(y) \not\equiv 0 \pmod p$. One has a Puiseux's type parametrization

$$\text{(I)} \quad \begin{cases} x = t^n \\ y = \sum_{i>0} a_i x^{i/n} = \sum_{i>0} a_i t^i \end{cases}$$

, and consider the Puiseux's exponents $\{\beta_0,\dots,\beta_g\}$ given by $\beta_0 = n$, and $\beta_{\nu+1} = \min \{ i \mid a_i \neq 0$ and g.c.d. $(\beta_0,\dots,\beta_\nu,i) <$ g.c.d. $(\beta_0,\dots,\beta_\nu)\}$, g being characterized by g.c.d. $(\beta_0,\dots,\beta_g) = 1$.

Theorem (Invariance).- Let $\tilde{A}_x$ the minimum subring of $\bar{A}$ containing A and verifying (P_x). Then $S(\tilde{A}_x)$ is the minimum subsemigroup of $\mathbb{Z}_+$ containing $\beta_0,\dots\beta_g$ and verifying (A_n). In particular $\beta_0,\dots,\beta_g$ only depend on A and x, and if $n \leq m$, $\beta_0,\dots,\beta_g$ only depend on A.

For the proof, let e_ν = g.c.d. $(\beta_0,\dots,\beta_\nu)$ and let $S' = \{ \beta_i + le_i \mid 0 \leq i \leq g$ and $l \geq 0 \} \cup \{0\}$. S' is the minimum semigroup saturated with respect to n containing $\beta_0,\dots,\beta_g$, and from the proposition one proves $S' = S(\tilde{A}_x)$. On the other hand, observe that $\beta_0 = n$ and $\beta_{\nu+1} = \min \{ \gamma \in S' - \{0\} \mid (\gamma, e_\nu) < e_\nu \}$, so $\beta_0,\dots,\beta_g$ only depends on $S(\tilde{A}_x)$. Finally, if $n \leq m$ $\tilde{A}_x$ does not depend on the element x with $v(x) = n = \min(S(A) - \{0\})$.

Take $t^* \in \bar{A}$ such that $y = (t^*)^m$ and set $x = \sum_{j>0} b_j y^{j/m}$. Let $\beta_0^*, \ldots, \beta_{g^*}^*$ the Puiseux's exponents of this parametrization. The main technical result is the following:

<u>Theorem</u>.- If $n \leq m$ then $\tilde{A}_y = k + \tilde{A}_x(y)$.

First of all, from the property (P_x), $k + x\tilde{A}_x(y)$ is a subring of $\bar{A}$ verifying (P_y) and containing A, so $\tilde{A}_y \subset k + x\tilde{A}_x(y)$. Set $S = S(\tilde{A}_x)$ $S^* = S(\tilde{A}_y)$, $S^{**} = S(k + x\tilde{A}_x(y)) = \{ n + \gamma - m \mid \gamma \geq m \text{ and } \gamma \in S\} \cup \{0\}$. Since $t^n \in A$, from the proposition one has $\tilde{A}_y = A_{S^*}$ and $k + x\tilde{A}_x(y) = A_{S^{**}}$ so it is sufficient to see that $S^{**} \subset S^*$. Assume, without loss of generality, that $y = t^m$ + higher order terms. One has $(yt^{-m})^n \in \tilde{A}_y(x)$, so $yt^{-m} \in \tilde{A}_y(x)$ and $xyt^{-m} \in \tilde{A}_y = A_{S^*}$. Now, since $S = \{ \beta_i + le_i \mid 0 \leq i \leq g, \ l \geq 0\} \cup \{0\}$ and $n + \beta_i - m \in S^*$ (look at the t-expansion of xyt^{-m}) one has $S^{**} \subset S^*$, as S^* is saturated with respect to m.

On the other hand $\beta_0^*, \ldots, \beta_{g^*}^*$, (resp. $\beta_0, \ldots, \beta_g$) can be obtained from S^* (resp. from S) as in the proof of the proposition. Since $S^* = S^{**} = \{n + \gamma - m \mid \gamma \geq m, \ \gamma \in S\} \cup \{0\}$, one has:

<u>Theorem (Inversion formula)</u>.- With the assumptions as in the above theorem one has

a) If $n \nmid m$ then $g^* = g$, $\beta_0^* = \beta_1$, $\beta_i^* = n + \beta_i - m$, $i = 1,2,\ldots,g$.

b) If $n \mid m$ then $g^* = g+1$, $\beta_0^* = m$, $\beta_1^* = n$, $\beta_i^* = n + \beta_{i-1} - m$, $i = 2,\ldots,g+1$

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